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Exponential inequalities and the law of the iterated logarithm in the unbounded forecasting game

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Abstract We study the law of the iterated logarithm in the framework of game-theoretic probability of Shafer and Vovk. We investigate hedges under which a game-theoretic version of the upper bound of the law of the iterated logarithm holds without any condition on Reality's moves in the unbounded forecasting game. We prove that in the unbounded forecasting game with an exponential hedge, Skeptic can force the upper bound of the law of the iterated logarithm without conditions on Reality's moves. We give two examples such a hedge. For proving these results we derive exponential inequalities in the game-theoretic framework which may be of independent interest. Finally, we give related results for measure-theoretic probability which improve the results of Liu and Watbled (Stochastic Processes and their Applications 119:3101–3132, 2009).

Keywords Exponential inequality · Game-theoretic probability · Law of the iterated logarithm

1 Introduction

In this paper, we investigate the law of the iterated logarithm in the framework of game-theoretic probability of Shafer and Vovk (2001). We consider the following protocol.

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THE UNBOUNDED FORECASTING GAME WITH A HEDGE

Protocol:
$\mathcal{K}_0 := 1.$
FOR $n = 1, 2, \ldots$:
Forecaster announces $v_n > 0$.
Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.
Reality announces $x_n \in \mathbb{R}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(h(x_n) - v_n)$.
END FOR

In this game, in order to prevent Skeptic from becoming infinitely rich, Reality is forced to behave probabilistically. In Chapter 4 of Shafer and Vovk (2001), a game-theoretic version of the strong law of large numbers is proved under the variance hedge (quadratic hedge) $h(x) = x^2$. Namely, in the unbounded forecasting game with the quadratic hedge $h(x) = x^2$, Skeptic has a strategy that does not risk bankruptcy and allows him to become infinitely rich if $(x_1 + \cdots + x_n)/n \not\to 0$. Kumon et al. (2007) established a game-theoretic version of the strong law of large numbers under other types of nonnegative symmetric hedges $h(x)$ such that $\sum_{n>c}^{\infty} 1/h(n) < \infty$ for some $c \geq 0$. In addition, Takazawa (2009) proposed an exponential inequality in the unbounded forecasting game with the hedge $h(x) = x^2$.

As regards the convergence rate of the strong law of large numbers in games with bounded variables, there exist various researches (see Horikoshi and Takemura (2008), Kumon and Takemura (2008), Kumon et al. (2008)). In Kumon and Takemura (2008), it is proved that in the bounded forecasting game a simple single strategy that is based only on the past average of Reality's moves weakly forces the strong law of large numbers with the convergence rate of $O(\sqrt{\log n/n})$. In Kumon et al. (2008), it is proved that the Bayesian strategy of Skeptic weakly forces the strong law of large numbers with the convergence rate of $O(\sqrt{\log n/n})$ in coin-tossing games. Horikoshi and Takemura (2008) considered the lower bound on the convergence rate of the strong law of large numbers in fair-coin game. Moreover, Vovk (2007) presented a game-theoretic version of Azuma–Hoeffding's inequality in the bounded forecasting game.

In Chapter 5 of Shafer and Vovk (2001), a game-theoretic version of the law of the iterated logarithm is proved under the variance hedge $h(x) = x^2$.

Theorem 1 (Theorem 5.2 of Shafer and Vovk (2001)) *In the unbounded forecasting game with the quadratic hedge $h(x) = x^2$, Skeptic can force*

$$A_n \to \infty \quad \text{and} \quad |x_n| = o\left(\sqrt{\frac{A_n}{\log\log A_n}}\right) \Rightarrow \limsup_{n\to\infty} \frac{S_n}{\sqrt{2A_n \log\log A_n}} \leq 1,$$

where $S_n = \sum_{i=1}^{n} x_i$ and $A_n = \sum_{i=1}^{n} v_i$.

This result corresponds to the upper bound of Kolmogorov's law of the iterated logarithm (see e.g. Petrov (1995)) and Stout's martingale version of the upper bound of the law of the iterated logarithm (see Stout (1970)). In Kolmogorov's and Stout's law of the iterated logarithm, the upper bound and the lower bound hold under the same conditions. On the other hand, in the unbounded forecasting game with the quadratic hedge $h(x) = x^2$, the lower bound of the law of the iterated logarithm does not hold (see Proposition 5.1 of Shafer and Vovk (2001)). In order to have the lower bound, we must make the unbounded forecasting protocol more favorable to Skeptic, that is, we must slightly restrict Reality's freedom of action and slightly increase Skeptic's freedom of action, as follows:

THE PREDICTABLY UNBOUNDED FORECASTING GAME

Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $v_n > 0$ and $c_n \geq 0$.
Skeptic announces $M_n \in \mathbb{R}$ and $V_n \in \mathbb{R}$.
Reality announces $|x_n| \leq c_n$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(x_n^2 - v_n)$.
END FOR

Theorem 2 (Theorem 5.1 of Shafer and Vovk (2001)) *In the predictably unbounded forecasting game, Skeptic can force*

$$A_n \to \infty \quad \textit{and} \quad c_n = o\left(\sqrt{\frac{A_n}{\log\log A_n}}\right) \Rightarrow \limsup_{n\to\infty} \frac{S_n}{\sqrt{2A_n \log\log A_n}} = 1,$$

where $S_n = \sum_{i=1}^n x_i$ *and* $A_n = \sum_{i=1}^n v_i$.

This is one of the differences between measure-theoretic probability and game-theoretic probability. In the game-theoretic framework, it is natural to consider the upper bound and lower bound of the law of the iterated logarithm under different conditions, because Skeptic can force the upper bound under weaker conditions than the lower bound.

In this paper, we investigate hedges under which a game-theoretic version of the upper bound of the law of the iterated logarithm holds without any condition on Reality's moves in the unbounded forecasting game. In the view of game-theoretic probability, the conditional statement of Theorem 1 is not desirable because Reality's moves and Forecaster's moves cannot necessarily be limited. Therefore, it is of interest to establish protocols under which the law of the iterated logarithm holds under conditions favorable to Reality. In particular, if we consider only the upper bound of the law of the iterated logarithm, it is natural to drop conditions on Reality's moves. The following theorem is our result in this direction.

Theorem 3 *Let* $S_n = \sum_{i=1}^n x_i$ *and* $A_n = \sum_{i=1}^n v_i$. *Then, in the unbounded forecasting game with the hedge* $h(x) = \mathrm{e}^{|x|} - 1$, *for any* $t \in (-1, 1)$, *the process*

$$\exp\left(tS_n - \frac{t^2}{1-|t|} \cdot \frac{A_n}{\mathrm{e}}\right)$$

is a game-theoretic supermartingale. Furthermore, set $B_n = 2\mathrm{e}^{-1} \sum_{i=1}^n v_i \, (= 2\mathrm{e}^{-1} A_n)$.

Then, Skeptic can force

$$\lim_{n\to\infty} B_n < \infty \Rightarrow \limsup_{n\to\infty} |S_n| < \infty,$$
$$\lim_{n\to\infty} B_n = \infty \Rightarrow \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2B_n \log\log B_n}} \le 1.$$

The notion of a game-theoretic supermartingale will be defined in Sect. 2. The first part of Theorem 3 says that in the unbounded forecasting game with the hedge $h(x) = \mathrm{e}^{|x|} - 1$, for any $t \in (-1, 1)$, Skeptic can guarantee

$$\mathcal{K}_n = \exp\left(tS_n - \frac{t^2}{1-|t|}\cdot\frac{A_n}{\mathrm{e}}\right),$$

if he is allowed to discard part of his capital at each step.

If we translate Theorem 3 in measure-theoretic terms, we have the following result.

Theorem 4 *Let* (X_i) *be a martingale differences sequence and* $S_n = \sum_{i=1}^n X_i$. *Set* $A_n = \sum_{i=1}^n E\left[\mathrm{e}^{|X_i|} - 1 \mid \mathcal{F}_{i-1}\right]$. *For all* $t \in (-1, 1)$, *let*

$$V_n = \exp\left(tS_n - \frac{t^2}{1-|t|}\cdot\frac{A_n}{\mathrm{e}}\right).$$

Then V_n *is a positive supermartingale with* $E[V_n] \le 1$. *Furthermore, set* $B_n = 2\mathrm{e}^{-1}A_n$. *If* $B_n < \infty$ *a.s. for all* n *and* $\lim_{n\to\infty} B_n = \infty$ *a.s., then*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2B_n \log\log B_n}} \le 1 \quad a.s.$$

At the same time, by the first part of Theorem 4, we have an exponential inequality in the measure-theoretic framework.

Theorem 5 *Let* (X_i) *be a martingale differences sequence and* $S_n = \sum_{i=1}^n X_i$. *Set* $A_n = \sum_{i=1}^n E\left[\mathrm{e}^{|X_i|} - 1 \mid \mathcal{F}_{i-1}\right]$. *Then, for all* $a, b > 0$,

$$P(|S_n| \ge a,\ A_n \le b) \le 2\exp\left\{-\left(\sqrt{a + \mathrm{e}^{-1}b} - \sqrt{\mathrm{e}^{-1}b}\right)^2\right\}. \tag{1}$$

In game-theoretic probability, we can define upper probability (see Sect. 2). Then, from Theorem 3, we have an exponential inequality in the game-theoretic framework.

Theorem 6 *In the unbounded forecasting game with the hedge* $h(x) = \mathrm{e}^{|x|} - 1$, *if all* v_n *are given in advance, then for all* $a > 0$,

$$\bar{P}(|S_n| \ge a) \le 2\exp\left\{-\left(\sqrt{a + \mathrm{e}^{-1}A_n} - \sqrt{\mathrm{e}^{-1}A_n}\right)^2\right\}, \tag{2}$$

where $S_n = \sum_{i=1}^n x_i$ *and* $A_n = \sum_{i=1}^n v_i$.

Note that although Theorem 6 includes the assumption that all v_n are given in advance, Theorem 3 does not include the assumption. We mention that our game-theoretic result (2) is simple compared with a corresponding measure-theoretic result (1). This represents effectiveness of game-theoretic probability.

Furthermore, in the case where $h(x) = \exp(x^2/2) - 1$, we have similar results to the case where $h(x) = \mathrm{e}^{|x|} - 1$.

We mention that the lower bound of the law of the iterated logarithm simply does not hold in the generality of the unbounded forecasting game with a hedge $h(x) = \mathrm{e}^{|x|} - 1$ and $h(x) = \exp(x^2/2) - 1$. This is because the argument in Shafer and Vovk (2001), Proposition 5.1, still applies. Therefore, this paper concentrates on the upper bound of the law of the iterated logarithm.

The organization of the rest of this paper is as follows: In Sect. 2, we give some notations on game-theoretic probability. In Sect. 3, we propose an exponential inequality and prove the upper bound of the law of the iterated logarithm in the unbounded forecasting game with the hedge $h(x) = \mathrm{e}^{|x|} - 1$. In Sect. 4, we propose an exponential inequality and prove the upper bound of the law of the iterated logarithm in the unbounded forecasting game with the hedge $h(x) = \exp(x^2/2) - 1$. Finally, in Sect. 5, we provide the related results for measure-theoretic probability.

2 Notation

In this section, we give some notations on game-theoretic probability.

THE UNBOUNDED FORECASTING GAME WITH A HEDGE

Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $v_n > 0$.
Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.
Reality announces $x_n \in \mathbb{R}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(h(x_n) - v_n)$.
END FOR

Skeptic starts with the initial capital of $\mathcal{K}_0 = 1$. In each round n, Forecaster announces $v_n > 0$ first. Next Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$ and Reality decides $x_n \in \mathbb{R}$ after seeing Skeptic's move M_n and V_n. In this protocol, we consider a hedge $h(x_n)$. It is nonnegative and has a finite price $0 < v_n < \infty$. Skeptic is only allowed to buy arbitrary amount of this hedge, that is, V_n is restricted to be nonnegative ($V_n \geq 0$).

The set Ω of all infinite sequences $v_1 x_1 v_2 x_2 \ldots$ of Forecaster's and Reality's moves is called sample space. $\omega = v_1 x_1 v_2 x_2 \ldots$ denotes an infinite sequence of Forecaster's and Reality's moves and $\omega^n = v_1 x_1 v_2 x_2 \ldots v_n x_n$ denotes a sequence of Forecaster's and Reality's moves up to round n. An event E is a subset of Ω.

We denote by $\mathcal{K}_n$ Skeptic's capital at the end of round n. For a strategy $\mathcal{P}$ of Skeptic, $\mathcal{K}_n^{\mathcal{P}}(\omega)$ denotes the capital process of $\mathcal{P}$. We say that $\mathcal{P}$ satisfies the collateral duty if its capital process is always nonnegative, that is, if

$$\mathcal{K}_n^{\mathcal{P}}(\omega) \geq 0, \quad \forall \omega \in \Omega, \quad \forall n \geq 0.$$

We also say that $\mathcal{P}$ is prudent if it satisfies the collateral duty. When $\mathcal{P}$ is prudent, the capital process $\mathcal{K}^{\mathcal{P}}$ is called a (game-theoretic) nonnegative martingale.

A process $\mathcal{S}$ is a real-valued function of $v_1 x_1 \ldots v_n x_n$ in the unbounded forecasting game. We say that a process $\mathcal{S}$ is a game-theoretic supermartingale if there is a strategy for Skeptic that guarantees

$$\mathcal{S}(v_1 x_1 \ldots v_n x_n) - \mathcal{S}(v_1 x_1 \ldots v_{n-1} x_{n-1}) \leq \mathcal{K}_n - \mathcal{K}_{n-1},$$

for all n and for all $v_1 x_1 v_2 x_2 \ldots$; in other words, a game-theoretic supermartingale is a possible capital process for Skeptic who is allowed to discard part of his capital at each step.

We say that Skeptic can weakly force an event E if there exists a prudent strategy $\mathcal{P}$ of Skeptic such that

$$\limsup_{n \to \infty} \mathcal{K}_n^{\mathcal{P}}(\omega) = \infty, \ \forall \omega \notin E.$$

Similarly, we say that Skeptic can force an event E if there exists a prudent strategy $\mathcal{P}$ of Skeptic such that

$$\lim_{n \to \infty} \mathcal{K}_n^{\mathcal{P}}(\omega) = \infty, \ \forall \omega \notin E.$$

Shafer and Vovk (2001) derived the following two lemmas.

Lemma 1 (Lemma 3.1 of Shafer and Vovk (2001)) *If Skeptic can weakly force E, then he can force E.*

Lemma 2 (Lemma 3.2 of Shafer and Vovk (2001)) *If Skeptic can weakly force each of a sequence $E_1, E_2, \ldots$ of events, then he can weakly force $\cap_{k=1}^{\infty} E_k$.*

Following Shafer and Vovk (2001), Takeuchi (2004), and Takemura et al. (2009), for an event E, we define upper probability $\bar{P}(E)$ as

$$\bar{P}(E) = \inf \Big\{ \alpha \geq 0 \mid \text{ There exists a prudent strategy } \mathcal{P} \text{ of Skeptic such that} \\ \sup_{n \geq 1} \mathcal{K}_n^{\mathcal{P}}(\omega) \geq 1/\alpha \text{ for all } \omega \in E \Big\}.$$

Note that $0 \leq \bar{P}(E) \leq 1$, because Skeptic can choose $M_n = 0$ and $V_n = 0$ for all n. In addition, if Skeptic can force E, then $\bar{P}(E^C) = 0$.

3 An exponential inequality and LIL in the unbounded forecasting game with the hedge $h(x) = \mathrm{e}^{|x|} - 1$

In this section, we consider the unbounded forecasting game with the hedge $h(x) = \mathrm{e}^{|x|} - 1$.

THE UNBOUNDED FORECASTING GAME I

Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \dots$:
Forecaster announces $v_n > 0$.
Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.
Reality announces $x_n \in \mathbb{R}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(\mathrm{e}^{|x_n|} - 1 - v_n)$.
END FOR

We start with the following theorem. This theorem is inspired by Lemma 2.6 in Liu and Watbled (2009).

Theorem 7 *In the unbounded forecasting game I, for any $t \in (-1, 1)$, the process*

$$\prod_{i=1}^{n} \exp\left(tx_i - \frac{t^2}{1-|t|} \cdot \frac{v_i}{\mathrm{e}}\right)$$

is a game-theoretic supermartingale.

Proof We consider the following strategy $\mathcal{P}$:

$$M_n = t \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v_n}{\mathrm{e}}\right) \mathcal{K}_{n-1},$$
$$V_n = \frac{1}{v_n} \left\{1 - \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v_n}{\mathrm{e}}\right)\right\} \mathcal{K}_{n-1}.$$

Under this strategy, Skeptic's capital process $\mathcal{K}_n^{\mathcal{P}}$, starting with the initial capital of $\mathcal{K}_0 = 1$, is written as

$$\mathcal{K}_n^{\mathcal{P}} = \prod_{i=1}^{n} \Bigg[\exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v_i}{\mathrm{e}}\right) + tx_i \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v_i}{\mathrm{e}}\right) + \frac{1}{v_i}\left(\mathrm{e}^{|x_i|} - 1\right)\left\{1 - \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v_i}{\mathrm{e}}\right)\right\}\Bigg].$$

In order to prove the theorem, it suffices to show that for any $t \in (-1, 1)$,

$$\exp\left(tx - \frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right) \leq \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right) + tx \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right) + \frac{1}{v}\left(\mathrm{e}^{|x|} - 1\right)\left\{1 - \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right)\right\}.$$

For $x \geq 0$ and $k \geq 0$, let

$$f_k(x) = \mathrm{e}^{x-1} - \mathrm{e}^{-1} - \frac{x^k}{k!}.$$

Then we have $f_k(0) = 0$ for all k, and for $k \geq 1$,

$$f_k'(x) = \mathrm{e}^{x-1} - \frac{x^{k-1}}{(k-1)!} = f_{k-1}(x) + \mathrm{e}^{-1}.$$

Since

$$f_1'(x) = \mathrm{e}^{x-1} - 1,$$

we have

$$f_2'(x) \geq f_1(1) + \mathrm{e}^{-1} = 0.$$

It follows that $f_2(x) \geq 0$ for any $x \geq 0$. Hence, for all $k \geq 2$, we have

$$f_k(x) \geq f_k(0) = 0$$

Therefore for $-1 < t < 1$,

$$\begin{aligned} \mathrm{e}^{tx} &= 1 + tx + \sum_{k=2}^{\infty} \frac{t^k x^k}{k!} \\ &\leq 1 + tx + \sum_{k=2}^{\infty} |t|^k \left(\mathrm{e}^{|x|-1} - \mathrm{e}^{-1}\right) \\ &= 1 + tx + \frac{t^2}{1-|t|} \left(\mathrm{e}^{|x|-1} - \mathrm{e}^{-1}\right). \end{aligned} \tag{3}$$

Since $y\mathrm{e}^{-y} \leq 1 - \mathrm{e}^{-y}$ for $y \geq 0$, we have

$$\begin{aligned} \exp\left(tx - \frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right) &\leq \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right) + tx \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right) \\ &\quad + \frac{t^2}{1-|t|} \left(\mathrm{e}^{|x|-1} - \mathrm{e}^{-1}\right) \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right) \\ &\leq \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right) + tx \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right) \\ &\quad + \frac{1}{v} \left(\mathrm{e}^{|x|} - 1\right) \left\{1 - \exp\left(-\frac{t^2}{1-|t|} \cdot \frac{v}{\mathrm{e}}\right)\right\}. \end{aligned}$$

Therefore, the process

$$\prod_{i=1}^{n} \exp\left(tx_i - \frac{t^2}{1-|t|} \cdot \frac{v_i}{\mathrm{e}}\right)$$

is a game-theoretic supermartingale. □

From Theorem 7, we have the following result similar to Theorem 4.1 of Liu and Watbled (2009).

Theorem 8 *Let $S_n = \sum_{i=1}^{n} x_i$ and $A_n = \sum_{i=1}^{n} v_i$. In the unbounded forecasting game with the hedge $h(x) = \mathrm{e}^{|x|} - 1$, if all v_n are given in advance, then for all $a > 0$,*

$$\bar{P}(|S_n| \geq a) \leq 2\exp\left\{-\left(\sqrt{a + \mathrm{e}^{-1}A_n} - \sqrt{\mathrm{e}^{-1}A_n}\right)^2\right\}.$$

Proof From Theorem 7, for $0 \leq t < 1$, Skeptic's capital $\mathcal{K}_n$ is bounded as

$$\begin{aligned}\mathcal{K}_n &\geq \prod_{i=1}^{n} \exp\left(tx_i - \frac{t^2}{1-t} \cdot \frac{v_i}{\mathrm{e}}\right)\\ &\geq \exp\left(ta - \frac{t^2}{1-t} \cdot \frac{A_n}{\mathrm{e}}\right),\end{aligned}$$

on the event $\{ S_n \geq a \}$. For $0 \leq t < 1$, let

$$f(t) = ta - \frac{t^2}{1-t} \cdot \frac{A_n}{\mathrm{e}}.$$

Then $f'(t) = 0$ if and only if

$$t = 1 - \frac{\sqrt{\mathrm{e}^{-1}A_n}}{\sqrt{a + \mathrm{e}^{-1}A_n}}.$$

Since

$$f\left(1 - \frac{\sqrt{\mathrm{e}^{-1}A_n}}{\sqrt{a + \mathrm{e}^{-1}A_n}}\right) = \left(\sqrt{a + \mathrm{e}^{-1}A_n} - \sqrt{\mathrm{e}^{-1}A_n}\right)^2,$$

we have

$$\mathcal{K}_n \geq \exp\left\{\left(\sqrt{a + \mathrm{e}^{-1}A_n} - \sqrt{\mathrm{e}^{-1}A_n}\right)^2\right\}.$$

Therefore we obtain

$$\bar{P}(S_n \geq a) \leq \exp\left\{-\left(\sqrt{a + \mathrm{e}^{-1}A_n} - \sqrt{\mathrm{e}^{-1}A_n}\right)^2\right\}.$$

We can find the same upper-bound for $\bar{P}(S_n \leq -a)$. Hence we have

$$\bar{P}(|S_n| \geq a) \leq 2\exp\left\{-\left(\sqrt{a+\mathrm{e}^{-1}A_n} - \sqrt{\mathrm{e}^{-1}A_n}\right)^2\right\}.$$

□

Remark 1 We mention that Theorem 8 is stronger than Liu and Watbled's (4.3) in Theorem 4.1. This is because Liu and Watbled's formula (4.3) can be written, in the same notation, as

$$P(|S_n| > a) \leq 2\exp\left\{-\left(\sqrt{a+A_n+n} - \sqrt{A_n+n}\right)^2\right\}.$$

Furthermore, by an argument similar to Theorem 5.2 of Shafer and Vovk (2001), we obtain the law of the iterated logarithm in the unbounded forecasting game with the hedge $h(x) = \mathrm{e}^{|x|} - 1$.

Theorem 9 *Let $S_n = \sum_{i=1}^n x_i$ and $B_n = 2\mathrm{e}^{-1}\sum_{i=1}^n v_i$. Then, in the unbounded forecasting game with the hedge $h(x) = \mathrm{e}^{|x|} - 1$, Skeptic can force*

$$\lim_{n\to\infty} B_n < \infty \Rightarrow \limsup_{n\to\infty} |S_n| < \infty,$$

$$\lim_{n\to\infty} B_n = \infty \Rightarrow \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2B_n \log\log B_n}} \leq 1.$$

Proof By Theorem 7, there exists a strategy for Skeptic $\mathcal{P}(1/2)$ such that

$$\mathcal{K}_n^{\mathcal{P}(1/2)} \geq \exp\left(\frac{1}{2}S_n - \frac{1}{4}B_n\right).$$

It follows that

$$\lim_{n\to\infty} B_n < \infty \quad \text{and} \quad \limsup_{n\to\infty} S_n = \infty \ \Rightarrow \ \limsup_{n\to\infty} \mathcal{K}_n^{\mathcal{P}(1/2)} = \infty.$$

Therefore Skeptic can force

$$\lim_{n\to\infty} B_n < \infty \Rightarrow \limsup_{n\to\infty} S_n < \infty.$$

Next, we will show that Skeptic has a prudent strategy $\mathcal{P}$ such that

$$\lim_{n\to\infty} B_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{S_n}{\sqrt{2B_n \log\log B_n}} > 1 \ \Rightarrow \ \lim_{n\to\infty} \mathcal{K}_n^{\mathcal{P}} = \infty. \tag{4}$$

Let $E = \{\omega \mid \lim_{n\to\infty} B_n = \infty\}$ and

$$E_0 = \left\{\omega \mid \limsup_{n\to\infty} \frac{S_n}{\sqrt{2B_n \log\log B_n}} > \frac{(1+\delta)^2}{\sqrt{1-\varepsilon}}\right\}.$$

In order to prove (4), it suffices to show that for any $\delta \in (0, 1)$ and for any $\varepsilon \in (0, 1)$, there exists a prudent strategy $\mathcal{P}$ of Skeptic such that for all $\omega \in E \cap E_0$,

$$\limsup_{n\to\infty} \mathcal{K}_n^{\mathcal{P}}(\omega) = \infty.$$

Fix temporarily a number $\varepsilon \in (0, 1)$. For any $\theta \in (0, \varepsilon)$, we consider the following strategy $\mathcal{P}^{(\theta)}$ with initial capital 1:

$$\begin{aligned} M_n^{(\theta)} &= \theta \exp\left(-\frac{\theta^2}{1-\theta} \cdot \frac{v_n}{\mathrm{e}}\right) \mathcal{K}_{n-1}^{(\theta)}, \\ V_n^{(\theta)} &= \frac{1}{v_n}\left\{1 - \exp\left(-\frac{\theta^2}{1-\theta} \cdot \frac{v_n}{\mathrm{e}}\right)\right\} \mathcal{K}_{n-1}^{(\theta)}. \end{aligned}$$

From Theorem 7, Skeptic's capital $\mathcal{K}_n^{(\theta)}$ satisfies

$$\begin{aligned} \mathcal{K}_n^{(\theta)} &\geq \exp\left(\theta S_n - \frac{\theta^2}{2(1-\theta)} B_n\right) \\ &\geq \exp\left(\theta S_n - \frac{\theta^2}{2(1-\varepsilon)} B_n\right). \end{aligned}$$

For any $\delta \in (0, 1)$, set

$$\begin{aligned} n_k &= \min\left\{n \mid B_n \geq (1+\delta)^k\right\}, \\ k(n) &= \left\lfloor \log_{1+\delta} B_n \right\rfloor. \end{aligned}$$

Then, for each n, there exists $r(n) \in [0, 1)$ such that $k(n) = \log_{1+\delta} B_n - r(n)$. Let

$$\theta(k) = \sqrt{2(1-\varepsilon)(1+\delta)^{-k} \log k}.$$

Then there exists a positive integer $N(\delta, \varepsilon)$ such that $\theta(k) < \varepsilon$ for all $k \geq N(\delta, \varepsilon)$. For $n_k \leq m < n_{k+1}$,

$$\begin{aligned} \sqrt{2(1-\varepsilon) B_m^{-1} \log\log \frac{B_m}{1+\delta}} &\leq \theta(k(m)) \\ &\leq \sqrt{2(1-\varepsilon)(1+\delta)^{r(m)} B_m^{-1} \log\left(\frac{\log B_m}{\log(1+\delta)}\right)}. \end{aligned}$$

Therefore for sufficiently large m,

$$\frac{1}{1+\delta}\sqrt{\frac{2(1-\varepsilon)\log\log B_m}{B_m}} \leq \theta(k(m)) \leq \sqrt{\frac{2(1-\varepsilon)(1+\delta)\log\log B_m}{B_m}}.$$

Hence we have

$$\begin{aligned}\mathcal{K}_m^{(\theta(k(m)))} &\geq \exp\left(\theta(k(m))S_m - \frac{\theta(k(m))^2}{2(1-\varepsilon)}B_m\right)\\ &\geq \exp\left(\frac{S_m}{1+\delta}\sqrt{\frac{2(1-\varepsilon)\log\log B_m}{B_m}} - (1+\delta)\log\log B_m\right)\\ &= \exp\left\{\left(\frac{2\sqrt{1-\varepsilon}}{(1+\delta)^2}\frac{S_m}{\sqrt{2B_m\log\log B_m}} - 1\right)(1+\delta)\log\log B_m\right\},\end{aligned}$$

for sufficiently large m. Set

$$E_k = \bigcup_{m=n_k}^{n_{k+1}-1}\left\{\omega \mid \frac{S_m}{\sqrt{2B_m\log\log B_m}} > \frac{(1+\delta)^2}{\sqrt{1-\varepsilon}}\right\}.$$

Define

$$\tau_k = \min\left\{n_k \leq m < n_{k+1} \mid \frac{S_m}{\sqrt{2B_m\log\log B_m}} > \frac{(1+\delta)^2}{\sqrt{1-\varepsilon}}\right\},$$

letting $\min\emptyset = n_{k+1}-1$. We consider the following strategy $\mathcal{P}^{[k]}$:

$$M_n^{[k]} = \begin{cases}\theta(k)\exp\left(-\frac{\theta(k)^2}{2(1-\theta(k))}v_n\right)\mathcal{K}_{n-1}^{(\theta(k))}, & n \leq \tau_k,\\ 0, & n > \tau_k,\end{cases}$$

$$V_n^{[k]} = \begin{cases}\frac{1}{v_n}\left\{1-\exp\left(-\frac{\theta(k)^2}{2(1-\theta(k))}v_n\right)\right\}\mathcal{K}_{n-1}^{(\theta(k))}, & n \leq \tau_k,\\ 0, & n > \tau_k.\end{cases}$$

For sufficiently large n, Skeptic's capital $\mathcal{K}_n^{[k]}$ satisfies

$$\begin{aligned}\mathcal{K}_n^{[k]} &\geq \exp\left((1+\delta)\log\log B_{n_k}\right)\\ &\geq \exp\left((1+\delta)\log\log(1+\delta)^k\right)\\ &= (k\log(1+\delta))^{1+\delta},\end{aligned}$$

on E_k. Let

$$K_0 = \sum_{k=N(\delta,\varepsilon)}^{\infty} k^{-(1+\delta)},$$

$$\mathcal{P}(\delta,\varepsilon) = \frac{1}{K_0} \sum_{k=N(\delta,\varepsilon)}^{\infty} k^{-(1+\delta)} \mathcal{P}^{[k]},$$

$$\mathcal{K}_n^{\mathcal{P}(\delta,\varepsilon)} = \frac{1}{K_0} \sum_{k=N(\delta,\varepsilon)}^{\infty} k^{-(1+\delta)} \mathcal{K}_n^{[k]}.$$

It follows that

$$\lim_{n\to\infty} B_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{S_n}{\sqrt{2B_n \log\log B_n}} > \frac{(1+\delta)^2}{\sqrt{1-\varepsilon}}$$
$$\Rightarrow \limsup_{n\to\infty} \mathcal{K}_n^{\mathcal{P}(\delta,\varepsilon)} = \infty.$$

Therefore, by Lemma 1 and Lemma 2, Skeptic can force

$$\lim_{n\to\infty} B_n = \infty \Rightarrow \limsup_{n\to\infty} \frac{S_n}{\sqrt{2B_n \log\log B_n}} \le 1.$$

The theorem is proved. □

4 An exponential inequality and LIL in the unbounded forecasting game with the hedge $h(x) = \exp(x^2/2) - 1$

In this section, we consider the unbounded forecasting game with the hedge $h(x) = \exp(x^2/2) - 1$.

THE UNBOUNDED FORECASTING GAME II

Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $v_n > 0$.
Skeptic announces $M_n \in \mathbb{R}$ and $V_n \ge 0$.
Reality announces $x_n \in \mathbb{R}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(\exp(x_n^2/2) - 1 - v_n)$.
END FOR

First we show the following theorem.

Theorem 10 *In the unbounded forecasting game II, for any $t \in (-1, 1)$, the process*

$$\prod_{i=1}^{n} \exp\left(t x_i - \frac{t^2}{1-|t|} v_i\right)$$

is a game-theoretic supermartingale.

Proof We consider the following strategy $\mathcal{P}$:

$$M_n = t\exp\left(-\frac{t^2}{1-|t|}v_n\right)\mathcal{K}_{n-1},\ \ V_n = \frac{1}{v_n}\left\{1-\exp\left(-\frac{t^2}{1-|t|}v_n\right)\right\}\mathcal{K}_{n-1}.$$

Under this strategy, Skeptic's capital process $\mathcal{K}_n^{\mathcal{P}}$, starting with the initial capital of $\mathcal{K}_0 = 1$, is written as

$$\begin{aligned}\mathcal{K}_n^{\mathcal{P}} = \prod_{i=1}^{n}\Bigg[&\exp\left(-\frac{t^2}{1-|t|}v_i\right) + x_i t\exp\left(-\frac{t^2}{1-|t|}v_i\right)\\ &+\frac{1}{v_i}\left\{\exp\left(\frac{x_i^2}{2}\right)-1\right\}\left\{1-\exp\left(-\frac{t^2}{1-|t|}v_i\right)\right\}\Bigg].\end{aligned}$$

In order to prove the theorem, it suffices to show that for any $t \in (-1, 1)$,

$$\begin{aligned}\exp\left(tx-\frac{t^2}{1-|t|}v\right) \le{}& \exp\left(-\frac{t^2}{1-|t|}v\right) + xt\exp\left(-\frac{t^2}{1-|t|}v\right)\\ &+\frac{1}{v}\left\{\exp\left(\frac{x^2}{2}\right)-1\right\}\left\{1-\exp\left(-\frac{t^2}{1-|t|}v\right)\right\}.\end{aligned}$$

For $x \ge 0$ and $k \ge 2$, let

$$g_k(x) = \exp\left(\frac{x^2}{2}\right) - 1 - \frac{x^k}{k!}.$$

Then, for all $j \ge 1$, we have

$$\begin{aligned}g'_{2j}(x) &= x\left\{\exp\left(\frac{x^2}{2}\right) - \frac{x^{2j-2}}{(2j-1)!}\right\}\\ &\ge x\left\{\exp\left(\frac{x^2}{2}\right) - \frac{1}{(j-1)!}\left(\frac{x^2}{2}\right)^{j-1}\right\} \ge 0,\end{aligned}$$

and

$$\begin{aligned}g'_{2j+1}(x) &= x\left\{\exp\left(\frac{x^2}{2}\right) - \frac{x^{2j-1}}{(2j)!}\right\}\\ &\ge \frac{x}{2}\left\{2\exp\left(\frac{x^2}{2}\right) - \frac{x^{2j}}{(2j)!} - \frac{x^{2j-2}}{(2j)!}\right\}\\ &\ge \frac{x}{2}\left\{2\exp\left(\frac{x^2}{2}\right) - \frac{1}{j!}\left(\frac{x^2}{2}\right)^{j} - \frac{1}{(j-1)!}\left(\frac{x^2}{2}\right)^{j-1}\right\} \ge 0.\end{aligned}$$

Hence, for all $k \geq 2$, we have

$$g_k(x) \geq g_k(0) = 0$$

Therefore for $-1 < t < 1$,

$$\begin{aligned} e^{tx} &= 1 + tx + \sum_{k=2}^{\infty} \frac{t^k x^k}{k!} \\ &\leq 1 + tx + \sum_{k=2}^{\infty} |t|^k \left\{ \exp\left(\frac{x^2}{2}\right) - 1 \right\} \\ &= 1 + tx + \frac{t^2}{1-|t|} \left\{ \exp\left(\frac{x^2}{2}\right) - 1 \right\}. \end{aligned}$$

Since $ye^{-y} \leq 1 - e^{-y}$ for $y \geq 0$, we have

$$\begin{aligned} \exp\left(tx - \frac{t^2}{1-|t|}v\right) &\leq \exp\left(-\frac{t^2}{1-|t|}v\right) + xt \exp\left(-\frac{t^2}{1-|t|}v\right) \\ &\quad + \frac{t^2}{1-|t|} \left\{ \exp\left(\frac{x^2}{2}\right) - 1 \right\} \exp\left(-\frac{t^2}{1-|t|}v\right) \\ &\leq \exp\left(-\frac{t^2}{1-|t|}v\right) + xt \exp\left(-\frac{t^2}{1-|t|}v\right) \\ &\quad + \frac{1}{v} \left\{ \exp\left(\frac{x^2}{2}\right) - 1 \right\} \left\{ 1 - \exp\left(-\frac{t^2}{1-|t|}v\right) \right\}. \end{aligned}$$

Therefore, the process

$$\prod_{i=1}^{n} \exp\left(tx_i - \frac{t^2}{1-|t|}v_i\right)$$

is a game-theoretic supermartingale. □

By the same arguments as Theorem 8 and Theorem 9, in the unbounded forecasting game with the hedge $h(x) = \exp(x^2/2) - 1$, we obtain an exponential inequality and the law of the iterated logarithm, respectively.

Theorem 11 *Let* $S_n = \sum_{i=1}^n x_i$ *and* $A_n = \sum_{i=1}^n v_i$. *In the unbounded forecasting game with the hedge* $h(x) = \exp(x^2/2) - 1$, *if all* v_n *are given in advance, then for all* $a > 0$,

$$\bar{P}(|S_n| \geq a) \leq 2 \exp\left\{ -\left(\sqrt{a + A_n} - \sqrt{A_n}\right)^2 \right\}.$$

Theorem 12 *Let* $S_n = \sum_{i=1}^n x_i$ *and* $C_n = 2\sum_{i=1}^n v_i$. *Then, in the unbounded forecasting game with the hedge* $h(x) = \exp(x^2/2) - 1$, *Skeptic can force*

$$\lim_{n\to\infty} C_n < \infty \Rightarrow \limsup_{n\to\infty} |S_n| < \infty,$$

$$\lim_{n\to\infty} C_n = \infty \Rightarrow \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2C_n \log\log C_n}} \leq 1.$$

5 The related results for measure-theoretic probability

In this section, we give exponential inequalities and the upper bound of the law of the iterated logarithm for a sequence of martingale differences. We obtain the following measure-theoretic results from Theorems 7–9.

Theorem 13 *Let* (X_i) *be a sequence of martingale differences and* $S_n = \sum_{i=1}^n X_i$. *Set*

$$A_n = \sum_{i=1}^n E\left[\mathrm{e}^{|X_i|} - 1 \mid \mathcal{F}_{i-1}\right],$$

and $B_n = 2\mathrm{e}^{-1} A_n$.

(1) *For all* $t \in (-1, 1)$, *let*

$$V_n = \exp\left(tS_n - \frac{t^2}{1-|t|} \cdot \frac{A_n}{\mathrm{e}}\right).$$

Then V_n *is a positive supermartingale with* $E[V_n] \leq 1$.

(2) *For all* $a, b > 0$,

$$P(|S_n| \geq a,\ A_n \leq b) \leq 2\exp\left\{-\left(\sqrt{a + \mathrm{e}^{-1}b} - \sqrt{\mathrm{e}^{-1}b}\right)^2\right\}.$$

(3) *If* $B_n < \infty$ *a.s. for all n and* $\lim_{n\to\infty} B_n = \infty$ *a.s., then*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2B_n \log\log B_n}} \leq 1 \quad a.s.$$

Theorem 13 can be derived from Theorems 7–9 along the lines of Corollaries 8.1–8.3 in Shafer and Vovk (2001). This is because strategies for Skeptic constructed in the proof of Theorems 7–9 are measurable.

Of course, we can provide the measure-theoretic proof of Theorem 13 independent of the game-theoretic probability. We can derive an exponential inequality for a sequence of martingale differences by an argument similar to Theorem 2.1 of Bercu and Touati (2008). Also, we can prove the upper bound of the law of the iterated logarithm for a sequence of martingale differences by an argument similar to Theorem 1.1 of Stout (1973), or by using Corollary 4.2 of de la Peña et al. (2004).

Remark 2 In Corollary 5.2 of Liu and Watbled (2009), it is proved that if (X_i) is a sequence of martingale differences such that for some constants $\lambda_i > 0$,

$$E[\mathrm{e}^{|X_i|} \mid \mathcal{F}_{i-1}] \leq \lambda_i \quad \text{a.s.,} \tag{5}$$

then we have

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{n \log n}} \leq 2\sqrt{L^*} \quad \text{a.s.,} \tag{6}$$

where $S_n = \sum_{i=1}^n X_i$ and $L^* = \limsup_{n\to\infty}(\lambda_1 + \cdots + \lambda_n)/n$. On the other hand, Theorem 13 derives the upper bound of the law of the iterated logarithm for a sequence of martingale differences without the boundedness condition (5) for the conditional exponential moments. In addition, this result is sharper than (6).

Similarly, by Theorems 10–12, we have the following theorem.

Theorem 14 *Let (X_i) be a sequence of martingale differences and $S_n = \sum_{i=1}^n X_i$. Set*

$$A_n = \sum_{i=1}^n E\left[\exp\left(\frac{X_i^2}{2}\right) - 1 \mid \mathcal{F}_{i-1}\right],$$

and $C_n = 2A_n$.

(1) *For all $t \in (-1, 1)$, let*

$$V_n = \exp\left(tS_n - \frac{t^2}{1-|t|}A_n\right).$$

Then V_n is a positive supermartingale with $E[V_n] \leq 1$.

(2) *For all $a, b > 0$,*

$$P(|S_n| \geq a,\ A_n \leq b) \leq 2\exp\left\{-\left(\sqrt{a+b} - \sqrt{b}\right)^2\right\}.$$

(3) *If $C_n < \infty$ a.s. for all n and $\lim_{n\to\infty} C_n = \infty$ a.s., then*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2C_n \log\log C_n}} \leq 1 \quad \textit{a.s.}$$

Remark 3 In Theorem 4.2 of Liu and Watbled (2009), it is proved that if (X_i) is a sequence of martingale differences such that for some constants $\lambda_i > 0$ and $R > 0$,

$$E\left[\exp(RX_i^2) \mid \mathcal{F}_{i-1}\right] \leq \lambda_i \quad \text{a.s.,} \tag{7}$$

then for each $L \geq (\lambda_1 + \cdots + \lambda_n)/n$, there exist a constant $c > 0$ depending only on L and R such that for all $t \in \mathbb{R}$,

$$E[e^{tS_n}] \leq \exp(cnt^2),$$

and for all $a > 0$,

$$P(|S_n| > a) \leq 2\exp\left(-\frac{a^2}{4cn}\right),$$

where $S_n = \sum_{i=1}^{n} X_i$. On the other hand, Theorem 14 derives an exponential inequality and the upper bound of the law of the iterated logarithm for a sequence of martingale differences without the boundedness condition (7) for the conditional exponential moments.

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