Ann Inst Stat Math (2012) 64:677–685
DOI 10.1007/s10463-011-0329-6

Directional dependence in multivariate distributions

Roger B. Nelsen · Manuel Úbeda-Flores

Received: 23 April 2010 / Revised: 5 November 2010 / Published online: 16 March 2011

Abstract In this paper, we develop some coefficients which can be used to detect dependence in multivariate distributions not detected by several known measures of multivariate association. Several examples illustrate our results.

Keywords Copula · Directional dependence · Measure of association

1 Introduction

In his article in the *Encyclopedia of Statistical Science on copulas*, Fisher (1997) writes: "Copulas [are] of interest to statisticians for two main reasons: First, as a way of studying scale-free measures of dependence; and secondly, as a starting point for constructing families of bivariate distributions..."

In Jogdeo's (1982) entry on concepts of dependence, we read: "Dependence relations between random variables is one of the most studied subjects in probability and statistics. The nature of the dependence can take a variety of forms, and unless some specific assumptions are made about the dependence, no meaningful statistical model can be contemplated."

R. B. Nelsen
Department of Mathematical Sciences, Lewis and Clark College,
0615 SW Palatine Hill Rd., Portland, OR 97219, USA
e-mail: nelsen@lclark.edu

M. Úbeda-Flores (✉)
Departamento de Estadística y Matemática Aplicada, Universidad de Almería,
Carretera de Sacramento s/n, 04120 La Cañada de San Urbano, Almería, Spain
e-mail: mubeda@ual.es

In this paper we use copulas to study a concept we call *directional dependence* in multivariate distributions, and introduce some coefficients to measure that dependence.

2 Preliminaries

2.1 Copulas and Sklar's theorem

Let $n \geq 2$ be a natural number. An *n-dimensional copula* (briefly, n-copula) is the restriction to $\mathbb{I}^n$ ($\mathbb{I} = [0, 1]$) of a continuous n-dimensional distribution function whose univariate margins are uniform on $\mathbb{I}$. Equivalently, an n-copula is a function C from $\mathbb{I}^n$ to $\mathbb{I}$ satisfying the following properties:

(i) For every $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in $\mathbb{I}^n$, $C(\mathbf{u}) = 0$ if at least one coordinate of $\mathbf{u}$ is 0, and $C(\mathbf{u}) = u_k$ whenever all coordinates of $\mathbf{u}$ are 1 except maybe u_k; and
(ii) for every $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ in $\mathbb{I}^n$ such that $a_k \leq b_k$ for all $k = 1, 2, \ldots, n$, $V_C([\mathbf{a}, \mathbf{b}]) = \sum \mathrm{sgn}(\mathbf{c}) \cdot C(\mathbf{c}) \geq 0$, where $[\mathbf{a}, \mathbf{b}]$ denotes the *n-box* $\times_{i=1}^{n}[a_i, b_i]$, the sum is taken over all the *vertices* $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ of $[\mathbf{a}, \mathbf{b}]$, i.e., each c_k is equal to either a_k or b_k, and $\mathrm{sgn}(\mathbf{c}) = 1$ if $c_k = a_k$ for an even number of k's, and $\mathrm{sgn}(\mathbf{c}) = -1$ otherwise.

The importance of copulas in statistics is described in the following result due to Sklar (1959): Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random n-vector with joint distribution function H and one-dimensional marginal distributions $F_1, F_2, \ldots, F_n$, respectively. Then there exists an n-copula C (which is uniquely determined on $\times_{i=1}^{n}\mathrm{Range}F_i$) such that $H(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$ for all $\mathbf{x} \in [-\infty, \infty]^n$. Thus, copulas link joint distribution functions to their one-dimensional margins. For example, Π^n is the n-copula for independent random variables, i.e., $\Pi^n(\mathbf{u}) = \prod_{i=1}^{n} u_i$; and M^n—the best-possible point-wise upper bound for the set of n-copulas—is an n-copula given by $M^n(\mathbf{u}) = \min(u_1, u_2, \ldots, u_n)$ for every $\mathbf{u}$ in $\mathbb{I}^n$. However, $W^n(\mathbf{u}) = \max\left(\sum_{i=1}^{n} u_i - n + 1, 0\right)$—the best-possible point-wise lower bound for the set of n-copulas—is an n-copula if and only if $n = 2$. For a complete survey on copulas, see Nelsen (2006).

If $\mathbf{U}$ is a vector of uniform $\mathbb{I}$ random variables whose distribution function is the n-copula C, then $\overline{C}$ denotes the *survival function* associated with C, i.e., $\overline{C}(\mathbf{u}) = \Pr[\mathbf{U} > \mathbf{u}]$, where $\mathbf{U} > \mathbf{u}$ denotes point-wise inequality; and $\hat{C}$ denotes the *survival copula* associated with C, i.e., $\hat{C}(\mathbf{u}) = \overline{C}(\mathbf{1} - \mathbf{u})$. Moreover, C_{ij}, with $1 \leq i < j \leq n$, will denote the (i, j)*-margin* of C, i.e., $C_{ij}(u_i, u_j) = C(1, \ldots, 1, u_i, 1, \ldots, 1, u_j, 1, \ldots, 1)$, which is a 2-copula.

2.2 Measures of association in the bivariate case

The population version of three of the most common nonparametric measures of association between the components of a continuous random pair (X, Y) are *Kendall's tau* (written τ_{XY}), *Spearman's rho* (written ρ_{XY}), and the medial correlation coefficient—or *Blomqvist's beta* (written β_{XY}). Such measures depend only on the copula C

associated with the pair (X, Y)—so they can also be written as $\tau(C)$, $\rho(C)$ and $\beta(C)$—and are given by $\tau(C) = 4\int_{\mathbb{I}^2} C(u,v)\,\mathrm{d}C(u,v) - 1 = 1 - 4\int_{\mathbb{I}^2} \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v}\,\mathrm{d}u\mathrm{d}v$, $\rho(C) = 12\int_{\mathbb{I}^2} C(u,v)\,\mathrm{d}u\mathrm{d}v - 3 = 12\int_{\mathbb{I}^2} uv\,\mathrm{d}C(u,v) - 3$, and $\beta(C) = 4\,C(\frac{1}{2}, \frac{1}{2}) - 1$, respectively. As a consequence, we can also assume that the random variables X and Y are uniform on $\mathbb{I}$ when studying properties of these measures. For a study of some of their properties, see Nelsen (2006) and the references therein.

In the bivariate case, a measure of association provides information about the *magnitude* and *direction* of the association between two random variables. When the measure is near to $+1$, large (small) values of the random variables tend to occur together; and when the measure is near -1, large values of one random variable tend to occur with small values of the other.

2.3 Association in the multivariate case

In the multivariate case, the situation is more complicated, and consequently we consider just the trivariate case. A typical trivariate measure of association is the average of the three pairwise measures, but such measure often fails to detect association among the three random variables. For instance, numerous examples exist of triples (X, Y, Z) which are pairwise independent but not mutually independent.

Example 1 Let (X, Y, Z) be a vector of continuous random variables uniform on $\mathbb{I}$ whose distribution function is the 3-copula C given by $C(u, v, w) = uvw\,[1 + \theta(1-u)(1-v)(1-w)]$, with $0 < |\theta| \le 1$. C is a member of the Farlie-Gumbel-Morgenstern family of 3-copulas (Johnson and Kotz 1972). Then $\frac{\tau_{XY}+\tau_{YZ}+\tau_{ZX}}{3} = \frac{\rho_{XY}+\rho_{YZ}+\rho_{ZX}}{3} = \frac{\beta_{XY}+\beta_{YZ}+\beta_{ZX}}{3} = 0$. However, (X, Y, Z) are not mutually independent, i.e., $C \ne \Pi^3$.

We will denote the measures in Example 1 by $\rho_3^* = \frac{\rho_{XY}+\rho_{YZ}+\rho_{ZX}}{3}$, and similarly for τ_3^* and β_3^*.

2.4 Generalizations of Spearman's rho

Let the copula C be the distribution function of the random vector (X, Y, Z). Two common trivariate generalizations of Spearman's rho are given by Joe (1990) and Nelsen (1996) $\rho_3^+(C) = 8\int_{\mathbb{I}^3} \overline{C}(u,v,w)\,\mathrm{d}u\mathrm{d}v\mathrm{d}w - 1 = 8\,\mathcal{E}[XYZ] - 1$, and $\rho_3^-(C) = 8\int_{\mathbb{I}^3} C(u,v,w)\,\mathrm{d}u\mathrm{d}v\mathrm{d}w - 1 = 8\,\mathcal{E}[(1-X)(1-Y)(1-Z)] - 1$, which are distinct from the average of the three pairwise version of Spearman's rho $\rho_3^*(C) = \frac{\rho_{XY}+\rho_{YZ}+\rho_{ZX}}{3}$. We also note that $\rho_3^* = \frac{\rho_3^+ + \rho_3^-}{2}$ (we will suppress the argument in the coefficients when the copula in question is understood).

Example 1 (continued) If C is the 3-copula given in Example 1, then $\rho_3^*(C) = 0$, but $\rho_3^+(C) = -\theta/27$ and $\rho_3^-(C) = \theta/27$. Notice that $P[(X, Y, Z) > \mathbf{1/2}] = 1/8 - \theta/64$ and $P[(X, Y, Z) \le \mathbf{1/2}] = 1/8 + \theta/64$, whereas for independent X, Y, Z we have $P[(X, Y, Z) > \mathbf{1/2}] = P[(X, Y, Z) \le \mathbf{1/2}] = 1/8$.

When one of the measures ρ_3^*, ρ_3^+, or ρ_3^- is near $+1$, then large (or small) values of the random variables tend to occur together, but as the following example shows, when ρ_3^+ or ρ_3^- or ρ_3^* is near 0, there may be dependence among the random variables undetected by the measures.

Example 2 Let (X, Y, Z) be a vector of continuous random variables uniform on $\mathbb{I}$ whose distribution function is the 3-copula C given by $C(u, v, w) = C_1(M^2(u, v), w)$, where C_1 is the 2-copula given by $C_1 = (\Pi^2 + W^2)/2$. Then it follows that a) $\rho_3^* = \rho_3^+ = \rho_3^- = 0$, and b) $P[X = Y = 1 - Z] = 1/2$, i.e., half the probability mass of C is (uniformly distributed) on the line joining the points $(0, 0, 1)$ and $(1, 1, 0)$; and this dependence is not detected by ρ_3^*, ρ_3^+ or ρ_3^-.

Here, we now develop some "coefficients of dependence" which reflect dependence in trivariate distributions not detected by known measures of association. These coefficients are based on "directional dependence."

Remark 1 We note that nothing is gained when $n = 2$ using the above procedure to create directional ρ-coefficients. If C is a 2-copula with $\rho(C) = \rho$ (see Sect. 2.2), then $\rho_2^{(1,1)} = \rho_2^{(-1,-1)} = \rho$ and $\rho_2^{(1,-1)} = \rho_2^{(-1,1)} = -\rho$.

3 Directional ρ-coefficients

The measures ρ_3^+ and ρ_3^- introduced earlier were obtained in Nelsen (1996) as follows: Let (X, Y, Z) be a vector of continuous random variables uniform on $\mathbb{I}$ whose distribution function is the 3-copula C (as we shall assume from now on). Then $\rho_3^+(C) = 8\int_{\mathbb{I}^3}[P(X > u, Y > v, Z > w) - P(X > u)P(Y > v)P(Z > w)]\,\mathrm{d}u\mathrm{d}v\mathrm{d}w$ and $\rho_3^-(C) = 8\int_{\mathbb{I}^3}[P(X \le u, Y \le v, Z \le w) - P(X \le u)P(Y \le v)P(Z \le w)]\,\mathrm{d}u\mathrm{d}v\mathrm{d}w$.

Now, consider the function $Q_{\alpha_1\alpha_2\alpha_3}(u, v, w)$ given by $P[\alpha_1 X > \alpha_1 u, \alpha_2 Y > \alpha_2 v, \alpha_3 Z > \alpha_3 w] - P[\alpha_1 X > \alpha_1 u]P[\alpha_2 Y > \alpha_2 v]P[\alpha_3 Z > \alpha_3 w]$ for u, v, w in $\mathbb{I}$ with $\alpha_i \in \{-1, 1\}$ for $i = 1, 2, 3$. Dependence properties derived from the fact that this difference can be greater or lesser than 0 can be found in Quesada-Molina et al (2011).

Each of the eight vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ determines the sense of each equality above, and so defines the "direction" in $\mathbb{I}^3$ in which we will measure dependence.

We now define a *directional ρ-coefficient* for each α in the following manner: $\rho_3^{(\alpha_1,\alpha_2,\alpha_3)}(C) = 8\int_{\mathbb{I}^3} Q_{\alpha_1\alpha_2\alpha_3}(u, v, w)\,\mathrm{d}u\mathrm{d}v\mathrm{d}w$. The constant 8 insures that the maximum value (over all possible 3-copulas C) of each coefficient ρ_3^α is 1. Observe that $\rho_3^{(1,1,1)} = \rho_3^+$ and $\rho_3^{(-1,-1,-1)} = \rho_3^-$.

We now consider the other directions, for example, $\alpha = (-1, -1, 1)$ (the procedure for other directions α is similar). Since $Q_{(-1,-1,1)}(u, v, w) = P[X < u, Y < v, Z > w] - uv(1 - w) = C_{12}(u, v) - C(u, v, w) - uv + uvw$, we have $\rho_3^{(-1,-1,1)} = 8\int_{\mathbb{I}^3}[C_{12}(u, v) - C(u, v, w) - uv + uvw]\,\mathrm{d}u\mathrm{d}v\mathrm{d}w = 8\left[\frac{\rho_{XY}+3}{12} - \frac{\rho_3^-+1}{8} - \frac{1}{4} + \frac{1}{8}\right] = \frac{2}{3}\rho_{XY} - \rho_3^-$. Thus, this coefficient (and each of the others) is simply a linear combination of measures of association encountered earlier.

To find a general pattern, we write $\rho_3^+ = \rho_3^* + \varepsilon_3$ and $\rho_3^- = \rho_3^* - \varepsilon_3$, recall that $\rho_3^+ + \rho_3^- = 2\rho_3^*$, with $\varepsilon_3 = (\rho_3^+ - \rho_3^-)/2$. Then we have $\rho_3^{(-1,-1,1)} = \frac{2}{3}\rho_{XY} - \rho_3^- = \frac{2}{3}\rho_{XY} - \frac{\rho_{XY}+\rho_{YZ}+\rho_{ZX}}{3} + \varepsilon_3 = \frac{\rho_{XY}-\rho_{YZ}-\rho_{ZX}}{3} + \frac{\rho_3^+-\rho_3^-}{2}$. Similar results hold for the other coefficients, yielding

Theorem 1 *Let* (X, Y, Z) *be a random vector with associated* 3*-copula* C*. Then, for each direction* $(\alpha_1, \alpha_2, \alpha_3)$*, we have*

$$\rho_3^{(\alpha_1,\alpha_2,\alpha_3)} = \frac{\alpha_1\alpha_2\rho_{XY} + \alpha_2\alpha_3\rho_{YZ} + \alpha_3\alpha_1\rho_{ZX})}{3} + \alpha_1\alpha_2\alpha_3\frac{\rho_3^+ - \rho_3^-}{2}. \tag{1}$$

Thus, each coefficient of directional dependence in (1) is a simple linear combination of the pairwise measures and the two measures ρ_3^+ and ρ_3^- of 3-variable association.

Example 2 (continued) Recall $C(u, v, w) = C_1(M^2(u, v), w)$ with $C_1 = (\Pi^2 + W^2)/2$, where we had $\rho_3^* = \rho_3^+ = \rho_3^- = 0$. Since $\rho_{XY} = 1$ and $\rho_{YZ} = \rho_{ZX} = -1/2$, it follows that $\rho_3^{(-1,-1,1)} = \rho_3^{(1,1,-1)} = 2/3$ and $\rho_3^{(-1,1,-1)} = \rho_3^{(1,-1,1)} = \rho_3^{(1,-1,-1)} = \rho_3^{(-1,1,1)} = -1/3$.

Remark 2 An intuitive interpretation of these coefficients goes as follows: If, say, $\rho_3^{(-1,-1,1)}$ or $\rho_3^{(1,1,-1)}$ is positive, then there is "positive dependence" in the direction determined by $(-1, -1, 1)$ or $(1, 1, -1)$. In this case, large (small) values of X and Y occur with small (large) values of Z; i.e., $\rho_{XY} > 0$ while $\rho_{YZ} < 0$ and $\rho_{ZX} < 0$, so that $\rho_{XY} - \rho_{YZ} - \rho_{ZX} > 0$. The presence of $\pm(\rho_3^+ - \rho_3^-)/2$ accounts for simultaneous 3-variable dependence not measured by the pairwise coefficients.

Remark 3 In general, ρ_3^α is not a multivariate measure of association.

Example 3 Trivariate Cuadras–Augé (1981) copulas are weighted geometric means of the 3-copulas Π^3 and M^3, i.e., $C_\theta(u, v, w) = (uvw)^{1-\theta}[\min(u, v, w)]^\theta$ for $(u, v, w) \in \mathbb{I}^3$ with $\theta \in [0, 1]$. Straightforward calculations yield $\rho_{XY} = \rho_{YZ} = \rho_{ZX} = \rho_3^* = \frac{3\theta}{4-\theta}$, $\rho_3^+ = \frac{\theta(11-5\theta)}{(3-\theta)(4-\theta)}$, and $\rho_3^- = \frac{\theta(7-\theta)}{(3-\theta)(4-\theta)}$, so that $\rho_3^{(-1,-1,1)} = \rho_3^{(-1,1,-1)} = \rho_3^{(1,-1,-1)} = \frac{-\theta(1+\theta)}{(3-\theta)(4-\theta)}$ and $\rho_3^{(1,1,-1)} = \rho_3^{(1,-1,1)} = \rho_3^{(-1,1,1)} = \frac{-\theta(5-3\theta)}{(3-\theta)(4-\theta)}$. Observe that for all $\theta > 0$ the coefficient ρ_3^α for $\alpha \neq (1, 1, 1)$ or $(-1, -1, -1)$ is negative. This is a consequence of the fact when $\theta > 0$, C_θ has a singular component on the main diagonal of $\mathbb{I}^3$.

Some salient properties of directional ρ-coefficients are summarized in the following result, whose proof is simple and we omit it. These results express the redundancy in the eight directional coefficients (since each is a function of only five other coefficients) and the symmetry present in the formula for $\rho_3^{(\alpha_1,\alpha_2,\alpha_3)}$ in Theorem 1.

Corollary 1 *Let* (X, Y, Z) *be a vector of continuous random variables uniform on* $\mathbb{I}$ *whose distribution function is the* 3*-copula* C*, and* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$*, where* $\alpha_i \in \{-1, 1\}$ *for* $i = 1, 2, 3$*. Then we have:*

1. $\rho_3^\alpha(\Pi^3) = 0$ *and* $\sum_\alpha \rho_3^\alpha(C) = 0$.
2. $\rho_3^\alpha(C) = \rho_3^{-\alpha}(\hat{C})$.
3. $\sum_{\alpha_1\alpha_2\alpha_3=1} \rho_3^\alpha(C) = 2(\rho_3^+ - \rho_3^-)$ *and* $\sum_{\alpha_1\alpha_2\alpha_3=-1} \rho_3^\alpha(C) = 2(\rho_3^- - \rho_3^+)$.
4. $\sum_{\alpha_i=1} \rho_3^\alpha(C) = 0 = \sum_{\alpha_i=-1} \rho_3^\alpha(C)$.
5. *If* $\rho_3^+(C) = \rho_3^-(C)$*, then*
   (a) $\rho_3^+(C) = \rho_3^-(C) = \rho_3^*(C)$*;*
   (b) $\rho_3^\alpha(C) = \rho_3^{-\alpha}(C) = (\alpha_1\alpha_2\rho_{XY} + \alpha_2\alpha_3\rho_{YZ} + \alpha_3\alpha_1\rho_{ZX})/3$*; and*
   (c) $\sum_{\alpha_1\alpha_2\alpha_3=1} \rho_3^\alpha(C) = 0 = \sum_{\alpha_1\alpha_2\alpha_3=-1} \rho_3^\alpha(C)$.
6. *If* $\rho_3^*(C) = 0$*, then* $\sum_{\alpha_1\alpha_2\alpha_3=1} \rho_3^\alpha(C) = 4\rho_3^+$, $\sum_{\alpha_1\alpha_2\alpha_3=-1} \rho_3^\alpha(C) = 4\rho_3^-$.

Although the directional ρ-coefficients may detect dependence undetected by ρ_3^*, ρ_3^+, and ρ_3^-, that is not always the case, as the following example illustrate.

Example 4 Let (X, Y, Z) be a random vector whose distribution function is the 3-copula $C(u, v, w) = \frac{M^3(u,v,w)+W^2(M^2(u,v),w)+W^2(M^2(u,w),v)+W^2(M^2(v,w),u)}{4}$ for all (u, v, w) in $\mathbb{I}^3$. This copula assigns probability mass uniformly on each of the four diagonals of $\mathbb{I}^3$. Each bivariate margin is the 2-copula $(M^2 + W^2)/2$. It is easy to show that we have $\rho_{XY} = \rho_{YZ} = \rho_{ZX} = 0$ so that the variables are pairwise uncorrelated but not independent; and $\rho_3^\alpha = 0$ for every direction α yet $P(X = Y = Z) = P(X = Y = 1 - Z) = P(X = 1 - Y = Z) = P(1 - X = Y = Z) = 1/4$.

4 Other directional coefficients

The population versions of the measures of association known as *Kendall's tau* and *Blomqvist's beta* are based on the notion of *concordance*: Two random n-vectors $\mathbf{X}$ and $\mathbf{Y}$ are *concordant* if $\mathbf{X} < \mathbf{Y}$ or $\mathbf{Y} < \mathbf{X}$ (component-wise). The idea of comparing the concordances of different n-uples of random variables is considered in, for example, Joe (1990, 1997) and Kimeldorf and Sampson (1987, 1989). We note that ρ_3^* is a measure of concordance—since it satisfies a set of axioms—but neither ρ_3^+ nor ρ_3^- is (Taylor (2007)).

If $\mathbf{X}$ and $\mathbf{Y}$ are two independent random vectors with a common n-copula C, an n-variate version of Kendall's tau is given by $\tau_n(C) = \frac{1}{2^{n-1}-1}\left[2^{n-1}P(\mathbf{X} < \mathbf{Y} \text{ or } \mathbf{Y} < \mathbf{X}) - 1\right] = \frac{1}{2^{n-1}-1}(2^n \int_{\mathbb{I}^n} C(\mathbf{u})\, \mathrm{d}C(\mathbf{u}) - 1)$ (Joe 1990; Nelsen 1996), and an n-variate version of Blomqvist's beta is given by $\beta_n(C) = \frac{1}{2^{n-1}-1}\left[2^{n-1}P(\mathbf{X} < \mathbf{1/2} \text{ or } \mathbf{X} > \mathbf{1/2}) - 1\right] = \frac{2^{n-1}[C(\mathbf{1/2})+\overline{C}(\mathbf{1/2})]-1}{2^{n-1}-1}$ (Úbeda-Flores 2005)—see Dolati and Úbeda-Flores (2006) and Taylor (2007, 2008) for some properties of these measures. However, when $n = 3$ we have $\tau_3(C) = \tau_3^*(C)$ and $\beta_3(C) = \beta_3^*(C)$.

Observe that $\tau_n(C) \geq \frac{-1}{2^{n-1}-1}$ and $\beta_n(C) \geq \frac{-1}{2^{n-1}-1}$, so that both $\tau_3(C)$ and $\beta_3(C)$ are at least $-1/3$.

Analogous to our work with directional ρ-coefficients, we can define *directional τ-coefficients* τ_3^α and *directional β-coefficients* β_3^α. Let $\mathbf{X} = (X_1, X_2, X_3)$, $\mathbf{Y} = (Y_1, Y_2, Y_3)$, and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ denote "direction" with $\alpha_i \in \{-1, 1\}$ for $i = 1, 2, 3$; and let juxtaposition of vectors denote component-wise multiplication, i.e., $\alpha\mathbf{X} = (\alpha_1 X_1, \alpha_2 X_2, \alpha_3 X_3)$. Thus $\tau_3^{\alpha}(C) = \frac{1}{3}\left[8\,P(\alpha\mathbf{X} > \alpha\mathbf{Y}) - 1\right]$, and $\beta_3^{\alpha}(C) = \frac{1}{3}\left\{4\left[P(\alpha\mathbf{X} > (\mathbf{1/2})\alpha) + P(\alpha\mathbf{X} \le (\mathbf{1/2})\alpha)\right] - 1\right\}$.

Theorem 2 *Let (X, Y, Z) be a random vector with 3-copula C. Then for each direction $(\alpha_1, \alpha_2, \alpha_3)$ we have*

$$\tau_3^{(\alpha_1,\alpha_2,\alpha_3)}(C) = \frac{\alpha_1\alpha_2\tau_{XY} + \alpha_2\alpha_3\tau_{YZ} + \alpha_3\alpha_1\tau_{ZX}}{3}, \tag{2}$$

$$\beta_3^{(\alpha_1,\alpha_2,\alpha_3)}(C) = \frac{\alpha_1\alpha_2\beta_{XY} + \alpha_2\alpha_3\beta_{YZ} + \alpha_3\alpha_1\beta_{ZX}}{3}. \tag{3}$$

Observe that $\tau_3(C)$ does not appear in the expression for τ_3^{α} in (2) since it is itself a function of the three pairwise coefficients (and similarly for β_3^{α} in 3).

In spite of the similarity in the expressions for ρ_3^{α}, τ_3^{α}, and β_3^{α}, their values are often different, as they measure different aspects of the dependence among X, Y, and Z.

Example 5 Let C be a member of the four-parameter Farlie-Gumbel-Morgenstern family of 3-copulas given by Johnson and Kotz (1972) $C(u, v, w) = uvw[1+\kappa(1-u)(1-v)+\lambda(1-u)(1-w)+\mu(1-v)(1-w)+\theta(1-u)(1-v)(1-w)]$ for u, v, w in $\mathbb{I}$ where $\kappa, \lambda, \mu, \theta \in [-1, 1]$ satisfying the inequalities $1 + \omega_1\kappa + \omega_2\lambda + \omega_3\mu \ge |\theta|$ for $\omega_i \in \{-1, 1\}$, $\omega_1\omega_2\omega_3 = 1$ [the 3-copula in Example 1 had $\kappa = \lambda = \mu = 0$]. Then $\rho_3^{*} = \frac{\kappa+\lambda+\mu}{9}$, $\rho_3^{+} = \frac{3(\kappa+\lambda+\mu)-\theta}{27}$, $\rho_3^{-} = \frac{3(\kappa+\lambda+\mu)+\theta}{27}$, $\tau_3 = \frac{2(\kappa+\lambda+\mu)}{27}$, and $\beta_3 = \frac{\kappa+\lambda+\mu}{12}$, so that for directions $\alpha = (-1, -1, 1)$ and $(1, 1, -1)$ we have $\rho_3^{(-1,-1,1)} = \frac{3(\kappa-\lambda-\mu)-\theta}{27}$ and $\rho_3^{(1,1,-1)} = \frac{3(\kappa-\lambda-\mu)+\theta}{27}$, but $\tau_3^{(-1,-1,1)} = \tau_3^{(1,1,-1)} = \frac{2(\kappa-\lambda-\mu)}{27}$ and $\beta_3^{(-1,-1,1)} = \beta_3^{(1,1,-1)} = \frac{\kappa-\lambda-\mu}{12}$. Results are similar for the other directions.

5 Bounds on directional ρ-, τ-, and β-coefficients

What are the maximum and minimum values for $\rho_3^{\alpha}(C)$, $\tau_3^{\alpha}(C)$, and $\beta_3^{\alpha}(C)$ as C ranges over the set of all 3-copulas? The maximum value for each is 1, as we chose the coefficients in the expression for each coefficient to make the maximum 1. We need only find the minima for one direction (say $\alpha = (1, 1, 1)$) since it is straightforward to rotate or reflect the mass distribution of a copula using symmetries of the cube [that is, given any copula C and direction α, one can find a copula C_α such that $\rho_3^{\alpha}(C) = \rho_3^{+}(C_\alpha)$].

As we noted earlier, $\tau_3(C) = \tau_3^{*}(C) \ge -1/3$ and $\beta_3(C) = \beta_3^{*}(C) \ge -1/3$, with equality when one of the 2-margins of C is W^2 (Úbeda-Flores 2005). But we can also have equality for copulas none of whose 2-margins is W^2.

Example 6 (a) The copula $C'(u, v, w) = \left[\max(u^{1/2} + v^{1/2} + w^{1/2} - 2, 0)\right]^2$ is a member of the Clayton family of Archimedean 3-copulas, with $\tau_{XY} = \tau_{YZ} = \tau_{ZX} = \tau_3^{*} = \frac{-1}{3}$. McNeil and Nešlehová (2009) have shown that for any Archimedean 3-copula C, $C \ge C'$.

(b) Let D be the 2-copula $D(u, v) = \max(0, u + v - 1, \min(u, v - 1/2))$, $(u, v) \in \mathbb{I}^2$. D is a *Shuffle of Min* (Mikusiński et al. 1992), and its mass is spread uniformly in $\mathbb{I}^2$ on two line segments joining the points $(0, 1/2)$ to $(1/2, 1)$, and $(1/2, 1/2)$ to $(1, 0)$. Now define the 3-copula $C(u, v, w) = wD(u, v)$ for every $(u, v, w) \in \mathbb{I}^3$. It is clear that the three bivariate margins are D, Π^2 and Π^2; so that $\beta_{XY} = -1$, $\beta_{XZ} = \beta_{YZ} = 0$, and consequently $\beta_3(C) = -1/3$.

We have the following related results about $\rho_3^*(C)$:

Theorem 3 *Let (X, Y, Z) be a random vector whose distribution function is the 3-copula C. If one of the 2-margins of C is W^2, then $\rho_3^*(C) = -1/3$.*

Proof Assume, without loss of generality, that $C(u, v, 1) = W^2(u, v)$. Then $\rho_{XY} = -1$. We will show that $\rho_{YZ} + \rho_{XZ} = 0$, which proves the theorem. Since $V_C([0, 1 - v] \times [0, v] \times [0, w]) = V_C([1 - v, 1] \times [v, 1] \times [0, w]) = 0$, we have the following chain of equalities: $C_{23}(v, w) = V_C([0, 1] \times [0, v] \times [0, w]) = V_C([1 - v, 1] \times [0, v] \times [0, w]) = V_C([1 - v, 1] \times [0, 1] \times [0, w]) = V_{C_{13}}([1 - v, 1] \times [0, w]) = w - C_{13}(1 - v, w)$. So $\int_{\mathbb{I}^2} C_{23}(v, w)\, dv dw = \frac{1}{2} - \int_{\mathbb{I}^2} C_{13}(1 - v, w)\, dv dw = \frac{1}{2} - \int_{\mathbb{I}^2} C_{13}(u, w)\, du dw$, and hence $\rho_{YZ} = 12 \int_{I^2} C_{23}(v, w)\, dv dw - 3 = 6 - 12 \int_{I^2} C_{13}(u, w)\, du dw - 3 = -\rho_{XZ}$, as desired. □

But unlike $\tau_3^*(C)$ and $\beta_3^*(C)$, $-1/3$ is *not* the minimum value of $\rho_3^*(C)$, as the following result shows.

Theorem 4 *Let (X, Y, Z) be a random vector whose distribution function is the 3-copula C. Then $\rho_3^*(C) \geq -1/2$; and $\rho_3^*(C) = -1/2$ if, and only if,* $\Pr[X + Y + Z = 3/2] = 1$.

Proof Observe that $\mathcal{E}[(X + Y + Z - 3/2)^2] = \mathcal{E}[X^2 + Y^2 + Z^2 - 3(X + Y + Z) + 2(XY + YZ + XZ) + 9/4] = 3(1/3) - 3(3/2) + 2(\frac{\rho_{XY}+3}{12} + \frac{\rho_{YZ}+3}{12} + \frac{\rho_{XZ}+3}{12}) + 9/4 = \frac{\rho_3^*(C)}{2} + 1/4$, whence the result follows. □

Only ρ_3^+ remains to be studied. The following example shows that the minimum is $-1/2$ or less.

Example 7 Let (X, Y, Z) be a random vector whose distribution function is the 3-copula C whose probability mass is distributed uniformly on the edges of the equilateral triangle in $\mathbb{I}^3$ with vertices $(0, 1/2, 1)$, $(1/2, 1, 0)$, and $(1, 0, 1/2)$ [note that the triangle lies in the plane $x + y + z = 3/2$ and none of the 2-margins are W^2]. Simply computations show that $\rho_{XY} = \rho_{YZ} = \rho_{ZX} = \rho_3^+ = \rho_3^- = -1/2$.

In general, we only know that for any 3-copula C, $\rho_3^+(C) > -2/3$ (Nelsen 1996).

6 Discussion

Given a random sample, estimators of pairwise measures such as ρ_{XY} are well known, from which one can estimate ρ_3^*. Estimators of ρ_3^+ and ρ_3^-(using ranks of the observations in the sample) can be found in Schmid and Schmidt (2007). Consequently, estimators of ρ_3^{α} for all α are also easily constructed.

In higher dimensions, say $n = 4$, we conjecture that each of the 16 coefficients $\rho_4^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ will be a linear combination of ρ_4^+, ρ_4^-, the 6 pairwise measures (ρ_{XY}, etc.) and the 8 three-wise measures (ρ_{XYZ}^+, ρ_{XYZ}^-, etc.); with similar results for $\tau_4^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ and $\beta_4^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$.

Acknowledgments The second author acknowledges the support by the Ministerio de Ciencia e Innovación (Spain) and FEDER, under research project MTM2009-08724, and the Consejería de Educación of the Junta de Andalucía (Spain).

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