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A sequential order statistics approach to step-stress testing

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Abstract For general step-stress experiments with arbitrary baseline distributions, wherein the stress levels change immediately after having observed pre-specified numbers of observations under each stress level, a sequential order statistics model is proposed and associated inferential issues are discussed. Maximum likelihood estimators (MLEs) of the mean lifetimes at different stress levels are derived, and some useful properties of the MLEs are established. Joint MLEs are also derived when an additional location parameter is introduced into the model, and estimation under order restriction of the parameters at different stress levels is finally discussed.

Keywords Accelerated life-testing · Step-stress experiment · Generalized order statistics · k-out-of-n system · Location-scale family of distributions · Maximum likelihood estimation · Order restricted inference

1 Introduction

Models and methods of accelerated life-testing are useful when technical systems under test tend to have long lifetimes. Under normal operating conditions, as systems usually last long, the corresponding life-tests become too time-consuming and expensive. In these cases, accelerated tests can be applied to reduce the experimental

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time and hence the cost; see, for example, Bagdonavicius and Nikulin (2002), Meeker and Escobar (1998), and Nelson (1980, 1990). One such accelerated life-testing is the step-stress testing wherein the units under test are successively exposed to increasing stress levels higher than under normal operating conditions. Then, additional model assumptions lead to inferential results for the underlying life-distribution based on observations taken under different stress levels. To begin with, we consider a simple step-stress test involving just two different stress levels.

In step-stress models, usually a cumulative exposure model is assumed (Sedyakin 1966; Bagdonavicius 1978; Nelson 1980; Miller and Nelson 1983), in which the remaining lifetime of a unit under test depends on the cumulative exposure up to the present time. In an exponential, simple step-stress setting with parameters θ_1 and θ_2 in the first and the second stress levels, respectively, and time τ at which the stress level changes, the underlying distribution of the lifetime of a unit is given by

$$G_{\theta_1,\theta_2}(t) = \begin{cases} F_1(t), & 0 < t < \tau \\ F_2\left(t + \tau(\frac{\theta_2}{\theta_1} - 1)\right), & t \geq \tau \end{cases}, \tag{1}$$

where

$$F_i(t) = 1 - e^{-t/\theta_i}, \quad t > 0, \ i = 1, 2 \quad ,$$

and the corresponding density function is

$$g_{\theta_1,\theta_2}(t) = \begin{cases} \dfrac{1}{\theta_1} e^{-t/\theta_1}, & 0 < t < \tau \\ \dfrac{1}{\theta_2} e^{-\frac{t-\tau}{\theta_2} - \frac{\tau}{\theta_1}}, & t \geq \tau \end{cases}.$$

This construction and its physical interpretation have been detailed by Miller and Nelson (1983) and Nelson (1990). Note that G is a continuous distribution function which becomes important when dealing with order statistics associated with the failures of systems. Moreover, in the intervals $(0, \tau)$ and $[\tau, \infty)$, G keeps the behaviour of F_1 and F_2 in terms of constant failure rates (i.e., hazard rates) $1/\theta_1$ and $1/\theta_2$, respectively. The point τ at which the stress level changes is simply the time point at which the failure rate changes. The same applies to general step-stress models with stress levels $x_1, \ldots, x_m$, and change points $\tau_1, \ldots, \tau_{m-1}$ with failure rates $\frac{1}{\theta_1}$ before τ_1, $1/\theta_i$ between τ_{i-1} and τ_i for $i = 2, \ldots, m-1$, and $\frac{1}{\theta_m}$ after τ_{m-1}.

Statistical inference for an exponential step-stress model has been developed by Xiong (1998), Xiong and Milliken (1999), and Balakrishnan et al. (2007, 2009a,b). For a recent overview of developments in this regard, one may refer to Balakrishnan (2009). In this paper, we will focus on the case when the available data are Type-II censored.

The multi-sample version of a simple step-stress experiment under Type-II censoring has been discussed by Kateri et al. (2009), wherein the MLEs of the parameters θ_1 and θ_2 as well as their exact distributions have been derived explicitly.

In most step-stress models discussed so far in the literature, stress levels change at pre-fixed time points. From this arises the problem that there may be no observations under some stress levels resulting in the non-existence of MLEs for the parameters corresponding to those stress levels. These probabilities of non-existence of the MLEs, which may even be large depending on the model parameters, have been examined, compared and illustrated in the case of a simple step-stress model by Kateri et al. (2010). For alleviating this problem as well as for some other reasons associated with the performance of a life-test, experimenters may consider increasing a stress level only after having observed a pre-specified number of observations at the current level of stress. Under this proposed alternative model, stress level changes are made right after failure times of a specified number of failures; evidently, the change times are random in this case. Compared to the original model, this approach ensures the existence of the MLEs of all the parameters in the model.

Xiong and Milliken (1999), Teng and Yeo (2002), Xiong et al. (2006), and Wang (2006) have all discussed step-stress models with random change times of stress. We comment on these works in Sect. 2.3 after introducing the basic description of this model. It should be mentioned that the step-stress model with level changes occurring at pre-fixed time points, although simple and intuitive, has the problem of not only the possible non-existence of the MLEs, but also quite complicated expressions for the distributions and the moments of the MLEs; see, for example, Balakrishnan et al. (2007, 2009b) for the single-sample case and Kateri et al. (2009) for the multi-sample case. This is the situation even in the case of a simple step-stress model (two levels of stress) and underlying exponential distributions. For example, even for this simple set-up, there is no closed form of the bias of the MLE of θ_1. Naturally, the general case involving multiple stress levels becomes much more arduous, with the ensuing exact inference becoming intractable unless some simplifying assumptions are made about the parameters. In the Weibull step-stress model, for example, no closed-form expressions are available for the MLEs even in the case of two stress levels, as shown by Kateri and Balakrishnan (2008).

These issues provided us the impetus for proposing an alternative model for general step-stress experiments, i.e., with two or more levels of stress, where, in the case of underlying exponential distributions, we stick to the idea of having a step-function for the failure rates over time. We adopt a general model in the analysis of ordered data called “sequential order statistics” (see Kamps 1995a,b) to the step-stress situation. This model, as the rest of the paper displays, enables exact inferential results within a wide class of distributions, including exponential distribution as a special case.

2 Model

2.1 Sampling situation

The life-test under consideration consists of n test units, and a number of $l \leq n$ stress levels are to be successively adopted. The parameters to be estimated based on the step-stress test are denoted by $\theta_1, \ldots, \theta_l > 0$, corresponding to the l stress levels.

The data may be Type-II censored resulting in $r \leq n$ complete lifetimes in total. The numbers of observations under different stress levels are pre-fixed.

Starting with $\rho_1 > 0$ observations under the first stress level with an underlying distribution function F_1 for the lifetimes given by

$$F_1(t) = 1 - \{1 - F(t)\}^{1/\theta_1}, \quad t > 0,$$

we successively obtain $\rho_j > 0$ observations under the jth stress level with F_j being the corresponding distribution function given by

$$F_j(t) = 1 - \{1 - F(t)\}^{1/\theta_j}, \quad \theta_j > 0, \ 2 \leq j \leq l.$$

In the representation of the level distributions, F is some baseline absolutely continuous distribution that is assumed to be known. So, $r = \rho_1 + \cdots + \rho_l$. With $\tilde{\rho}_j = \sum_{i=1}^{j} \rho_i$, the experiment then involves the jth stress level being set at the time of the $\tilde{\rho}_{j-1}$th failure, resulting in a change of distribution from F_{j-1} to F_j, for $2 \leq j \leq l$.

Example 1 i) Let the baseline distribution be standard exponential, i.e.,

$$F(t) = 1 - e^{-t}, \quad t > 0, \ \text{then } F_j \equiv Exp(\theta_j), \quad 1 \leq j \leq l.$$

In this case, we obtain the same situation as under the cumulative exposure model (1) in the sense that the hazard rate in each step of the experiment is constant.

ii) More generally, the baseline distribution could be taken as

$$F(t) = 1 - \exp\{-g(t) + \mu\}, \quad t \geq g^{-1}(\mu), \quad \mu \in \mathbb{R},$$

with g being differentiable and strictly increasing on $(g^{-1}(\mu), \infty)$. This model, which provides great flexibility (cf. Cramer and Kamps 2001b), includes Weibull, special Pearson Type-I, Pareto and Lomax distributions as special cases. In this case, the F_j's are given by

$$F_j(t) = 1 - \exp\left\{-\frac{g(t) - \mu}{\theta_j}\right\}, \quad 1 \leq j \leq l.$$

2.2 Setting in terms of sequential order statistics

Sequential order statistics (SOSs) were introduced by Kamps (1995a,b) as an extension of the usual order statistics in connection with generalized order statistics for modelling sequential k-out-of-n systems, wherein the failures of components possibly affect the lifetime distributions of the remaining ones. In this setting, a more realistic model results for k-out-of-n system, wherein, upon the failure of a component, an increased stress is put on the remaining active components. For more details on the model and its properties and associated inferential results, one may refer to Cramer and Kamps (1996, 2001a,b, 2003) and Balakrishnan et al. (2008).

SOSs can be formally defined as follows. Let $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n > 0$ be parameters and $B_i \sim Pow(\alpha_i(n-i+1))$, $1 \le i \le n$, be independent random variables, where $Pow(\alpha)$ denotes a power function distribution with distribution function u^α, $0 < u < 1$. Then, $X_*^{(r)} \overset{d}{=} F^{-1}\left(1 - \prod_{i=1}^r B_i\right)$, $1 \le r \le n$, are called SOSs based on the distribution function F. Here, we assume F to be absolutely continuous with density function f. If, in particular, we choose $\alpha_1 = \cdots = \alpha_n = 1$, we end up with the usual order statistics from distribution function F.

The joint density function of the SOSs $X_*^{(1)}, \ldots, X_*^{(r)}$ is then given by

$$
\begin{aligned}
&f^{X_*^{(1)},\ldots,X_*^{(r)}}(x_1, \ldots, x_r) \\
&= \frac{n!}{(n-r)!}\left(\prod_{i=1}^r \alpha_i\right)\left(\prod_{j=1}^{r-1}(1-F(x_j))^{m_j} f(x_j)\right)(1-F(x_r))^{\alpha_r(n-r+1)-1} f(x_r), \\
&\quad \text{on the cone } F^{-1}(0+) < x_1 \le \cdots \le x_r < F^{-1}(1), \quad r \le n, \\
&\quad \text{with } m_j = \gamma_j - \gamma_{j+1} - 1, \quad 1 \le j \le r-1, \\
&\quad \gamma_j = (n-j+1)\,\alpha_j, \quad 1 \le j \le r.
\end{aligned} \tag{2}
$$

The latter parameters are associated with generalized order statistics (cf. Kamps 1995a,b).

SOSs can be interpreted as describing successive failures, where, after the failure of some component, the lifetime distributions of the remaining components is allowed to change, i.e., the conditional distribution of $X_*^{(i)}$, given the observation of the previous failure, is

$$
P\left(X_*^{(i)} > t | X_*^{(i-1)} = s\right) = \left(\frac{1 - G_i(t)}{1 - G_i(s)}\right)^{n-i+1}, \quad t > s,\ 1 \le i \le r,
$$

with absolutely continuous distribution functions $G_1, G_2, \ldots$.

In our situation, if G_i is chosen to be $G_i(t) = 1 - \{1 - F(t)\}^{\alpha_i}$, we simply obtain

$$
P\left(X_*^{(i)} > t | X_*^{(i-1)} = s\right) = \left(\frac{1 - F(t)}{1 - F(s)}\right)^{\alpha_i(n-i+1)}, \quad 1 \le i \le r.
$$

The distribution is then assumed to stay the same within each level of stress. In this setup, we then have r observations (as described earlier in Sect. 2.1) which are supposed to be realizations of SOSs $X_*^{(1)}, \ldots, X_*^{(r)}$ based on F and the choice of parameters

$$
\alpha_1 = \cdots = \alpha_{\rho_1} = \frac{1}{\theta_1}, \quad \alpha_{\tilde{\rho}_j+1} = \cdots = \alpha_{\tilde{\rho}_{j+1}} = \frac{1}{\theta_{j+1}} \quad \text{for } 1 \le j \le l-1; \tag{3}
$$

see Kamps (1995a,b) and Cramer and Kamps (2001b, 2003). We follow the conventional notation that $\sum_{j=a}^b * = 0$ and $\prod_{j=a}^b * = 1$ for $a > b$.

Hence,

$$m_i = \begin{cases} \dfrac{1}{\theta_1} - 1, & 1 \le i \le \rho_1 - 1 \\ \dfrac{n-\rho_1+1}{\theta_1} - \dfrac{n-\rho_1}{\theta_2} - 1, & i = \rho_1 \\ \dfrac{1}{\theta_{j+1}} - 1, & \tilde{\rho}_j + 1 \le i \le \tilde{\rho}_{j+1} - 1, \quad 1 \le j \le l-1 \\ \dfrac{n-\tilde{\rho}_{j+1}+1}{\theta_{j+1}} - \dfrac{n-\tilde{\rho}_{j+1}}{\theta_{j+2}} - 1, & i = \tilde{\rho}_{j+1}, \quad 1 \le j \le l-2 \end{cases}.$$

With this, the joint density function of $X_*^{(1)}, \ldots, X_*^{(r)}$, for $r \le n$ [see (2)], and $F(x_0) = 0$ (say), is given by

$$\begin{aligned} & f^{X_*^{(1)},\ldots,X_*^{(r)}}(x_1, \ldots, x_r) \qquad (4) \\ &= \frac{n!}{(n-r)!} \left(\prod_{i=1}^{r} \alpha_i \right) \left(\prod_{i=1}^{r-1} \bar{F}^{m_i}(x_i) f(x_i) \right) \bar{F}^{\gamma_r - 1}(x_r) f(x_r) \\ &= \frac{n!}{(n-r)!} \left(\prod_{j=1}^{l} \theta_j^{-\rho_j} \right) \left(\prod_{i=1}^{\rho_1 - 1} \bar{F}^{\frac{1}{\theta_1} - 1}(x_i) f(x_i) \right) \bar{F}^{\frac{n-\rho_1+1}{\theta_1} - \frac{n-\rho_1}{\theta_2} - 1}(x_{\rho_1}) f(x_{\rho_1}) \\ &\times \left\{ \prod_{j=2}^{l-1} \left(\prod_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j - 1} \bar{F}^{\frac{1}{\theta_j} - 1}(x_i) f(x_i) \right) \bar{F}^{\frac{n-\tilde{\rho}_j+1}{\theta_j} - \frac{n-\tilde{\rho}_j}{\theta_{j+1}} - 1}(x_{\tilde{\rho}_j}) f(x_{\tilde{\rho}_j}) \right\} \\ &\times \left(\prod_{i=\tilde{\rho}_{l-1}+1}^{r-1} \bar{F}^{\frac{1}{\theta_l} - 1}(x_i) f(x_i) \right) \bar{F}^{\frac{n-r+1}{\theta_l} - 1}(x_r) f(x_r), \end{aligned}$$

where $\bar{F} = 1 - F$ denotes the survival function.

On jth stress level, $1 \le j \le l$, the conditional distribution of $X_*^{(i)}$, given $X_*^{(i-1)} = s$, in the above setting, is given by

$$\begin{aligned} & P\left(X_*^{(i)} > t | X_*^{(i-1)} = s\right) = \left(\frac{\bar{F}_j(t)}{\bar{F}_j(s)} \right)^{n-i+1}, \quad t > s, \\ & \tilde{\rho}_{j-1} + 1 \le i \le \tilde{\rho}_j, \quad \tilde{\rho}_0 = 0. \qquad (5) \end{aligned}$$

Hence, the failures under the jth stress level can be viewed as the usual order statistics from the distribution F_j. The failure rate of F_j is

$$\frac{f_j(t)}{\bar{F}_j(t)} = \frac{1}{\theta_j}\,\frac{f(t)}{\bar{F}(t)}, \quad 1 \le j \le l,$$

which is proportional to the failure rate of the baseline distribution.

If we assume exponential life-times, then, in the situation of Example 1 i) with a standard exponential distribution as baseline distribution having constant failure rate 1, we are actually assuming that the failures on the jth stress level are realizations of the usual order statistics from an exponential distribution with parameter θ_j, with constant failure rate $1/\theta_j$. SOSs, therefore, serve as a suitable stochastic model for the description of a general step-stress model.

In the setting of Example 1 ii), the failure rate of F_j is given by $\frac{1}{\theta_j} g'(t)$, which gives rise to a model involving an adjustment of $1/\theta_j$ over time.

Generally, the model of SOSs reflects the conditional nature of the step-stress model under consideration. Due to its Markovian property with the transition probabilities as shown in (5), we are in the situation of *a* cumulative exposure model, since the history of the experiment is not recorded.

2.3 Existing literature

As mentioned earlier in the Introduction, there exists limited work on random change times of stress levels in step-stress experiments, and we shall make some comments on them here.

Xiong and Milliken (1999) considered random stress change times in a general step-stress experiment. Integral representations of the lifetime distribution of the experiments were worked out for simple as well as for general step-stress models. With an underlying exponential distribution, an explicit representation of the lifetime distribution was presented in the simple step-stress model. In addition to illustrating their inferential methods with a numerical example, they also discussed optimal test plan under simple step-stress.

Teng and Yeo (2002) treated general step-stress experiments with failure-censored structure and exponential failures, while assuming a log-linear life-stress relationship. They further applied a least-squares approach for estimating the model parameters and presented illustrative numerical examples.

Wang (2006) considered a step-stress situation wherein some stress level changes when a pre-specified number of observations had been observed. From the description of the experiment, Wang arrived at a joint density of the failure times as a Weinman multivariate exponential distribution for ordered quantities [see Kotz et al. (2000, p. 388)], which in turn may be viewed as one of the SOSs given in (2) corresponding to the exponential case (Kamps 1995a, p. 54). Thus, the structure of SOSs is implicitly present in this work. In the present notation, Wang introduced the statistics

$$T_1 = \sum_{i=1}^{\rho_1} X_*^{(i)} + (n - \rho_1)\, X_*^{(\rho_1)}$$

and

$$T_{j+1} = \sum_{i=\tilde{\rho}_j+1}^{\tilde{\rho}_{j+1}} \left(X_*^{(i)} - X_*^{(\tilde{\rho}_j)} \right) + \left(n - \tilde{\rho}_{j+1}\right) \left(X_*^{(\tilde{\rho}_{j+1})} - X_*^{(\tilde{\rho}_j)} \right), \quad 1 \le j \le l-1,$$

where $X_*^{(1)}, X_*^{(2)}, \ldots$ denote the SOSs based on a standard exponential distribution with parameter setting as in (3). Upon rewriting these statistics in terms of spacings of SOSs, we end up with

$$T_1 = \sum_{i=1}^{\rho_1} (n-i+1) \left(X_*^{(i)} - X_*^{(i-1)} \right), \quad X_*^{(0)} = 0,$$

and

$$T_{j+1} = \sum_{i=\tilde{\rho}_j+1}^{\tilde{\rho}_{j+1}} (n-i+1) \left(X_*^{(i)} - X_*^{(i-1)} \right), \quad 1 \le j \le l-1.$$

Wang proved that $T_1, \ldots, T_l$ are independent and gamma distributed. It will be shown later in Sect. 3 that the quantities $T_j/\rho_j,\ 1 \le j \le l$, are in fact the MLEs of $\theta_1, \ldots, \theta_l$ (cf. Theorem 1). Their independence and distribution in the general case is presented in Theorem 2. Wang (2006) further assumed that the logarithm of the mean lifetime of a test unit is a linear function of the stress level, and then discussed the estimation of the intercept and slope parameters.

Xiong et al. (2006) discussed a simple step-stress test where again the change of the stress level is made immediately after the failure with a pre-specified number. They then assumed to observe only this particular failure and a pre-fixed failure terminating the experiment, and presented estimators of the corresponding parameters when the mean lifetime was once again assumed to be a log-linear function of the stress level.

3 Maximum likelihood estimation

In this section, we discuss the joint estimation of the model parameters without and with an order restriction, as well as the joint estimation of a location parameter.

3.1 Maximum likelihood estimators of $\theta_1, \ldots, \theta_l$

Now, on the basis of the joint density function in (4), the MLEs of $\theta_1, \ldots, \theta_l$ can be obtained as given in the following theorem.

Theorem 1 *The unique maximum likelihood estimators* $\hat{\theta}_1, \ldots, \hat{\theta}_l$ *of* $\theta_1, \ldots, \theta_l$*, respectively, are given by*

$$\hat{\theta}_1 = -\frac{1}{\rho_1} \sum_{i=1}^{\rho_1} (n - i + 1)(\log \bar{F}(x_i) - \log \bar{F}(x_{i-1}))$$

and

$$\hat{\theta}_j = -\frac{1}{\rho_j} \sum_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j} (n - i + 1)(\log \bar{F}(x_i) - \log \bar{F}(x_{i-1})), \quad 2 \leq j \leq l.$$

Proof The log-likelihood function is

$$\begin{aligned}
l(&\theta_1, \ldots, \theta_l; x_1, \ldots, x_r) = \log L(\theta_1, \ldots, \theta_l; x_1, \ldots, x_r) \\
&= \log \frac{n!}{(n-r)!} - \sum_{j=1}^{l} \rho_j \log \theta_j \\
&\quad + \frac{1}{\theta_1} \left(\sum_{i=1}^{\rho_1 - 1} \log \bar{F}(x_i) + (n - \rho_1 + 1) \log \bar{F}(x_{\rho_1}) \right) \\
&\quad - \sum_{i=1}^{\rho_1 - 1} \log \bar{F}(x_i) - \left(\frac{n - \rho_1}{\theta_2} + 1 \right) \log \bar{F}(x_{\rho_1}) \\
&\quad + \sum_{j=2}^{l-1} \left\{ \frac{1}{\theta_j} \left(\sum_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j - 1} \log \bar{F}(x_i) + (n - \tilde{\rho}_j + 1) \log \bar{F}(x_{\tilde{\rho}_j}) \right) \right. \\
&\quad \left. - \sum_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j - 1} \log \bar{F}(x_i) - \left(\frac{n - \tilde{\rho}_j}{\theta_{j+1}} + 1 \right) \log \bar{F}(x_{\tilde{\rho}_j}) \right\} \\
&\quad + \sum_{i=\tilde{\rho}_{l-1}+1}^{r-1} (\frac{1}{\theta_l} - 1) \log \bar{F}(x_i) + \left(\frac{n - r + 1}{\theta_l} - 1 \right) \log \bar{F}(x_r) + \sum_{i=1}^{r} \log f(x_i)
\end{aligned}$$

Rewriting this log-likelihood function in terms of $\hat{\theta}_1, \ldots, \hat{\theta}_l$, we get

$$\begin{aligned}
l(&\theta_1, \ldots, \theta_l; x_1, \ldots, x_r) \\
&= \log \frac{n!}{(n-r)!} - \sum_{j=1}^{l} \rho_j \log \theta_j - \frac{\rho_1 \hat{\theta}_1}{\theta_1} \\
&\quad - \sum_{i=1}^{\rho_1 - 1} \log \bar{F}(x_i) - \left(\frac{n - \rho_1}{\theta_2} + 1 \right) \log \bar{F}(x_{\rho_1})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=2}^{l-1}\frac{1}{\theta_j}\left(-\rho_j\hat{\theta}_j+(n-\tilde{\rho}_{j-1})\log\bar{F}(x_{\tilde{\rho}_{j-1}})\right)\\
&-\sum_{j=2}^{l-1}\left(\sum_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j-1}\log\bar{F}(x_i)-\left(\frac{n-\tilde{\rho}_j}{\theta_{j+1}}+1\right)\log\bar{F}(x_{\tilde{\rho}_j})\right)\\
&+\frac{1}{\theta_l}\left(\sum_{i=\tilde{\rho}_{l-1}+1}^{r-1}\log\bar{F}(x_i)+(n-r+1)\log\bar{F}(x_r)\right)\\
&-\sum_{i=\tilde{\rho}_{l-1}+1}^{r-1}\log\bar{F}(x_i)-\log\bar{F}(x_r)+\sum_{i=1}^{r}\log f(x_i)\\
=&\log\frac{n!}{(n-r)!}+\sum_{j=1}^{l}\rho_j\log\frac{1}{\theta_j}-\sum_{j=1}^{l}\frac{\rho_j\hat{\theta}_j}{\theta_j}+\sum_{i=1}^{r}\log\frac{f(x_i)}{\bar{F}(x_i)}.
\end{aligned}
$$

Since $\log\frac{1}{\theta_j}=\log\frac{\hat{\theta}_j}{\theta_j}-\log\hat{\theta}_j\le\frac{\hat{\theta}_j}{\theta_j}-1-\log\hat{\theta}_j$, the latter expression is bounded from above by

$$
\log\frac{n!}{(n-r)!}-r-\sum_{j=1}^{l}\rho_j\log\hat{\theta}_j+\sum_{i=1}^{r}\log\frac{f(x_i)}{\bar{F}(x_i)}
$$

with equality iff $\theta_j=\hat{\theta}_j,\quad 1\le j\le l$. Hence, the result. □

Theorem 1 presents the MLEs $\hat{\theta}_1,\ldots,\hat{\theta}_l$ of $\theta_1,\ldots,\theta_l$, respectively, in a general step-stress situation with arbitrary number of stress levels as well as for an arbitrary absolutely continuous baseline distribution.

Remark 1 In the particular case of only one observation at each stress level, i.e., $\rho_j=1$ for all $1\le j\le l$, the MLEs are as presented by Cramer and Kamps (1996).

Then, by adopting the ideas of Cramer and Kamps (1996), it is possible to derive several useful properties of the MLEs in Theorem 1. It is easily seen that the rv's $-\log\bar{F}(X_*^{(i)}),\quad 1\le i\le r$, are SOSs based on a standard exponential distribution. Moreover, from Theorem 3.3.5 of Kamps (1995a, p. 81), it is known that the normalized spacings

$$
-n\alpha_1\log\bar{F}(X_*^{(1)})\quad\text{and}\ -(n-i+1)\alpha_i(\log\bar{F}(X_*^{(i)})-\log\bar{F}(X_*^{(i-1)})),\ 2\le i\le r,
$$

are iid rv's with standard exponential distribution. So, with $V_1,\ldots,V_r$ denoting iid *Exp*(1) random variables, we have the stochastic representations for the MLEs as

$$\hat{\theta}_1 \sim \frac{\theta_1}{\rho_1} \sum_{i=1}^{\rho_1} V_i \quad \text{and} \quad \hat{\theta}_j \sim \frac{\theta_j}{\rho_j} \sum_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j} V_i, \quad 2 \le j \le l.$$

Evidently, the MLEs are gamma distributed.

We shall now use the notation $X \sim \Gamma(a, b)$ for the random variable X having a gamma distribution with parameters $a > 0$ and $b > 0$, i.e., with density function

$$f^X(x) = \frac{1}{\Gamma(a)} b^{-a} x^{a-1} e^{-x/b}, \quad x > 0.$$

Note for $c > 0$, we have $cX \sim \Gamma(a, bc)$.

Theorem 2 *The MLEs $\hat{\theta}_1, \ldots, \hat{\theta}_l$ of $\theta_1, \ldots, \theta_l$, respectively, possess the following properties:*

i) *$\hat{\theta}_1, \ldots, \hat{\theta}_l$ are stochastically independent;*
ii) *$\hat{\theta}_j \sim \Gamma(\rho_j, \frac{\theta_j}{\rho_j})$ with $E\hat{\theta}_j = \theta_j$, $1 \le j \le l$, i.e., the MLEs are unbiased, and $Var(\hat{\theta}_j) = \frac{\theta_j^2}{\rho_j}$, $1 \le j \le l$;*
iii) *$(\hat{\theta}_1, \ldots, \hat{\theta}_l)$ is a complete and sufficient statistic for $(\theta_1, \ldots, \theta_l)$;*
iv) *$(\hat{\theta}_1, \ldots, \hat{\theta}_l)$ is the UMVUE of $(\theta_1, \ldots, \theta_l)$;*
v) *The sequence of estimators $(\hat{\theta}_j(\rho_j))_{\rho_j}$ is strongly consistent for θ_j, i.e., $\hat{\theta}_j(\rho_j) \to \theta_j$ a.e. with respect to $\rho_j \to \infty$ for $1 \le j \le l$;*
vi) *$\hat{\theta}_1, \ldots, \hat{\theta}_l$ are asymptotically normal, i.e., $\sqrt{\rho_j}\left(\frac{\hat{\theta}_j}{\theta_j} - 1\right) \xrightarrow{d} N(0, 1)$, $\rho_j \to \infty$, $1 \le j \le l$.*

Proof i) and ii) are obvious.

iii) Sufficiency is directly obtained by Fisher–Neyman factorization.
To prove completeness, let h be a measurable function; then, the equation

$$\begin{aligned} & E_{\theta_1,\ldots,\theta_l} h\,(\hat{\theta}_1, \ldots, \hat{\theta}_l) \\ & = \int_0^\infty \ldots \int_0^\infty h(x_1, \ldots, x_l) \prod_{j=1}^{l} \frac{1}{(\rho_j - 1)!} \left(\frac{\theta_j}{\rho_j}\right)^{-\rho_j} x_j^{\rho_j - 1} \\ & \quad \times \exp\left(-\frac{x_j \rho_j}{\theta_j}\right) dx_1 \ldots dx_l \\ & = 0 \quad \text{for all} \quad \theta_1, \ldots, \theta_l > 0 \end{aligned}$$

implies $h \equiv 0$ a.e. following the arguments of Chiou and Cohen (1984).
iv) In view of ii) and iii), the assertion is an immediate consequence of Lehmann–Scheffé theorem. □

In the special case of a standard exponential distribution as baseline distribution [cf. Example 1 i)], the MLEs are linear estimators and are therefore the same as the best linear unbiased estimators (BLUEs). Since $\hat{\theta}_1, \ldots, \hat{\theta}_l$ are independent gamma distributed random variables, standard routines may be used to construct confidence intervals and also carry out statistical tests of hypotheses. It is also of great interest to note that the properties of the MLEs presented in Theorem 2 are independent of the baseline distribution function F!

3.2 Order restricted inference

The underlying hazard rates in the step-stress experiment are given by $\frac{1}{\theta_j}$, $1 \leq j \leq l$, which, in view of increasing stress put on the experimental units, may be assumed to be non-decreasing as well. In other words, the distributions F_j, $1 \leq j \leq l$, are decreasingly ordered. We may, therefore, consider estimating $\theta_1, \ldots, \theta_l$ under the order restriction $\theta_1 \geq \cdots \geq \theta_l$. In the common situation of SOSs, order restricted inference has been discussed by Balakrishnan et al. (2008, 2009a) in the context of exponential step-stress experiments.

Theorem 3 *The MLEs* $\hat{\theta}_1^{(o)}, \ldots, \hat{\theta}_l^{(o)}$ *of* $\theta_1, \ldots, \theta_l$, *respectively, under the simple order restriction* $\theta_1 \geq \cdots \geq \theta_l$, *are given by*

$$\theta_j^{(o)} = \min_{p \leq j} \max_{q \geq j} \frac{\sum_{k=p}^{q} \rho_k \hat{\theta}_k}{\sum\limits_{k=p}^{q} \rho_k}, \quad 1 \leq j \leq l,$$

where $\hat{\theta}_1, \ldots, \hat{\theta}_l$ *denote the MLEs presented in Theorem 1.*

Proof The likelihood function in (4) (see also the proof of Theorem 1) can be expressed as

$$L(\theta_1, \ldots, \theta_l; x_1, \ldots, x_r) = \frac{n!}{(n-r)!} \left(\prod_{i=1}^{r} \frac{f(x_i)}{\bar{F}(x_i)} \right) \left(\prod_{j=1}^{l} \theta_j^{-\rho_j} \right) \prod_{j=1}^{l} \exp\left\{ -\rho_j \frac{\hat{\theta}_j}{\theta_j} \right\}.$$

Thus, with respect to isotonic regression and except for reversed order, we are in the situation of Ex. 1.9 of Barlow et al. (1972, p. 45) for exponential distributions [cf. Balakrishnan et al. (2008, 2009a)]. The weights in the isotonic regression are given by $\omega(j) = \rho_j$, $1 \leq j \leq l$. Hence, the result. □

3.3 Joint estimation of a location parameter

If the baseline distribution is chosen according to Example 1 ii), it is easily seen that the location parameter μ may be simultaneously estimated as well.

Theorem 4 *Let F be as given in Example 1 ii). Then, the joint MLEs $\tilde{\mu}, \tilde{\theta}_1, \ldots, \tilde{\theta}_l$ of $\mu, \theta_1, \ldots, \theta_l$, respectively, are given by*

$$\tilde{\mu} = g\left(X_*^{(1)}\right),$$
$$\tilde{\theta}_1 = \frac{1}{\rho_1}\sum_{i=2}^{\rho_1}(n-i+1)\left(g(X_*^{(i)}) - g(X_*^{(i-1)})\right)$$

and

$$\tilde{\theta}_j = \frac{1}{\rho_j}\sum_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j}(n-i+1)\left(g(X_*^{(i)}) - g(X_*^{(i-1)})\right), \quad 2 \le j \le l.$$

Proof From the representation of the log-likelihood function in the proof of Theorem 1, we find the log-likelihood function, in our situation with $\log \bar{F}(x) = \mu - g(x)$, as

$$\begin{aligned}
&l(\mu, \theta_1, \ldots, \theta_l; x_1, \ldots, x_r)\\
&= \log\frac{n!}{(n-r)!} - \sum_{j=1}^{l}\rho_j \log\theta_j + \frac{n}{\theta_1}(\mu - g(x_1))\\
&\quad -\frac{1}{\theta_1}\sum_{i=2}^{\rho_1}(n-i+1)\,(g(x_i) - g(x_{i-1}))\\
&\quad -\sum_{j=2}^{l}\frac{1}{\theta_j}\sum_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j}(n-i+1)\,(g(x_i) - g(x_{i-1})) + \sum_{i=1}^{r}\log g'(x_i)\\
&= \log\frac{n!}{(n-r)!} - \sum_{j=1}^{l}\rho_j \log\theta_j + \frac{n}{\theta_1}(\mu - g(x_1)) - \sum_{j=1}^{l}\frac{\rho_j}{\theta_j}\tilde{\theta}_j + \sum_{i=1}^{r}\log g'(x_i).
\end{aligned}$$

Regarding μ, since $\mu \le g(x_1)$, this expression attains its global maximum at $g(x_1) = \tilde{\mu}$, say. Hence, we need to maximize

$$\begin{aligned}
&l(\tilde{\mu}, \theta_1, \ldots, \theta_l; x_1, \ldots, x_r)\\
&\quad = \log\frac{n!}{(n-r)!} + \sum_{j=1}^{l}\rho_j \log\frac{1}{\theta_j} - \sum_{j=1}^{l}\frac{\rho_j}{\theta_j}\tilde{\theta}_j + \sum_{i=1}^{r}\log g'(x_i)
\end{aligned}$$

with respect to $\theta_1, \ldots, \theta_l$. Now, formally, we have arrived at the same expression as in the proof of Theorem 1, and consequently $\tilde{\theta}_1, \ldots, \tilde{\theta}_l$ are the required MLEs. Hence, the result. □

Upon rewriting the MLEs $\tilde{\mu}, \tilde{\theta}_1, \dots, \tilde{\theta}_l$ in terms of F with $\bar{F}(x) = \exp\{-g(x)+\mu\}$, we obtain

$$\tilde{\mu} = \mu - \log \bar{F}(X_*^{(1)}),$$
$$\tilde{\theta}_1 = -\frac{1}{\rho_1} \sum_{i=2}^{\rho_1} (n-i+1) \left(\log \bar{F}(X_*^{(i)}) - \log \bar{F}(X_*^{(i-1)})\right)$$

and

$$\tilde{\theta}_j = -\frac{1}{\rho_j} \sum_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j} (n-i+1) \left(\log \bar{F}(X_*^{(i)}) - \log \bar{F}(X_*^{(i-1)})\right), \quad 2 \le j \le l.$$

Using the fact that the normalized spacings of generalized order statistics from a standard exponential distribution are iid random variables with standard exponential distributions (see Sect. 3.1), the above MLEs may again be stochastically represented as

$$\tilde{\mu} \sim \mu + \frac{\theta_1}{n} V_1,$$
$$\tilde{\theta}_1 \sim \frac{\theta_1}{\rho_1} \sum_{i=2}^{\rho_1} V_i,$$

and

$$\tilde{\theta}_j \sim \frac{\theta_j}{\rho_j} \sum_{i=\tilde{\rho}_{j-1}+1}^{\tilde{\rho}_j} V_i,$$

with $V_1, \dots, V_r$ being iid $Exp(1)$ random variables. Furthermore, by adopting the arguments in the proof of Theorem 2 with respect to completeness and sufficiency, we readily obtain the following theorem.

Theorem 5 *The MLEs $\tilde{\mu}, \tilde{\theta}_1, \dots, \tilde{\theta}_l$ of $\mu, \theta_1, \dots, \theta_l$, respectively, possess the following properties:*

i) *$\tilde{\mu}, \tilde{\theta}_1, \dots, \tilde{\theta}_l$ are stochastically independent;*
ii) *$\tilde{\mu}$ has a two-parameter exponential $Exp(\mu, \frac{\theta_1}{n})$ distribution, with distribution function*
   $F^{\tilde{\mu}}(x) = 1 - \exp\left\{-\frac{n}{\theta_1}(x-\mu)\right\}, \quad x > \mu;$
iii) *$\tilde{\theta}_1 \sim \Gamma\left(\rho_1 - 1, \frac{\theta_1}{\rho_1}\right)$ with $E\tilde{\theta}_1 = \frac{\rho_1-1}{\rho_1}\theta_1$ and $Var(\tilde{\theta}_1) = \frac{\rho_1-1}{\rho_1^2}\theta_1^2$, and*
   $\tilde{\theta}_j \sim \Gamma\left(\rho_j, \frac{\theta_j}{\rho_j}\right)$ with $E\tilde{\theta}_j = \theta_j$ and $Var\tilde{\theta}_j = \frac{\theta_j^2}{\rho_j}, \quad 2 \le j \le l;$
iv) *$(\tilde{\mu}, \tilde{\theta}_1, \dots, \tilde{\theta}_l)$ is a complete sufficient statistic for $(\mu, \theta_1, \dots, \theta_l)$.*

Since $E\tilde{\mu} = \mu + \frac{\theta_1}{n}$, $\tilde{\mu}$ is biased, but we find $\tilde{\tilde{\mu}} = \tilde{\mu} - \frac{\rho_1}{n(\rho_1 - 1)}\tilde{\theta}_1$ to be an unbiased estimator of μ, since $\tilde{\tilde{\theta}} = \frac{\rho_1}{\rho_1 - 1}\tilde{\theta}_1$ is an unbiased estimator of θ_1. Consistency and asymptotic normality of the estimators $\tilde{\theta}_1, \ldots, \tilde{\theta}_l$ also follow easily [cf. the arguments in Cramer and Kamps (2001a,b) and Section 3.2]. Moreover, the above unbiased estimators $\tilde{\tilde{\mu}}$ and $\tilde{\tilde{\theta}}$ can also be shown to be UMVUEs.

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