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Asymptotic conditional distribution of exceedance counts: fragility index with different margins

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Abstract We consider a random vector $\boldsymbol{X}$, whose components are neither necessarily independent nor identically distributed. The fragility index (FI), if it exists, is defined as the limit of the expected number of exceedances among the components of $\boldsymbol{X}$ above a high threshold, given that there is at least one exceedance. It measures the asymptotic stability of the system of components. The system is called stable if the FI is one and fragile otherwise. In this paper, we show that the asymptotic conditional distribution of exceedance counts exists, if the copula of $\boldsymbol{X}$ is in the domain of attraction of a multivariate extreme value distribution, and if the marginal distribution functions satisfy an appropriate tail condition. This enables the computation of the FI corresponding to $\boldsymbol{X}$ and of the extended FI as well as of the asymptotic distribution of the exceedance cluster length also in that case, where the components of $\boldsymbol{X}$ are not identically distributed.

Keywords Exceedance over high threshold · Fragility index · Extended fragility index · Multivariate extreme value theory · Peaks-over-threshold approach · Copula · Exceedance cluster length

1 Introduction

Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random vector (rv), whose components are identically distributed but not necessarily independent. The number of exceedances among

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$X_1, \ldots, X_d$ above the threshold s is denoted by $N_s := \sum_{i=1}^d 1_{(s,\infty)}(X_i)$. The fragility index (FI) corresponding to $\boldsymbol{X}$ is the asymptotic conditional expected number of exceedances, given that there is at least one exceedance, i.e., $\mathrm{FI} = \lim_{s\nearrow} E(N_s \mid N_s > 0)$. The FI was introduced in Geluk et al. (2007) to measure the stability of the stochastic system $\{X_1, \ldots, X_d\}$. The system is called *stable* if $\mathrm{FI} = 1$, otherwise it is called *fragile*.

In the two-dimensional case, the FI is directly linked to the upper tail dependence coefficient $\lambda^{\mathrm{up}} := \lim_{t\downarrow 0} P(X_2 > F_2^{-1}(1-t) \mid X_1 > F_1^{-1}(1-t))$, which goes back to Geffroy (1958, 1959) and Sibuya (1960). We have $\mathrm{FI} = 2/(2-\lambda^{\mathrm{up}})$, provided the df F_1, F_2 of X_1, X_2 are continuous and λ^{up} exists. In contrast to the upper tail dependence coefficient, the FI presents a measure for tail dependence in arbitrary dimensions.

In Falk and Tichy (2012), the asymptotic conditional distribution $p_k := \lim_{s\nearrow} P(N_s = k \mid N_s > 0)$ of the number of exceedances was investigated, given that there is at least one exceedance, $1 \le k \le d$.

It turned out that this *asymptotic conditional distribution of exceedance counts* (ACDEC) exists, if the copula C corresponding to $\boldsymbol{X}$ is in the domain of attraction of a (multivariate) extreme value distribution (EVD) G, denoted by $C \in D(G)$, i.e. $C^n((1+\frac{x_1}{n}, \ldots, 1+\frac{x_d}{n})) \to_{n\to\infty} G(\boldsymbol{x}), \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$.

In this paper, we investigate the ACDEC, dropping the assumption that the margins X_i, $1 \le i \le d$, are identically distributed. This will be done in Sect. 2. If the ACDEC exists then the FI exists and we have, in particular, $\mathrm{FI} = \sum_{k=1}^d kp_k$. In Sect. 3 we will compute the FI under quite general conditions on $\boldsymbol{X}$.

The extended fragility index $\mathrm{FI}(m)$ is the extension of the $\mathrm{FI} = \mathrm{FI}(1)$ through the condition that there are at least $m \ge 1$ exceedances, i.e.,

$$\mathrm{FI}(m) = \lim_{s\nearrow} E(N_s \mid N_s \ge m) = \frac{\sum_{k=m}^d kp_k}{\sum_{k=m}^d p_k},$$

if the ACDEC exists. But now we encounter the problem that the denominator in the definition of $\mathrm{FI}(m)$ may vanish: $\sum_{k=m}^d p_k = 0$. In Sect. 4 we will establish a characterization of $\sum_{k=m}^d p_k = 0$ in terms of tools from multivariate extreme value theory.

The total number of sequential time points at which a stochastic process exceeds a high threshold is an *exceedance cluster length*. The asymptotic distribution as the threshold increases of the remaining exceedance cluster length, conditional on the assumption that there is an exceedance at index $\kappa \in \{1, \ldots, d\}$, will be computed for $\boldsymbol{X} = (X_1, \ldots, X_d)$ in Sect. 5. It turns out that this can be expressed in terms of the minimum of a *generator* of the D-norm, cf. Eq. (2).

2 ACDEC

By Sklar's Theorem (see, for example, Nelsen 2006, Theorem 2.10.9) we can assume the representation $(X_1, \ldots, X_d) = (F_1^{-1}(U_1), \ldots, F_d^{-1}(U_d))$, where F_i is the (univariate) distribution function (df) of X_i, $1 \le i \le d$, and the rv $\boldsymbol{U} = (U_1, \ldots, U_d)$

follows a copula on $\mathbb{R}^d$, i.e., each U_i is uniformly on $(0, 1)$ distributed, $1 \le i \le d$. By $F^{-1}(q) := \inf\{t \in \mathbb{R} : F(t) \ge q\}, q \in (0, 1)$, we denote the generalized inverse of a df F.

The following condition is crucial for the present paper. It substitutes the condition of equal margins $F_1 = \cdots = F_d$ in Falk and Tichy (2012). By $\omega(F) := \sup\{F^{-1}(q) : q \in (0, 1)\} = \sup\{t \in \mathbb{R} : F(t) < 1\}$ we denote the upper endpoint of a df F.

We require throughout the existence of an index $\kappa \in \{1, \ldots, d\}$ with $\omega(F_\kappa) =: \omega^*$, such that

$$\lim_{s \uparrow \omega^*} \frac{1 - F_i(s)}{1 - F_\kappa(s)} = \gamma_i \in [0, \infty), \quad 1 \le i \le d. \tag{C}$$

Note that condition (C) implies $\omega(F_i) \le \omega^*$ for each i, since otherwise we would get $\gamma_i = \infty$, which is excluded. We, thus, have $\omega^* = \max_{i \le d} \omega(F_i)$.

The following result is taken from Aulbach et al. (2011). By $\boldsymbol{e}_i$ we denote the i-th unit vector in $\mathbb{R}^d$, $1 \le i \le d$; all operations on vectors such as $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$ are meant componentwise.

Proposition 1 *An arbitrary copula C on $\mathbb{R}^d$ is in the domain of attraction of an EVD G if and only if there exists a norm $\|\cdot\|_D$ on $\mathbb{R}^d$ with $\|\boldsymbol{e}_i\|_D = 1$, $1 \le i \le d$, such that*

$$C(\boldsymbol{y}) = 1 - \|\boldsymbol{y} - \boldsymbol{1}\|_D + o\left(\|\boldsymbol{y} - \boldsymbol{1}\|_D\right),$$

uniformly for $\boldsymbol{y} \in [0, 1]^d$. In this case $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$.

The following result is an immediate consequence of Proposition 1 and the equivalence $F^{-1}(q) \le t \iff q \le F(t), q \in (0, 1), t \in \mathbb{R}$, which holds for an arbitrary df F.

Corollary 1 *Suppose that the copula C corresponding to the rv $\boldsymbol{X}$ is in the domain of attraction of an EVD G and that condition (C) is satisfied. Then there exists a norm $\|\cdot\|_D$ on $\mathbb{R}^d$ with $\|\boldsymbol{e}_i\|_D = 1$, $1 \le i \le d$, such that for any nonempty index set $K \subset \{1, \ldots, d\}$*

$$P(X_k \le s, k \in K) = 1 - (1 - F_\kappa(s)) \left\| \sum_{k \in K} \gamma_k \boldsymbol{e}_k \right\|_D + o(1 - F_\kappa(s))$$

as $s \uparrow \omega^*$.

The following result provides the asymptotic unconditional distribution of exceedance counts.

Lemma 1 *Under the conditions of Corollary 1 we obtain with $c := 1 - F_\kappa(s)$*

$$\begin{aligned} a_k &:= \lim_{s \uparrow \omega^*} \frac{P(N_s = k)}{c} \\ &= \sum_{0 \le j \le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{\emptyset \ne T \subset \{1,\ldots,d\} \\ |T| = d-j}} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D \end{aligned}$$

for $1 \le k \le d$, *and*

$$a_0 := \lim_{s \uparrow \omega^*} \frac{1 - P(N_s = 0)}{c} = \left\| \sum_{j=1}^{d} \gamma_j \boldsymbol{e}_j \right\|_D .$$

Proof Corollary 1 implies

$$P(N_s = 0) = 1 - c \left\| \sum_{j=1}^{d} \gamma_j \boldsymbol{e}_j \right\|_D + o(c),$$

for $s \uparrow \omega^*$.

From the well-known additivity formula, Corollary 1 and the equality $\sum_{\emptyset \ne T \subset S} (-1)^{|T|+1} = 1$ for any nonempty subset $S \subset \{1, \ldots, d\}$, we obtain for $1 \le k \le d$ as $s \uparrow \omega^*$

$$\begin{aligned}
&P(N_s = k)\\
&= \sum_{\substack{S \subset \{1,\ldots,d\}\\ |S|=k}} P\left(X_i > s,\ i \in S,\ X_j \le s,\ j \in S^{\complement}\right)\\
&= \sum_{\substack{S \subset \{1,\ldots,d\}\\ |S|=k}} P\left(X_i > s,\ i \in S \mid X_j \le s,\ j \in S^{\complement}\right) P\left(X_j \le s,\ j \in S^{\complement}\right)\\
&= \sum_{\substack{S \subset \{1,\ldots,d\}\\ |S|=k}} \left(1 - \sum_{\emptyset \ne T \subset S} (-1)^{|T|+1} P\left(X_i \le s,\ i \in T \mid X_j \le s,\ j \in S^{\complement}\right)\right)\\
&\quad \times P\left(X_j \le s,\ j \in S^{\complement}\right)\\
&= \sum_{\substack{S \subset \{1,\ldots,d\}\\ |S|=k}} \left(P\left(X_j \le s,\ j \in S^{\complement}\right) - \sum_{\emptyset \ne T \subset S} (-1)^{|T|+1} P\left(X_i \le s,\ i \in T \cup S^{\complement}\right)\right)\\
&= \sum_{\substack{S \subset \{1,\ldots,d\}\\ |S|=k}} \left(1 - c \left\| \sum_{j \in S^{\complement}} \gamma_j \boldsymbol{e}_j \right\|_D - \sum_{\emptyset \ne T \subset S} (-1)^{|T|+1} \left(1 - c \left\| \sum_{j \in T \cup S^{\complement}} \gamma_j \boldsymbol{e}_j \right\|_D\right)\right)\\
&\quad + o(c)\\
&= c \sum_{\substack{S \subset \{1,\ldots,d\}\\ |S|=k}} \sum_{T \subset S} (-1)^{|T|+1} \left\| \sum_{j \in T \cup S^{\complement}} \gamma_j \boldsymbol{e}_j \right\|_D + o(c).
\end{aligned}$$

With a suitable index transformation we get

$$
\begin{aligned}
P(N_s = k) &= c \sum_{\substack{S\subset\{1,\dots,d\}\\|S|=k}} \sum_{0\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subset S\\|K|=r}} \left\| \sum_{\substack{i\in K\cup S^{\complement}:=T\\|T|=r+d-k}} \gamma_i \boldsymbol{e}_i \right\|_D + o(c)\\
&= c \sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T\subset\{1,\dots,d\}\\|T|=d-j}} \left\| \sum_{j\in T} \gamma_j \boldsymbol{e}_j \right\|_D + o(c),
\end{aligned}
$$

which completes the proof of Lemma 1. □

Note that $a_0 > 0$ as $\gamma_k = 1$ and that $a_k \ge 0$, $1 \le k \le d$, in Lemma 1. The following main result of this section is, therefore, an immediate consequence of Lemma 1. It provides the ACDEC also in the case, where the components X_i of the rv $\boldsymbol{X} = (X_1, \dots, X_d)$ are not identically distributed.

Theorem 1 (ACDEC) *Under the conditions of Corollary 1 we have that the limits*

$$
p_k := \lim_{s\uparrow\omega^*} P(N_s = k \mid N_s > 0) = \frac{a_k}{a_0}, \quad 1 \le k \le d,
$$

exist and that they define a probability distribution on $\{1, \dots, d\}$.

fOR the usual λ-norm $\|\boldsymbol{x}\|_\lambda = (\sum_{1\le i\le d} |x_i|^\lambda)^{1/\lambda}$, $\boldsymbol{x} \in \mathbb{R}^d$, $\lambda \in [1, \infty)$, we obtain, for example, $a_0 = (\sum_{1\le i\le d} \gamma_i^\lambda)^{1/\lambda}$ and

$$
a_k = \sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{\emptyset\ne T\subset\{1,\dots,d\}\\|T|=d-j}} \left(\sum_{i\in T} \gamma_i^\lambda \right)^{1/\lambda}, \quad 1 \le k \le d.
$$

For $\lambda = 1$, which is the case of independent margins of G, we obtain, in particular, $a_0 = \sum_{1\le i\le d} \gamma_i = a_1$, $a_k = 0$, $2 \le k \le d$, and, thus, $p_1 = 1$, $p_k = 0$, $2 \le k \le d$.

3 The fragility index

The following theorem is the main result of this section.

Theorem 2 *Under the conditions of Corollary 1 we have*

$$
\mathrm{FI} = \frac{\sum_{i=1}^d \gamma_i}{\left\| \sum_{i=1}^d \gamma_i \boldsymbol{e}_i \right\|_D} \in [1, d].
$$

Proof We have

$$\begin{aligned}
E(N_s \mid N_s > 0) &= \sum_{i=1}^{d} E\left(1_{(s,\infty)}(X_i) \mid N_s > 0\right) \\
&= \sum_{i=1}^{d} \frac{P(X_i > s)}{1 - P(N_s = 0)} \\
&= \sum_{i=1}^{d} \frac{1 - F_i(s)}{1 - F_\kappa(s)} \frac{1 - F_\kappa(s)}{1 - P(N_s = 0)} \\
&\to_{s \uparrow \omega^*} \frac{\sum_{i=1}^{d} \gamma_i}{\left\| \sum_{i=1}^{d} \gamma_i \boldsymbol{e}_i \right\|_D}.
\end{aligned}$$

by Lemma 1 and condition (C). □

It is well known that an arbitrary D-norm satisfies the inequality $\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_D \le \|\boldsymbol{x}\|_1$, $\boldsymbol{x} \ge \boldsymbol{0} \in \mathbb{R}^d$; see, for example Falk et al. (2010, 4.37). The range of the FI in Theorem 2 is, consequently, $[1, d]$.

Suppose that $\gamma_i > 0$, $1 \le i \le d$. Then it follows from Takahashi (1988) that

$$\left\| \sum_{i=1}^{d} \gamma_i \boldsymbol{e}_i \right\|_D = \sum_{i=1}^{d} \gamma_i \iff \|\cdot\|_D = \|\cdot\|_1,$$

where $\|\cdot\|_D = \|\cdot\|_1$ is the case of independence of the margins of G. We, thus, obtain in case $\gamma_i > 0$, $1 \le i \le d$,

$$\mathrm{FI} = 1 \iff \|\cdot\|_D = \|\cdot\|_1 \iff \text{independence of the margins of } G.$$

In case of complete dependence of G, i.e., if $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_\infty = \max_{1 \le i \le d} |x_i|$, we obtain for general $\gamma_i \ge 0$ that $\mathrm{FI} = \sum_{i=1}^{d} \gamma_i / \max_{1 \le i \le d} \gamma_i$.

Example 1 (Weighted Pareto) Let $Y_1, \ldots, Y_m$ be independent and identically Pareto distributed rv with parameter $\alpha > 0$. Put $X_i := \sum_{j=1}^{m} \lambda_{ij} Y_j$, $1 \le i \le d$, where the weights λ_{ij} are nonnegative and satisfy $\sum_{j=1}^{m} \lambda_{ij}^{\alpha} = 1$, $1 \le i \le d$.

The df of the rv $\boldsymbol{X} = (X_1, \ldots, X_d)$ is in the domain of attraction of the EVD

$$G^*(\boldsymbol{s}) = \exp\left(- \sum_{j=1}^{m} \max_{i \le d} \left(\frac{\lambda_{ij}}{s_i} \right)^{\alpha} \right), \quad \boldsymbol{s} = (s_1, \ldots, s_d) > \boldsymbol{0},$$

with standard Fréchet margins $G_k(s) = \exp(-s^{-\alpha})$, $s > 0$, $1 \le k \le d$. This can be seen by proving that for $\boldsymbol{s} > \boldsymbol{0} \in \mathbb{R}^d$

$$P\left(\sum_{j=1}^{m}\lambda_{ij}Y_j \le n^{1/\alpha}s_i,\ 1\le i\le d\right) = 1-\frac{1}{n}\left(\sum_{j=1}^{m}\max_{i\le d}\left(\frac{\lambda_{ij}}{s_i}\right)^{\alpha}+o(1)\right),$$

which follows from tedious but elementary computations, using conditioning on $Y_j = y_j$, $j = 2,\dots,m$.

As a consequence we obtain that the copula pertaining to $\boldsymbol{X}$ is in the domain of attraction of $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x}\le\boldsymbol{0}\in\mathbb{R}^d$, where $\|\boldsymbol{x}\|_D := \sum_{j=1}^{m}(\max_{i\le d}(\lambda_{ij}^{\alpha}|x_i|))$, $\boldsymbol{x}\in\mathbb{R}^d$.

From Embrechts et al. (1997, Lemma A 3.26) we obtain that the df F_i of X_i satisfies $1-F_i(s)\sim s^{-\alpha}\sum_{j=1}^{m}\lambda_{ij}^{\alpha} = s^{-\alpha}$, $1\le i\le d$, as $s\to\infty$ and, thus,

$$\gamma_i = \lim_{s\to\infty}\frac{1-F_i(s)}{1-F_\kappa(s)} = 1,\quad 1\le i\le d,$$

where $\kappa\in\{1,\dots,d\}$ can be chosen arbitrarily. As a consequence we obtain for the fragility index

$$\mathrm{FI} = \frac{\sum_{i=1}^{d}\gamma_i}{\left\|\sum_{i=1}^{d}\gamma_i\boldsymbol{e}_i\right\|_D} = \frac{d}{\sum_{j=1}^{m}\max_{i\le d}\lambda_{ij}^{\alpha}}.$$

Example 2 (GPD-Copula) Take an arbitrary rv $\boldsymbol{Z}$ that realizes in $[0,c]^d$ and which satisfies $E(Z_i) = 1, 1\le i\le d$. Choose $\beta_1,\dots,\beta_d > 0$ and let U be a rv, which is uniformly on $(0,1)$ distributed and that is independent of $\boldsymbol{Z}$. Put $\boldsymbol{X} := (\beta_1 Z_1,\dots,\beta_d Z_d)/U$. Then $F_i(x) = P(X_i\le x) = 1-\frac{\beta_i}{x}$, $x\ge c\beta_i$, $1\le i\le d$, and the copula of $\boldsymbol{X}$ is in the domain of attraction of the EVD $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x}\le\boldsymbol{0}\in\mathbb{R}^d$, with $\|\boldsymbol{x}\|_D = E(\max_{1\le i\le d}(|x_i|\,Z_i))$, $\boldsymbol{x}\in\mathbb{R}^d$.

Let $\beta_\kappa = \max_{1\le i\le d}\beta_i$. Then we have

$$\frac{1-F_i(s)}{1-F_\kappa(s)} = \frac{\beta_i}{\beta_\kappa} =: \gamma_i,\quad s\ge c\beta_\kappa,\ 1\le i\le d,$$

and we obtain for the fragility index corresponding to $\boldsymbol{X}$

$$\mathrm{FI} = \frac{\sum_{i=1}^{d}\gamma_i}{E\left(\max_{1\le i\le d}\gamma_i Z_i\right)}.$$

Note that the copula C of $\boldsymbol{X}$ is actually a *GPD copula* ((multivariate) generalized Pareto distribution), characterized by the equation $C(\boldsymbol{u}) = 1-\|\boldsymbol{1}-\boldsymbol{u}\|_D$ for $\boldsymbol{u}\in[0,1]^d$ close to $\boldsymbol{1}$, see Aulbach et al. (2011). If $Z_1 = \cdots = Z_d$ a.s., then we obtain the maximum-norm $\|\boldsymbol{x}\|_D = \max_{1\le i\le d}|x_i|$, and $\mathrm{FI} = \sum_{i=1}^{d}\gamma_i/\max_{1\le i\le d}\gamma_i$.

4 The extended fragility index

The extended FI is the asymptotic expected number of exceedances above a high threshold, conditional on the assumption that there are at least $m \geq 1$ exceedances:

$$\mathrm{FI}(m) := \lim_{s \nearrow} E(N_s \mid N_s \geq m), \quad 1 \leq m \leq d.$$

If the ACDEC corresponding to $X_1, \dots X_d$ exists, then, obviously,

$$\mathrm{FI}(m) = \frac{\sum_{k=m}^{d} k p_k}{\sum_{k=m}^{d} p_k}, \quad 1 \leq m \leq d. \tag{1}$$

But now we encounter the problem that we might divide by 0 in (1), i.e., $\sum_{k=m}^{d} p_k$ can vanish if $m \geq 2$. This is, for example, true for the L_1-norm. But there are other norms in dimension $d \geq 3$ such that $\sum_{k=m}^{d} p_k = 0$, see Falk and Tichy (2012). In this section we establish a characterization of $\sum_{k=m}^{d} p_k = 0$ also in that case, where the initial $X_1, \dots, X_d$ follow different distributions.

Lemma 2 *Assume the conditions of Corollary 1 and put* $I := \{i \in \{1, \dots, d\} : \gamma_i = 0\}$. *Then we obtain* $\sum_{k=m}^{d} p_k = 0$ *for* $m > m^* := \left| I^{\complement} \right| = d - |I|$.

Proof Without loss of generality we can assume that $I \neq \emptyset$. Recall, moreover, that $\gamma_\kappa = 1$, i.e., $I \neq \{1, \dots, d\}$ as well. We have

$$a_k = \lim_{s \uparrow \omega^*} \frac{P(N_s = k)}{1 - F_\kappa(s)} = \lim_{s \uparrow \omega^*} \sum_{\substack{S \subset \{1,\dots,d\} \\ |S| = k}} \frac{P\left(X_i > s,\ i \in S,\ X_j \leq s,\ j \in S^{\complement}\right)}{1 - F_\kappa(s)}.$$

If $|S| = k \geq m^* + 1$, then S must contain an index i_S, say, with $i_S \in I$. We, thus, obtain for $k \geq m^* + 1$

$$a_k \leq \limsup_{s \uparrow \omega^*} \sum_{\substack{S \subset \{1,\dots,d\} \\ abs S = k}} \frac{P(X_{i_S} > s)}{1 - F_\kappa(s)} = \sum_{\substack{S \subset \{1,\dots,d\} \\ |S| = k}} \lim_{s \uparrow \omega^*} \frac{1 - F_{i_S}(s)}{1 - F_\kappa(s)} = 0.$$

□

The following characterization is the main result of this section. It is formulated in terms of different representations of a multivariate EVD G on $\mathbb{R}^d$ with standard negative exponential margins $G(x\boldsymbol{e}_i) = \exp(x), x \leq 0, 1 \leq i \leq d$. We have for $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$

$$G(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_D\right) \quad \text{(Hofmann)}$$

$$= \exp\left(-\int_{S_d} \max(-u_i x_i)\, \mu(\mathrm{d}\boldsymbol{u})\right) \quad \text{(Pickands–de Haan-Resnick)}$$

$$= \exp\left(-\nu\left([-\infty, \boldsymbol{x}]^{\complement}\right)\right), \qquad \text{(Balkema–Resnick)}$$

where $\|\cdot\|_D$ is some norm on $\mathbb{R}^d$ with $\|\boldsymbol{e}_i\|_D = 1, 1 \le i \le d$, μ is the *angular measure* on the unit simplex $S_d = \left\{\boldsymbol{u} \in [0,1]^d : \sum_{i \le d} u_i = 1\right\}$, satisfying $\mu(S_d) = d$, $\int_{S_d} u_i \, \mu(d\boldsymbol{u}) = 1, 1 \le i \le d$, and ν is the σ-finite *exponent measure* on $[-\infty, 0]^d \setminus \{\infty\}$; for details we refer to Falk et al. (2010). We also include the fact that each D-norm can be generated by nonnegative and bounded rv $Z_1, \ldots, Z_d$ with $E(Z_i) = 1, 1 \le i \le d$, as

$$\|\boldsymbol{x}\|_D = E\left(\max_{1 \le i \le d}(|x_i|\, Z_i)\right), \quad \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d. \tag{2}$$

This is a consequence of the Pickands–de Haan-Resnick representation. The rv $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ is called *generator* of $\|\cdot\|_D$. Note that each rv $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ of nonnegative and bounded rv Z_i with $E(Z_i) = 1$ generates a D-norm via Eq. (2).

Proposition 2 *Assume the conditions of Corollary 1 and put* $I = \{i \in \{1, \ldots, d\} : \gamma_i = 0\}$. *Then we have* $\sum_{k=m}^{d} p_k = 0$ *for some* $m \le m^* = \left|I^{\complement}\right|$ *if and only if we have for each subset* $K \subset I^{\complement}$ *with at least* m *elements*

$$\lim_{s \uparrow \omega^*} \frac{P(X_k > s,\ k \in K)}{1 - F_\kappa(s)} = 0 \tag{3}$$

$$\begin{aligned}
&\iff \sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} x_i \boldsymbol{e}_i \right\|_D = 0 \quad \textit{for all } \boldsymbol{x} \ge \boldsymbol{0} \in \mathbb{R}^d \\
&\iff \sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D = 0 \\
&\iff \min_{k \in K} Z_k = 0 \quad a.s. \qquad\qquad (4)\\
&\iff \mu\left(\left\{\boldsymbol{u} \in S_d : \min_{i \in K} u_i > 0\right\}\right) = 0 \\
&\iff \nu\left(\times_{k \in K}(-\infty, 0] \times_{i \notin K} [-\infty, 0]\right) = 0,
\end{aligned}$$

i.e., the projection $\nu_K := \nu * (\pi_i,\ i \in K)$ *of the exponent measure* ν *onto its components* $i \in K$ *is the null measure on* $(-\infty, 0]^{|K|}$.

While in the (bivariate) case $K = \{k_1, k_2\}$ the condition

$$\begin{aligned}
&\sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D = 0 \\
&\iff \left\|\boldsymbol{e}_{k_1}\right\|_D + \left\|\boldsymbol{e}_{k_2}\right\|_D - \left\|\boldsymbol{e}_{k_1} + \boldsymbol{e}_{k_2}\right\|_D = 0 \\
&\iff \left\|\boldsymbol{e}_{k_1} + \boldsymbol{e}_{k_2}\right\|_D = 2 = \left\|\boldsymbol{e}_{k_1} + \boldsymbol{e}_{k_2}\right\|_1
\end{aligned}$$

implies by Takahashi's Theorem (1988) independence of the marginal distributions k_1, k_2 of the EVD $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D), \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, this is no longer true for $|K| \ge 3$. Take, for example, a rv ξ that attains only the values 1; 2; 3 with probability 1/6; 1/3; 1/2 and put

$$Z_1 := \begin{cases} 0 & \text{if } \xi = 1 \\ \frac{6}{5} & \text{elsewhere} \end{cases}, \quad Z_2 := \begin{cases} 0 & \text{if } \xi = 2 \\ \frac{3}{2} & \text{elsewhere} \end{cases}, \quad Z_3 := \begin{cases} 0 & \text{if } \xi = 3 \\ 2 & \text{elsewhere} \end{cases}.$$

Then $E(Z_i) = 1, i = 1, 2, 3, \min_{1 \le i \le 3} Z_i = 0, E(\max_{1 \le i \le 3} Z_i) < 3$ as well as $E(\max(Z_i, Z_j)) < 2$ for all $1 \le i \ne j \le 3$, i.e., there is no marginal independence among Z_1, Z_2, Z_3.

Proof We have by Theorem 1 and Lemma 1

$$\begin{aligned} &\sum_{k=m}^{d} p_k = 0 \\ &\iff \lim_{s \uparrow \omega^*} \frac{P(N_s \ge m)}{1 - F_\kappa(s)} = 0 \\ &\iff \lim_{s \uparrow \omega^*} \frac{P\left(\bigcup_{\substack{K \subset \{1,\dots,d\} \\ |K| \ge m}} \{X_k > s,\ k \in K\}\right)}{1 - F_\kappa(s)} = 0 \\ &\iff \lim_{s \uparrow \omega^*} \frac{P(X_k > s,\ k \in K)}{1 - F_\kappa(s)} = 0 \text{ for any } K \subset \{1, \dots, d\} \text{ with } |K| \ge m \\ &\iff \lim_{s \uparrow \omega^*} \frac{P(X_k > s,\ k \in K)}{1 - F_\kappa(s)} = 0 \text{ for any } K \subset I^{\complement} \text{ with } |K| \ge m, \end{aligned}$$

which is equivalence (3). Note that $\sum_{T \subset K} (-1)^{|T|-1} \max_{i \in T} a_i = \min_{k \in K} a_k$ for any set $\{a_k : k \in K\}$ of real numbers, which can be seen by induction. We, consequently, have

$$\sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D = \sum_{T \subset K} (-1)^{|T|-1} E\left(\max_{i \in T} Z_i\right) = E\left(\min_{i \in T} Z_i\right)$$

and, thus,

$$\sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D = 0 \iff E\left(\min_{i \in T} Z_i\right) = 0 \iff \min_{k \in K} Z_k = 0 \; a.s.$$

The other equivalences follow from Proposition 5.2 in Falk and Tichy (2012). $\square$

5 Exceedance cluster lengths

The total number of sequential time points at which a stochastic process exceeds a high threshold is an exceedance cluster length. The mathematical tools developed in the preceding sections enable the computation of its distribution as well. Precisely, denote by $L_\kappa(s)$ the number of sequential exceedances above the threshold s, if we have an exceedance at $\kappa \in \{1, \dots, d\}$, i.e.

$$L_\kappa(s) := \sum_{k=0}^{d-\kappa} k 1 \left(X_\kappa > s, \dots, X_{\kappa+k} > s, X_{\kappa+k+1} \le s\right).$$

We have, in particular, $L_d(s) = 0 = L_\kappa(s)$, if $X_{\kappa+1} \le s$. We suppose throughout this section that condition (C) holds for the index $\kappa \in \{1, \dots, d\}$. The following auxiliary result will be crucial.

Lemma 3 *Assume the conditions of Corollary 1. Then we obtain for $\kappa \in \{1, \dots, d\}$ as $s \nearrow \omega^*$*

$$\begin{aligned} P\left(L_\kappa(s) \ge k \mid X_\kappa > s\right) &= P\left(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s\right) \\ &= \sum_{\emptyset \ne T \subset \{\kappa, \dots, \kappa+k\}} (-1)^{|T|+1} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(1) \\ &=: s_\kappa(k) + o(1), \quad 0 \le k \le d - \kappa. \end{aligned}$$

Proof From the additivity formula we obtain

$$\begin{aligned} &P\left(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s\right) \\ &= \frac{1 - P\left(\bigcup_{0 \le i \le k} \{X_{\kappa+i} \le s\}\right)}{1 - F_\kappa(s)} \\ &= \frac{1 - \sum_{\emptyset \ne T \subset \{\kappa, \dots, \kappa+k\}} (-1)^{|T|+1} P\left(X_i \le s,\ i \in T\right)}{1 - F_\kappa(s)} \\ &= \frac{1 - \sum_{\emptyset \ne T \subset \{\kappa, \dots, \kappa+k\}} (-1)^{|T|+1} \left(1 - c \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D\right) + o(1 - F_\kappa(s))}{1 - F_\kappa(s)} \\ &= \sum_{\emptyset \ne T \subset \{\kappa, \dots, \kappa+k\}} (-1)^{|T|+1} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(1). \end{aligned}$$

□

Corollary 2 *Suppose in addition to the assumptions in Corollary 1 that $\mathbf{Z}$ is a generator of the D-norm $\|\cdot\|_D$. Then we obtain for $\kappa \in \{1, \dots, d\}$ as $s \nearrow \omega^*$*

$$P(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s) = E\left(\min_{\kappa \le i \le \kappa+k} (\gamma_i Z_i)\right) + o(1),$$

for $0 \le k \le d - \kappa$.

Though the distribution of a generator of a D-norm is not uniquely determined, the preceding result entails that the numbers $E(\min_{\kappa\le i\le\kappa+k}(\gamma_i Z_i))$, $0\le k\le d-\kappa$, are uniquely determined by the D-norm.

The asymptotic distribution of the exceedance cluster length, conditional on the assumption that there is an exceedance at time point $\kappa\in\{1,\dots,d\}$, is an immediate consequence of Lemma 3. It follows from the equation

$$P(L_\kappa(s)=k\mid X_\kappa>s)=P(L_\kappa(s)\ge k\mid X_\kappa>s)-P(L_\kappa(s)\ge k+1\mid X_\kappa>s).$$

Note, moreover, that $P(L_\kappa(s)=0\mid X_\kappa>s)=1$ for $\kappa=d$.

Proposition 3 *Assume the conditions of Corollary 1. Then we have for $\kappa<d$ as $s\nearrow\omega^*$*

$$P(L_\kappa(s)=k\mid X_\kappa>s)=\begin{cases}\sum_{\emptyset\ne T\subset\{\kappa,\dots,d\}}(-1)^{|T|+1}\left\|\sum_{i\in T}\gamma_i\boldsymbol{e}_i\right\|_D+o(1), & k=d-\kappa,\\ \sum_{T\subset\{\kappa,\dots,\kappa+k\}}(-1)^{|T|+1}\left\|\gamma_{\kappa+k+1}\boldsymbol{e}_{\kappa+k+1}+\sum_{i\in T}\gamma_i\boldsymbol{e}_i\right\|_D+o(1), & 0\le k<d-\kappa.\end{cases}$$

We obtain, for example, for $\kappa<d$

$$P(L_\kappa(s)=0\mid X_\kappa>s)=\|\boldsymbol{e}_\kappa+\gamma_{\kappa+1}\boldsymbol{e}_{\kappa+1}\|_D-1+o(1),$$

which converges to $\gamma_{\kappa+1}$ if $\|\cdot\|_D=\|\cdot\|_1$. Recall that $\gamma_\kappa=1$.

In terms of a generator $\mathbf{Z}$ of a D-norm, Proposition 3 becomes the following result.

Corollary 3 *Assume in addition to the conditions of Corollary 1 that $\mathbf{Z}$ is a generator of the D-norm $\|\cdot\|_D$. Then we have for $\kappa<d$ as $s\nearrow\omega^*$*

(i) $P(L_\kappa(s)=k\mid X_\kappa>s)$

$$=\begin{cases}E\left(\min_{\kappa\le i\le d}(\gamma_i Z_i)\right)+o(1), & k=d-\kappa\\ E\left(\min_{\kappa\le i\le\kappa+k}(\gamma_i Z_i)-\min_{\kappa\le i\le\kappa+k+1}(\gamma_i Z_i)\right)+o(1), & 0\le k<d-\kappa.\end{cases}$$

(ii) $P(L_\kappa(s)\le k\mid X_\kappa>s)$

$$=\begin{cases}1, & k=d-\kappa\\ 1-E\left(\min_{\kappa\le i\le\kappa+k+1}(\gamma_i Z_i)\right)+o(1), & 0\le k<d-\kappa.\end{cases}$$

We, thus, obtain the limit distribution of the exceedance cluster length:

$$Q_\kappa([0,k]):=\lim_{s\nearrow\omega^*}P(L_\kappa(s)\le k\mid X_\kappa>s)=\begin{cases}1, & k=d-\kappa\\ 1-E\left(\min_{\kappa\le i\le\kappa+k+1}(\gamma_i Z_i)\right), & 0\le k<d-\kappa.\end{cases}$$

Take, for example, the generator $\mathbf{Z} = 2(U_1, \dots, U_d)$, where the U_i are independent and uniformly on (0, 1) distributed rv. If, in addition, $\gamma_i = 1, \kappa \le i \le d$, then we obtain

$$Q_\kappa([0,k]) = \begin{cases} 1, & k = d-\kappa \\ 1 - \frac{2}{k+3}, & 0 \le k < d-\kappa. \end{cases}$$

Next we compute the asymptotic mean exceedance cluster length.

Proposition 4 *Assume the conditions of Corollary 1 and let* $\mathbf{Z}$ *be a generator of the D-norm* $\|\cdot\|_D$. *Then we have for* $1 \le \kappa \le d$

$$\begin{aligned} E\left(L_\kappa(s) \mid X_\kappa > s\right) &= \begin{cases} 0, & \text{if } \kappa = d \\ \sum_{k=1}^{d-\kappa} s_\kappa(k) + o(1) & \text{else} \end{cases} \\ &= \begin{cases} 0, & \text{if } \kappa = d \\ \sum_{k=1}^{d-\kappa} E\left(\min_{\kappa \le i \le \kappa+k}(\gamma_i Z_i)\right) + o(1) & \text{else.} \end{cases} \end{aligned}$$

Proof Since $L_\kappa(s)$ attains only nonnegative values, we have for $\kappa < d$

$$\begin{aligned} E\left(L_\kappa(s) \mid X_\kappa > s\right) &= \int_0^\infty P\left(L_\kappa(s) \ge t \mid X_\kappa > s\right) \mathrm{d}t \\ &= \sum_{k=1}^{d-\kappa} P\left(L_\kappa(s) \ge k \mid X_\kappa > s\right) \\ &= \sum_{k=1}^{d-\kappa} P\left(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s\right) \\ &= \sum_{k=1}^{d-\kappa} s_\kappa(k) + o(1). \end{aligned}$$

□

Corollary 4 *Under the conditions of the preceding result we have for* $\kappa < d$, *if* $\gamma_k > 0, 1 \le k \le d$,

$$\lim_{s \uparrow \omega^*} E(L_\kappa(s) \mid X_\kappa > s) = 0$$

if and only if $\|x\mathbf{e}_\kappa + y\mathbf{e}_{\kappa+1}\|_D = \|x\mathbf{e}_\kappa + y\mathbf{e}_{\kappa+1}\|_1 = x + y, x, y \ge 0.$

Proof Note that $s_\kappa(1) \ge \dots \ge s_\kappa(d-\kappa)$. We, thus, obtain from Proposition 4

$$\lim_{s \uparrow \omega^*} E(L_\kappa(s) \mid X_\kappa > s) = 0 \iff s_\kappa(1) = 0.$$

The assertion is now a consequence of Proposition 6.1 in Falk and Tichy (2012). □

Suppose in addition to the assumptions of Corollary 1 that the components $X_1, \ldots, X_d$ of the rv $\boldsymbol{X}$ are exchangeable. Then we have $\gamma_1 = \cdots = \gamma_d = 1$, as well as $\|\sum_{i\in T} \boldsymbol{e}_i\|_D = \|\sum_{i=1}^{|T|} \boldsymbol{e}_i\|_D$ for any nonempty subset $T \subset \{1, \ldots, d\}$. As a consequence we obtain

$$s_\kappa(k) = \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k+1}{j} \left\| \sum_{i=1}^{j} \boldsymbol{e}_i \right\|_D, \quad 0 \le k \le d-\kappa,$$

and, thus, by rearranging sums,

$$\begin{aligned}
\lim_{s\nearrow} E\left(L_\kappa(s) \mid X_\kappa > s\right) &= \sum_{k=1}^{d-\kappa} s_\kappa(k) \\
&= \sum_{j=1}^{d-\kappa+1} (-1)^{j+1} \left\| \sum_{i=1}^{j} \boldsymbol{e}_i \right\|_D \sum_{k=\max(1,j-1)}^{d-\kappa} \binom{k+1}{j} \\
&= -1 + \sum_{j=1}^{d-\kappa+1} (-1)^{j+1} \binom{d-\kappa+2}{j+1} \left\| \sum_{i=1}^{j} \boldsymbol{e}_i \right\|_D, \qquad (5)
\end{aligned}$$

where the final equality follows from the general equation $\sum_{r=n}^{N} \binom{r}{n} = \binom{N+1}{n+1}$.

Example 3 (Marshall-Olkin D-norm) The Marshall-Olkin D-norm is the convex combination of the maximum-norm and the L_1-norm:

$$\|\boldsymbol{x}\|_{\mathrm{MO}} = \vartheta \, \|\boldsymbol{x}\|_1 + (1-\vartheta) \, \|\boldsymbol{x}\|_\infty, \quad \boldsymbol{x} \in \mathbb{R}^d, \ \vartheta \in [0, 1],$$

see Falk et al. (2010, Example 4.3.4). In this case we obtain from Eq. (5)

$$\lim_{s\nearrow} E\left(L_\kappa(s) \mid X_\kappa > s\right) = (1-\vartheta)(d-\kappa),$$

where we have used the general equation $\sum_{j=0}^{m} (-1)^j \binom{m}{j} = 0$.

In the case $\vartheta = 0$ of complete tail dependence of the margins we, therefore, obtain $\lim_{s\nearrow} E(L_\kappa(s) \mid X_\kappa > s) = d - \kappa$, which is the full possible length, whereas in the tail independence case $\vartheta = 1$ we obtain the shortest length $\lim_{s\nearrow} E(L_\kappa(s) \mid X_\kappa > s) = 0$, which is in complete accordance with Corollary 4.

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