Ann Inst Stat Math (2012) 64:343–357
DOI 10.1007/s10463-010-0308-3

Estimating the ratio of two scale parameters: a simple approach

Panayiotis Bobotas · George Iliopoulos · Stavros Kourouklis

Received: 30 November 2009 / Revised: 17 May 2010 / Published online: 31 July 2010

Abstract We describe a simple approach for estimating the ratio $\rho = \sigma_2/\sigma_1$ of the scale parameters of two populations from a decision theoretic point of view. We show that if the loss function satisfies a certain condition, then the estimation of ρ reduces to separately estimating σ_2 and $1/\sigma_1$. This implies that the standard estimator of ρ can be improved by just employing an improved estimator of σ_2 or $1/\sigma_1$. Moreover, in the case where the loss function is convex in some function of its argument, we prove that such improved estimators of ρ are further dominated by corresponding ones that use all the available data. Using this result, we construct new classes of double-adjustment improved estimators for several well-known convex as well as non-convex loss functions. In particular, Strawderman-type estimators of ρ in general models are given whereas Shinozaki-type estimators of the ratio of two normal variances are briefly treated.

Keywords Decision theory · Improved estimation of a scale parameter · Improved estimation of ratio of scale parameters · Stein's and Brewster and Zidek's techniques · Strawderman's technique · Kubokawa's approach

P. Bobotas · S. Kourouklis
Department of Mathematics, University of Patras, 26504 Patras, Greece
e-mail: pb@math.upatras.gr

S. Kourouklis
e-mail: stavros@math.upatras.gr

G. Iliopoulos (✉)
Department of Statistics and Insurance Science, University of Piraeus,
80 Karaoli & Dimitriou str., 18534 Piraeus, Greece
e-mail: geh@unipi.gr

1 Introduction

The problem of estimating a scale parameter in the presence of another nuisance parameter from a decision theoretic point of view has been extensively studied in the literature. The methods of Stein (1964), Brown (1968), Brewster and Zidek (1974), Strawderman (1974) and Kubokawa (1994a) for deriving better estimators than the standard one are now classical in this context which encompasses the estimation of the variance of a normal distribution with unknown mean as a typical case. These improved estimators are scale equivariant. Note that in the case of a normal variance, modifications of the above methods that yield improved non-scale equivariant estimators have been developed by Shinozaki (1995). The review papers by Maatta and Casella (1990), Pal et al. (1998), Kubokawa (1999) are very good sources for tracing the origin and development of this problem.

The study of the analogous problem of estimating the ratio $\rho = \sigma_2/\sigma_1$ of the scale parameters σ_1 and σ_2 of two populations, with dominant case that of estimating the ratio of the variances of two normal populations with unknown means, followed naturally from the respective one-sample problem, with considerable delay though. Gelfand and Dey (1988) extended arguments of Stein (1964) and Brown (1968) and obtained improved estimators of a normal variance ratio. For the same parameter, Madi (1995) gave smooth improved estimators by adapting Brewster and Zidek (1974) method, while Ghosh and Kundu (1996) constructed generalized Bayes estimators and proved their dominance by establishing a two-sample extension of Kubokawa (1994a). Also, Kubokawa (1994b) and Kubokawa and Srivastava (1996) presented improved estimators for ρ in general models extending the one-sample method of Kubokawa (1994a).

The purpose of this work is to propose a simple alternative approach for deriving improved estimators of ρ. Let δ_{01}, δ_{02} and δ_0 be the standard (i.e., the best equivariant) estimators of $1/\sigma_1$, σ_2 and ρ respectively under a loss function L. Also, suppose that δ_1 and δ_2 are typical improved estimators of $1/\sigma_1$ and σ_2 at our disposal. Then we set as a goal in this paper to answer the question: how can δ_1 and δ_2 be used to produce an improved estimator of ρ? We first show that if the loss satisfies

$$L'(xy) = \lambda_1(x)L'(y) + \lambda_2(y)L'(x), \quad \forall x, y > 0 \tag{1}$$

for some functions λ_1 and λ_2 then

$$\delta_1\delta_{02} \quad \text{and} \quad \delta_{01}\delta_2 \tag{2}$$

are better than δ_0. Condition (1) holds for several commonly used convex as well as non-convex loss functions such as squared error, entropy and Brown (1968) losses (see Table 1). In this paper, $\delta_1\delta_{02}$ and $\delta_{01}\delta_2$ are referred to as single-adjustment improved estimators as they use an improved estimator of only one of the parameters involved in the ratio ρ, adjustment meaning shrinkage or expansion, accordingly. If, in addition to (1), $L(t)$ is convex in $\ln t$ then a stronger result is established, namely, that

$$\delta = \delta_1\delta_2 \tag{3}$$

is better than both $\delta_1\delta_{02}$ and $\delta_{01}\delta_2$, and hence δ_0. Estimators such as δ which use improved estimators of both $1/\sigma_1$ and σ_2 will be referred to as double-adjustment improved estimators. Even more generally, it is shown that if $L(t)$ is convex in *some* monotone function $u(t)$ with inverse $v(t)$ then

$$\delta_u = v\big(u(\delta_1\delta_{02}) + u(\delta_{01}\delta_2) - u(\delta_{01}\delta_{02})\big) \tag{4}$$

is better than $\delta_1\delta_{02}$, $\delta_{01}\delta_2$ and δ_0. As a function may be convex in more than one (monotone) functions, it is conceivable that for the same loss we may end up with more than one double-adjustment improved estimators of the form (4), for instance this is the case with both squared error and entropy losses (see Table 2). The conditions we require on δ_1, δ_2 for (2), (3) and (4) to hold are minimal.

This work is closely related to Kubokawa (1994b) and Kubokawa and Srivastava (1996), where the loss is assumed to be (strictly) convex and the improved estimators δ_1, δ_2 are (essentially) required to obey Kubokawa (1994a) sufficient conditions (also stated in Sect. 3). Here, not only we relax the assumption of convexity of L to convexity in $u(t)$, but also succeed in obtaining a new and quite rich class of estimators, that in (4), allowing δ_1 and δ_2 even to violate Kubokawa (1994a) conditions as long as they dominate δ_{01} and δ_{02}, respectively. Also, in our work, δ_1 and δ_2 are not necessarily scale equivariant. Furthermore, if $L(t)$ is convex then $u(t) = t$ and (4) gives, as a special case, Kubokawa (1994b) class of estimators, $\delta_1\delta_{02} + \delta_{01}\delta_2 - \delta_{01}\delta_{02}$. On the other hand, if $L(t)$ is convex in $u(t) = \ln t$ then (4) simplifies to $\delta_1\delta_2$ in (3). Note that Kubokawa and Srivastava (1996) have proven the superiority of this estimator over $\delta_{01}\delta_2$ and hence over $\delta_{01}\delta_{02}$ by imposing some further conditions in the case of convex $L(t)$. The results of the present paper also extend Iliopoulos and Kourouklis (1999) work on the estimation of the ratio of two normal generalized variances.

The advantage of our approach is transparent. Condition (1) guarantees that the search for an estimator of ρ reduces to the estimation of $1/\sigma_1$ and σ_2, the latter being one-sample problems which are simpler to deal with, not to mention the fact that the literature is rich in estimators δ_1 and δ_2 ready to be plugged into (2), (3) and (4). This way, it is *unnecessary* to resort to two-sample extensions of one-sample arguments in order to estimate ρ. Besides and unlike previous research, the approach allows for non-convex loss functions and works for many types of improved estimators δ_1 and δ_2, that is, Stein-type, Brown-type, Brewster and Zidek-type, Kubokawa-type and Strawderman-type, thus producing estimators of ρ in an integrated manner. In particular, Strawderman-type estimators of ρ in general models having monotone likelihood ratio properties are exhibited for the first time in the literature. In addition, in the case of two normal populations, Shinozaki-type estimators of the ratio of the variances are given under the squared error loss by modifying respective Strawderman-type estimators.

Section 2 contains the model, conditions and main results. In Sect. 3, several results available in the literature are reobtained in a much simpler way as special cases of our findings and new results that cannot be handled by previous work are given, as applications of our approach. Finally, an Appendix contains some technical results.

2 Main results

Let S_1, S_2, T_1, T_2 be independent statistics such that for $i = 1, 2$, S_i/σ_i and T_i/σ_i have probability density functions

$$g_i(s)I_{(0,\infty)}(s) \quad \text{and} \quad h_i(t;\mu_i,\sigma_i)I_{[\kappa_i(\mu_i,\sigma_i),\infty)}(t), \tag{5}$$

respectively, where $\sigma_i > 0$, μ_i is a nuisance parameter and $\kappa_i(\mu_i, \sigma_i)$ is some real function of μ_i and σ_i such that $\kappa_i(\mu_{0i}, \sigma_{0i}) = 0$, for some μ_{0i} and $\sigma_{0i} > 0$. It is well-known that scale equivariant estimators of σ_i^α and $\rho = \sigma_2/\sigma_1$ have the form cS_i^α and cS_2/S_1, respectively. Moreover, under the condition

(A) for any $0 < a < b$ the ratio $g_i(as)/g_i(bs)$ is strictly increasing in $s > 0$,

the best scale equivariant estimator of σ_i^α with respect to any absolutely continuous and strictly bowl-shaped loss function of the form $L(\delta/\sigma_i^\alpha)$ is $\delta_{i,\alpha} = c_{i,\alpha} S_i^\alpha$, where $c_{i,\alpha}$ is the unique solution to the equation

$$\mathsf{E}_{\sigma_i=1}\{L'(cS_i^\alpha)S_i^\alpha\} = 0 \tag{6}$$

with respect to c, provided the expectation exists and is finite. The existence and uniqueness of $c_{i,\alpha}$ follows from the fact that under condition (A) the functions $c \mapsto \mathsf{E}_{\sigma_i=1}\{L(cS_i^\alpha)\}$ are strictly bowl-shaped as well (see Brewster and Zidek 1974, or Kubokawa 1994a). The same holds true for the estimation problem of ρ since it is the scale parameter of the distribution of S_2/S_1. For any absolutely continuous and strictly bowl-shaped loss function of the form $L(\delta/\rho)$, condition (A) implies that the function $c \mapsto \mathsf{E}_{\sigma_1=\sigma_2=1}\{L(cS_2/S_1)\}$ is also strictly bowl-shaped and so the best scale equivariant estimator of S_2/S_1 is $\delta_0 = c_0 S_2/S_1$, where c_0 is the unique solution to the equation

$$\mathsf{E}_{\sigma_1=\sigma_2=1}\{L'(cS_2/S_1)S_2/S_1\} = 0. \tag{7}$$

For notational convenience in what follows we will drop the subscripts from all expectations and whenever allowable (e.g., by invariance) we will take without loss of generality $\sigma_1 = \sigma_2 = 1$. Moreover, since we are interested in the estimation of ρ we set $c_{01} = c_{1,-1}$, $c_{02} = c_{2,1}$ and $\delta_{01} = c_{01}S_1^{-1}$, $\delta_{02} = c_{02}S_2$.

Theorem 1 *Assume that the loss function satisfies* (1) *and* $\mathsf{E}\{\lambda_1(c_{01}S_1^{-1})S_1^{-1}\}$, $\mathsf{E}\{\lambda_2(c_{02}S_2)S_2\}$ *are finite. Then it holds* $c_0 = c_{01}c_{02}$*, or, equivalently,* $\delta_0 = \delta_{01}\delta_{02}$.

Proof Equation (1), the independence of S_1 and S_2 and Eqs. (6), (7) give

$$\begin{aligned}\mathsf{E}\left\{L'\left(c_{01}c_{02}\frac{S_2}{S_1}\right)\frac{S_2}{S_1}\right\} &= \mathsf{E}\left\{\lambda_1\left(\frac{c_{01}}{S_1}\right)\frac{1}{S_1}\right\}\mathsf{E}\{L'(c_{02}S_2)S_2\} \\ &\quad + \mathsf{E}\{\lambda_2(c_{02}S_2)S_2\}\mathsf{E}\left\{L'\left(\frac{c_{01}}{S_1}\right)\frac{1}{S_1}\right\} = 0\end{aligned}$$

and the result follows from the uniqueness of c_0. □

Remark 1 Condition (1) is satisfied by several popular loss functions such as the losses (a)–(f) given in Table 1. Specifically, (a) is the squared error loss, (b) is the entropy (or Stein's) loss introduced in James and Stein (1961), (c) is Brown (1968) loss, (d) and (e) are modifications of the squared error and entropy losses, respectively, used by Pal (1988) and (f) is a symmetric loss which results as the sum of (b) and (e) and has been treated by Pal (1988) and Kubokawa and Konno (1990).

Table 1 Popular bowl-shaped loss functions that can be written in the form (1)

	$L(t)$	$L'(t)$	$\lambda_1(t)$	$\lambda_2(t)$	Convex in t	Convex in $\ln t$
(a)	$(t-1)^2$	$2(t-1)$	t	1	✓	–
(b)	$t-\ln t-1$	$1-\frac{1}{t}$	$\frac{1}{t}$	1	✓	✓
(c)	$(\ln t)^2$	$\frac{2\ln t}{t}$	$\frac{1}{t}$	$\frac{1}{t}$	–	✓
(d)	$(\frac{1}{t}-1)^2$	$\frac{2(t-1)}{t^3}$	$\frac{1}{t^2}$	$\frac{1}{t^3}$	–	–
(e)	$\frac{1}{t}+\ln t-1$	$\frac{t-1}{t^2}$	$\frac{1}{t^2}$	$\frac{1}{t}$	–	✓
(f)	$t+\frac{1}{t}-2$	$1-\frac{1}{t^2}$	$\frac{1}{t^2}$	1	✓	✓

Theorem 1 tells us that, under (1), the best scale equivariant estimator of ρ equals the product of the best scale equivariant estimators of $1/\sigma_1$ and σ_2.

In order to prove our next result we will need the following.

Lemma 1 *Under the conditions of Theorem 1, both* $\mathsf{E}\{\lambda_1(c_{01}S_1^{-1})S_1^{-1}\}$ *and* $\mathsf{E}\{\lambda_2(c_{02}S_2)S_2\}$ *are positive.*

Proof Let $c_2' > c_{02}$ so that $c_{01}c_2' > c_{01}c_{02} = c_0$. As $c \mapsto \mathsf{E}\{L(cS_2/S_1)\}$ and $c \mapsto \mathsf{E}\{L(cS_2)\}$ are strictly bowl-shaped in c we have

$$\begin{aligned}
0 &= \mathsf{E}\{L'(c_{01}c_{02}S_2/S_1)S_2/S_1\} < \mathsf{E}\{L'(c_{01}c_2'S_2/S_1)S_2/S_1\}\\
&= \mathsf{E}\{\lambda_1(c_{01}S_1^{-1})S_1^{-1}L'(c_2'S_2)S_2 + \lambda_2(c_2'S_2)S_2L'(c_{01}S_1^{-1})S_1^{-1}\}\\
&= \mathsf{E}\{\lambda_1(c_{01}S_1^{-1})S_1^{-1}\}\mathsf{E}\{L'(c_2'S_2)S_2\} + \mathsf{E}\{\lambda_2(c_2'S_2)S_2\}\mathsf{E}\{L'(c_{01}S_1^{-1})S_1^{-1}\}\\
&= \mathsf{E}\{\lambda_1(c_{01}S_1^{-1})S_1^{-1}\}\mathsf{E}\{L'(c_2'S_2)S_2\},
\end{aligned}$$

since $\mathsf{E}\{L'(c_{01}S_1^{-1})S_1^{-1}\} = 0$. But $c_2' > c_{02}$ implies that $\mathsf{E}\{L'(c_2'S_2)S_2\} > 0$ and thus $\mathsf{E}\{\lambda_1(c_{01}S_1^{-1})S_1^{-1}\} > 0$ too. The proof for $\mathsf{E}\{\lambda_2(c_{02}S_2)S_2\}$ is similar. □

Remark 2 Although one may think that the functions λ_1 and λ_2 are positive (as those displayed in Table 1), this is not always the case. In fact these functions are not unique. It is easy to see that if (1) holds with some λ_1 and λ_2, then it also holds for $\lambda_1^* = \lambda_1 + \beta L'$, $\lambda_2^* = \lambda_2 - \beta L'$ and any $\beta \in \mathbb{R}$.

We now seek to improve upon δ_0 by first using estimators of σ_2 or $1/\sigma_1$ of the following form. For $i = 1, 2$, let $\phi_i(w_i)$ be absolutely continuous functions on $(0, \infty)$ and define

$$\delta_{\phi_1} = \begin{cases} \phi_1(W_1)S_1^{-1} & \text{if } W_1 > 0,\\ \delta_{01} & \text{otherwise,}\end{cases} \qquad \delta_{\phi_2} = \begin{cases} \phi_2(W_2)S_2 & \text{if } W_2 > 0,\\ \delta_{02} & \text{otherwise,}\end{cases} \tag{8}$$

where $W_i = T_i/S_i$. In location-scale models, δ_{ϕ_i} is a typical scale equivariant estimator.

Theorem 2 *Assume that* (1) *holds and* $\lim_{w_i\to\infty}\phi_i(w_i) = c_{0i}, i = 1, 2$. *Then we have the following.*

(i) $\delta_{\phi_1}\delta_{02}$ *improves on* $\delta_0 = \delta_{01}\delta_{02}$ *if and only if* δ_{ϕ_1} *improves on* δ_{01}.
(ii) $\delta_{01}\delta_{\phi_2}$ *improves on* $\delta_0 = \delta_{01}\delta_{02}$ *if and only if* δ_{ϕ_2} *improves on* δ_{02}.

Proof (i) Without loss of generality we can again take $\sigma_1 = \sigma_2 = 1$. Then using Kubokawa (1994b), Condition (1) and Theorem 1 we have

$$\begin{aligned}
\mathsf{E}\{L(\delta_0)\} - \mathsf{E}\{L(\delta_{\phi_1}\delta_{02})\} &= \mathsf{E}\left\{\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}t} L\left(c_{02}\phi_1(tW_1)\frac{S_2}{S_1}\right)\mathrm{d}t\, I(W_1>0)\right\} \\
&= \mathsf{E}\left\{\int_1^\infty L'\left(c_{02}\phi_1(tW_1)\frac{S_2}{S_1}\right)c_{02}W_1\phi_1'(tW_1)\frac{S_2}{S_1}\mathrm{d}t\, I(W_1>0)\right\} \\
&= \mathsf{E}\left\{\int_1^\infty \lambda_1(\phi_1(tW_1)S_1^{-1})L'(c_{02}S_2)c_{02}W_1\phi_1'(tW_1)\frac{S_2}{S_1}\mathrm{d}t\, I(W_1>0)\right\} \\
&\quad + \mathsf{E}\left\{\int_1^\infty \lambda_2(c_{02}S_2)L'(\phi_1(tW_1)S_1^{-1})c_{02}W_1\phi_1'(tW_1)\frac{S_2}{S_1}\mathrm{d}t\, I(W_1>0)\right\} \\
&= c_{02}\mathsf{E}\left\{L'(c_{02}S_2)S_2\right\}\mathsf{E}\left\{\int_1^\infty \lambda_1(\phi_1(tW_1)S_1^{-1})W_1\phi_1'(tW_1)S_1^{-1}\mathrm{d}t\, I(W_1>0)\right\} \\
&\quad + c_{02}\mathsf{E}\{\lambda_2(c_{02}S_2)S_2\}\mathsf{E}\left\{\int_1^\infty L'(\phi_1(tW_1)S_1^{-1})W_1\phi_1'(tW_1)S_1^{-1}\mathrm{d}t\, I(W_1>0)\right\} \\
&= c_{02}\mathsf{E}\{\lambda_2(c_{02}S_2)S_2\}\mathsf{E}\left\{\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}t}L(\phi_1(tW_1)S_1^{-1})\mathrm{d}t\, I(W_1>0)\right\} \\
&= c_{02}\mathsf{E}\{\lambda_2(c_{02}S_2)S_2\}\left\{\mathsf{E}\left\{L(c_{01}S_1^{-1})\right\} - \mathsf{E}\left\{L(\phi_1(W_1)S_1^{-1})\right\}\right\} \\
&= c_{02}\mathsf{E}\{\lambda_2(c_{02}S_2)S_2\}\left\{\mathsf{E}\{L(\delta_{01})\} - \mathsf{E}\{L(\delta_{\phi_1})\}\right\},
\end{aligned}$$

where the fifth equality holds by the definition of c_{02}. The result now follows from Lemma 1.

(ii) It is similar to the proof of (i). □

In Theorem 2, we have proven that under the above conditions in order to improve on $\delta_0 = \delta_{01}\delta_{02}$ it suffices to improve separately on either δ_{01} or δ_{02} using a corresponding estimator δ_{ϕ_1} or δ_{ϕ_2}, say, and then take the product $\delta_{\phi_1}\delta_{02}$ or $\delta_{01}\delta_{\phi_2}$. Naturally, one could ask whether the product $\delta_{\phi_1}\delta_{\phi_2}$ of these improved estimators dominates the previous ones. This question was considered by Kubokawa and Srivastava (1996) assuming (strictly) convex loss. Under a set of conditions SG1.a, SG1.b, SG2.a, SG2.b, the authors showed that $\delta_{\phi_1}\delta_{\phi_2}$ dominates $\delta_{01}\delta_{\phi_2}$ (only) and thus δ_0. It is noted that under (1) these conditions simplify to Kubokawa (1994a) one-sample conditions (also given in Sect. 3). Below, using a very simple argument we establish the dominance of $\delta_{\phi_1}\delta_{\phi_2}$ over $\delta_{01}\delta_{\phi_2}$ and $\delta_{\phi_1}\delta_{02}$ for a loss that is convex in $\ln t$ without imposing SG1.b and SG2.b. The same argument was used by Iliopoulos and Kourouklis (1999) in the special case of estimating the ratio of generalized variances of two multivariate normal populations.

Theorem 3 *Let* δ_{ϕ_i} *be as in* (8), $i = 1, 2$. *Assume that* ϕ_1 *is non-increasing on* $(0,\infty)$ *with* $\lim_{w_1\to\infty}\phi_1(w_1) = c_{01}$, ϕ_2 *is non-decreasing on* $(0,\infty)$ *with* $\lim_{w_2\to\infty}$ $\phi_2(w_2) = c_{02}$ *and the estimators* $\delta_{\phi_1}\delta_{02}$, $\delta_{01}\delta_{\phi_2}$ *dominate* $\delta_0 = \delta_{01}\delta_{02}$ *with respect* *to the loss function L. If $tL'(t)$ is non-decreasing in $t > 0$ (i.e., L is convex in $\ln t$) then $\delta_{\phi_1}\delta_{\phi_2}$ dominates both $\delta_{\phi_1}\delta_{02}$ and $\delta_{01}\delta_{\phi_2}$.*

Proof The above assumptions imply that $\lim_{W_1\to\infty}\big\{L(\delta_{\phi_1}\delta_{\phi_2}) - L(\delta_{\phi_1}\delta_{02})\big\} = L(\delta_{01}\delta_{\phi_2}) - L(\delta_{01}\delta_{02})$ and that

$$\frac{\partial}{\partial W_1}\big\{L(\delta_{\phi_1}\delta_{\phi_2}) - L(\delta_{\phi_1}\delta_{02})\big\} = \frac{\phi_1'(W_1)}{\phi_1(W_1)}\big\{\delta_{\phi_1}\delta_{\phi_2}L'(\delta_{\phi_1}\delta_{\phi_2}) - \delta_{\phi_1}\delta_{02}L'(\delta_{\phi_1}\delta_{02})\big\}.$$

This partial derivative is non-negative since $\phi_1' \leqslant 0$ and the quantity in curly brackets is non-positive as well. This follows from the facts that $tL'(t)$ is non-decreasing and $\phi_2 \leqslant c_{02}$ which means that $\delta_{\phi_1}\delta_{\phi_2} \leqslant \delta_{\phi_1}\delta_{02}$. Thus, $L(\delta_{\phi_1}\delta_{\phi_2}) - L(\delta_{\phi_1}\delta_{02}) \leqslant L(\delta_{01}\delta_{\phi_2}) - L(\delta_{01}\delta_{02})$. By taking expectations and using the fact that $\delta_{01}\delta_{\phi_2}$ dominates δ_0 we get that $\delta_{\phi_1}\delta_{\phi_2}$ dominates $\delta_{\phi_1}\delta_{02}$. The proof for $\delta_{01}\delta_{\phi_2}$ is similar. □

In the case that L satisfies (1) and is also convex in $\ln t$, by combining Theorems 2 and 3, we immediately obtain the following result.

Corollary 1 *Let $\delta_{\phi_1}, \delta_{\phi_2}$ in (8) be improved estimators of $1/\sigma_1, \sigma_2$ satisfying*

(i) $\phi_1(w_1)$ *is non-increasing in* $w_1 > 0$ *and* $\lim_{w_1\to\infty}\phi_1(w_1) = c_{01}$,
(ii) $\phi_2(w_2)$ *is non-decreasing in* $w_2 > 0$ *and* $\lim_{w_2\to\infty}\phi_2(w_2) = c_{02}$.

Assume that (1) holds and L is convex in $\ln t$. Then $\delta_{\phi_1}\delta_{\phi_2}$ dominates both $\delta_{\phi_1}\delta_{02}$ and $\delta_{01}\delta_{\phi_2}$.

By taking a look at Table 1 we can see that the entropy loss as well as some other popular loss functions are indeed convex in $\ln t$ and hence Corollary 1 applies. More generally, we get the following result.

Theorem 4 *Suppose that the assumptions of Theorem 3 hold for $\delta_{\phi_1}, \delta_{\phi_2}, \delta_0$ and L is convex in some monotone absolutely continuous function $u(t)$ with inverse $v(t)$. Then $v\big(u(\delta_{\phi_1}\delta_{02}) + u(\delta_{01}\delta_{\phi_2}) - u(\delta_{01}\delta_{02})\big)$ dominates both $\delta_{\phi_1}\delta_{02}$ and $\delta_{01}\delta_{\phi_2}$.*

Proof The condition that L is convex in $u(t)$ means that $u'(t)^{-1}L'(t) \equiv v'(u(t))L'(t)$ and $u(t)$ have the same kind of monotonicity. Similarly to the proof of Theorem 3, observe that $\lim_{W_1\to\infty}\{L(v(u(\delta_{\phi_1}\delta_{02}) + u(\delta_{01}\delta_{\phi_2}) - u(\delta_{01}\delta_{02}))) - L(\delta_{\phi_1}\delta_{02})\} = L(\delta_{01}\delta_{\phi_2}) - L(\delta_{01}\delta_{02})$ and that

$$\begin{aligned}
&\frac{\partial}{\partial W_1}\Big\{L\Big(v\big(u(\delta_{\phi_1}\delta_{02}) + u(\delta_{01}\delta_{\phi_2}) - u(\delta_{01}\delta_{02})\big)\Big) - L(\delta_{\phi_1}\delta_{02})\Big\} \\
&= u'(\delta_{\phi_1}\delta_{02})\frac{\phi_1'(W_1)}{\phi_1(W_1)}\delta_{\phi_1}\delta_{02}\Big\{v'\big(u(\delta_{\phi_1}\delta_{02}) + u(\delta_{01}\delta_{\phi_2}) - u(\delta_{01}\delta_{02})\big) \\
&\quad\times L'\Big(v\big(u(\delta_{\phi_1}\delta_{02}) + u(\delta_{01}\delta_{\phi_2}) - u(\delta_{01}\delta_{02})\big)\Big) - v'\big(u(\delta_{\phi_1}\delta_{02})\big)L'(\delta_{\phi_1}\delta_{02})\Big\}.
\end{aligned}$$

The proof now proceeds as in Theorem 3. □

The next result generalizes Corollary 1.

Corollary 2 *Let $\delta_{\phi_1}, \delta_{\phi_2}$ in (8) be improved estimators of $1/\sigma_1, \sigma_2$ satisfying*

(i) *$\phi_1(w_1)$ is non-increasing in $w_1 > 0$ and $\lim_{w_1\to\infty}\phi_1(w_1) = c_{01}$,*
(ii) *$\phi_2(w_2)$ is non-decreasing in $w_2 > 0$ and $\lim_{w_2\to\infty}\phi_2(w_2) = c_{02}$.*

Assume that (1) holds and L is convex in some monotone absolutely continuous function $u(t)$ with inverse $v(t)$. Then $v(u(\delta_{\phi_1}\delta_{02})+u(\delta_{01}\delta_{\phi_2})-u(\delta_{01}\delta_{02}))$ dominates both $\delta_{\phi_1}\delta_{02}$ and $\delta_{01}\delta_{\phi_2}$.

Theorem 3 is a special case of Theorem 4. Indeed, by taking $u(t) = \ln t$ we have $v(t) = e^t$ and so, $v(u(\delta_{\phi_1}\delta_{02}) + u(\delta_{01}\delta_{\phi_2}) - u(\delta_{01}\delta_{02})) = \exp\{\ln(\delta_{\phi_1}\delta_{02}) + \ln(\delta_{01}\delta_{\phi_2}) - \ln(\delta_{01}\delta_{02})\} = \delta_{\phi_1}\delta_{\phi_2}$. The squared error loss $L(t) = (t-1)^2$ is convex in $u(t) = t^k, 1 \leqslant k \leqslant 2$. Then, $v(t) = t^{1/k}$ and thus, the estimator $v(u(\delta_{\phi_1}\delta_{02}) + u(\delta_{01}\delta_{\phi_2}) - u(\delta_{01}\delta_{02})) = \{(\delta_{\phi_1}\delta_{02})^k + (\delta_{01}\delta_{\phi_2})^k - (\delta_{01}\delta_{02})^k\}^{1/k}$ dominates both $\delta_{\phi_1}\delta_{02}$ and $\delta_{01}\delta_{\phi_2}$. For $k = 1$, this is just Kubokawa (1994b) estimator. As an additional example consider the loss function $L(t) = (\frac{1}{t} - 1)^2$ which is clearly convex in $u(t) = t^{-k}, 1 \leqslant k \leqslant 2$. By Corollary 2, under this loss function, the estimator $v(u(\delta_{\phi_1}\delta_{02})+u(\delta_{01}\delta_{\phi_2})-u(\delta_{01}\delta_{02})) = \{(\delta_{\phi_1}\delta_{02})^{-k} + (\delta_{01}\delta_{\phi_2})^{-k} - (\delta_{01}\delta_{02})^{-k}\}^{-1/k}$ dominates both $\delta_{\phi_1}\delta_{02}$ and $\delta_{01}\delta_{\phi_2}$.

The above results can be extended to a broader class of estimators of σ_2 or $1/\sigma_1$. For $i = 1, 2$, let $\psi_i(w_i, s_i)$ be absolutely continuous functions with respect to w_i, defined on $(0,\infty)\times(0,\infty)$, and set

$$\delta_{\psi_1} = \begin{cases} \psi_1(W_1, S_1)S_1^{-1} & \text{if } W_1 > 0,\\ \delta_{01} & \text{otherwise,}\end{cases} \qquad \delta_{\psi_2} = \begin{cases} \psi_2(W_2, S_2)S_2 & \text{if } W_2 > 0,\\ \delta_{02} & \text{otherwise.}\end{cases} \tag{9}$$

The next result provides single-adjustment improved estimators of ρ based on δ_{ψ_i}.

Theorem 5 *Assume that (1) holds and $\lim_{w_i\to\infty}\psi_i(w_i, s_i) = c_{0i}$, for every $s_i > 0$, $i = 1, 2$. Then we have the following.*

(i) *$\delta_{\psi_1}\delta_{02}$ improves on $\delta_0 = \delta_{01}\delta_{02}$ if and only if δ_{ψ_1} improves on δ_{01}.*
(ii) *$\delta_{01}\delta_{\psi_2}$ improves on $\delta_0 = \delta_{01}\delta_{02}$ if and only if δ_{ψ_2} improves on δ_{02}.*

Proof Write $\mathsf{E}\{L(\delta_0/\rho)\} - \mathsf{E}\{L(\delta_{\psi_1}\delta_{02}/\rho)\} = \mathsf{E}\{\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}t}L\big(c_{02}\psi_1(tW_1, S_1)\frac{S_2\sigma_1}{S_1\sigma_2}\big)\, \mathrm{d}t\, I(W_1 > 0)\}$ and follow the proof of Theorem 2. □

Analogously to Theorem 4, in the following theorem we obtain double-adjustment improved estimators based on $\delta_{\psi_i}, i = 1, 2$. Its proof is similar to those of Theorems 3 and 4 and therefore is omitted.

Theorem 6 *Let δ_{ψ_i} be as in (9), $i = 1, 2$. Assume that ψ_1 is non-increasing in $w_1 > 0$ with $\lim_{w_1\to\infty}\psi_1(w_1, s_1) = c_{01}$ for every $s_1 > 0$, ψ_2 is non-decreasing in $w_2 > 0$ with $\lim_{w_2\to\infty}\psi_2(w_2, s_2) = c_{02}$ for every $s_2 > 0$ and the estimators $\delta_{\psi_1}\delta_{02}$, $\delta_{01}\delta_{\psi_2}$ dominate $\delta_0 = \delta_{01}\delta_{02}$ with respect to the loss function L. Assume also that L is convex in some monotone absolutely continuous function $u(t)$ with inverse $v(t)$. Then $v\big(u(\delta_{\psi_1}\delta_{02}) + u(\delta_{01}\delta_{\psi_2}) - u(\delta_{01}\delta_{02})\big)$ dominates both $\delta_{\psi_1}\delta_{02}$ and $\delta_{01}\delta_{\psi_2}$.*

It is evident, that combining Theorems 5 and 6 we can get similar results to those in Corollaries 1 and 2.

The above results demonstrate that, under the conditions stated, the problem of estimating ρ reduces simply to the estimation of $1/\sigma_1$ and σ_2. That is, if one has available or can derive improved estimators $\delta_{\phi_1}, \delta_{\phi_2}$ or $\delta_{\psi_1}, \delta_{\psi_2}$ then he/she can immediately construct double-adjustment improved estimators for ρ through our approach, rather than trying to extend an existing specific one-sample method of estimating a single scale parameter. Conditions (i) and (ii) of Corollaries 1 and 2 (also appearing in Theorems 3, 4) are mild and hold for typical improved estimators δ_{ϕ_1} and δ_{ϕ_2}, such as Stein-type, Brown-type, Brewster and Zidek-type, Kubokawa-type and Strawderman-type procedures. Thus, respective estimators of ρ are then easily found. For estimating a normal variance, improved non-scale equivariant estimators of the form δ_{ψ_2} in (9) have been presented by Shinozaki (1995). Following his approach, improved Shinozaki-type estimators of the form δ_{ψ_1} in (9) for a normal precision under the squared error loss can also be constructed (see Subsection 3.3). Thus, by invoking Theorems 5 and 6, non-scale equivariant single as well as double-adjustment improved estimators of the ratio of two normal variances are immediately obtained.

3 Applications

In this section we present several applications of our main results. We will need, in addition to (A), the following conditions. From Sect. 2, we recall that $\kappa_i(\mu_{0i}, \sigma_{0i}) = 0$ and set $h_i(t) = h_i(t; \mu_{0i}, \sigma_{0i})$, $t > 0$ and $V_i = S_i/\sigma_i$, $i = 1, 2$.

(B) For $i = 1, 2$, $h_i(t; \mu_i, \sigma_i)/h_i(t)$ is non-decreasing in $t > 0$.
(C) For $0 < a < b$, $i = 1, 2$, $h_i(at)/h_i(bt)$ is strictly increasing in $t > 0$.

3.1 Improved estimators of ρ based on Kubokawa's one-sample method

For a strictly bowl-shaped loss L, Kubokawa (1994a) showed that under (A) and (B) the estimators δ_{ϕ_1} and δ_{ϕ_2} of $1/\sigma_1$ and σ_2 given by (8) improve on δ_{01} and δ_{02}, respectively, if

(i) $\phi_1(w_1)$ is non-increasing and $\lim_{w_1\to\infty} \phi_1(w_1) = c_{01}$,
(ii) $\mathsf{E}\left\{L'(\phi_1(w_1)V_1^{-1})V_1^{-1} \mid W_1 \leqslant w_1\right\} \leqslant 0, \forall w_1 > 0$,
(iii) $\phi_2(w_2)$ is non-decreasing and $\lim_{w_2\to\infty} \phi_2(w_2) = c_{02}$,
(iv) $\mathsf{E}\left\{L'(\phi_2(w_2)V_2)V_2 \mid W_2 \leqslant w_2\right\} \leqslant 0, \forall w_2 > 0$,

where expectations are evaluated at $\mu_i = \mu_{0i}$ and $\sigma_i = \sigma_{0i}$. For instance, under (A), (B) and (C), the Stein-type and the Brewster and Zidek-type estimators of $1/\sigma_1$ and σ_2 satisfy (i)–(iv) (see Kubokawa 1994a). Conditions (A), (B) and (C) are known to hold, in particular, for normal, exponential (and hence lognormal and Pareto) as well as inverse Gaussian distributions. Consequently, using δ_{ϕ_1} and δ_{ϕ_2} we can immediately derive double-adjustment improved estimators for ρ by employing Corollaries 1 and 2. A summary of these improved estimators for certain well-known loss functions is given in Table 2. The loss functions marked with an asterisk are

Table 2 Well-known convex and non-convex losses and respective classes of double-adjustment improved estimators ($*$: non-convex loss)

$L(t)$	$u(t)$	Improved estimator
$(t-1)^2$	$t^k, 1 \leqslant k \leqslant 2$	$\left\{(\delta_{\phi_1}\delta_{02})^k + (\delta_{01}\delta_{\phi_2})^k - (\delta_{01}\delta_{02})^k\right\}^{1/k}$
$t - \ln t - 1$	$t^k, 0 < k \leqslant 1$	$\left\{(\delta_{\phi_1}\delta_{02})^k + (\delta_{01}\delta_{\phi_2})^k - (\delta_{01}\delta_{02})^k\right\}^{1/k}$
	$\ln t$	$\delta_{\phi_1}\delta_{\phi_2}$
$(\ln t)^2 *$	$\ln t$	$\delta_{\phi_1}\delta_{\phi_2}$
$(\frac{1}{t} - 1)^2 *$	$t^{-k}, 1 \leqslant k \leqslant 2$	$\left\{(\delta_{\phi_1}\delta_{02})^{-k} + (\delta_{01}\delta_{\phi_2})^{-k} - (\delta_{01}\delta_{02})^{-k}\right\}^{-1/k}$
$\frac{1}{t} + \ln t - 1*$	$t^{-k}, 0 < k \leqslant 1$	$\left\{(\delta_{\phi_1}\delta_{02})^{-k} + (\delta_{01}\delta_{\phi_2})^{-k} - (\delta_{01}\delta_{02})^{-k}\right\}^{-1/k}$
	$\ln t$	$\delta_{\phi_1}\delta_{\phi_2}$
$t + \frac{1}{t} - 2$	$t^k, -1 \leqslant k \leqslant 1, k \neq 0$	$\left\{(\delta_{\phi_1}\delta_{02})^k + (\delta_{01}\delta_{\phi_2})^k - (\delta_{01}\delta_{02})^k\right\}^{1/k}$
	$\ln t$	$\delta_{\phi_1}\delta_{\phi_2}$

Analogous estimators are obtained using δ_{ψ_i}

non-convex and hence the respective estimators cannot be derived from the existing literature. In the case of squared error loss, our class of estimators extends Kubokawa (1994b) class $\delta_{\phi_1}\delta_{02} + \delta_{01}\delta_{\phi_2} - \delta_{01}\delta_{02}$. For the entropy loss, the improved estimators $\delta_{\phi_1}\delta_{02} + \delta_{01}\delta_{\phi_2} - \delta_{01}\delta_{02}$ and $\delta_{\phi_1}\delta_{\phi_2}$ were also obtained by Kubokawa (1994b) and Kubokawa and Srivastava (1996), respectively, the latter, however, by exploiting the convexity of $t - \ln t - 1$, whereas we employ its convexity in $\ln t$. Especially, taking δ_{ϕ_1} and δ_{ϕ_2} to be the Stein-type or the Brewster and Zidek-type improved estimators of $1/\sigma_1$ and σ_2, we get Stein-type and Brewster and Zidek-type estimators of ρ for each of the above mentioned models and for each loss of Table 2. Also, Ghosh and Kundu (1996) generalized Bayes estimators of the ratio of two normal variances are single-adjustment improved estimators of the form $\delta_{\phi_1}\delta_{02}$ or $\delta_{01}\delta_{\phi_2}$ with δ_{ϕ_1} and δ_{ϕ_2} satisfying (i)–(iv) and hence their dominance follows from Theorem 2.

3.2 Improved estimators of ρ based on Strawderman's method

Using a different method than Stein (1964), Brown (1968) and Brewster and Zidek (1974), Strawderman (1974) and Maruyama and Strawderman (2006) obtained improved estimators for the variance of normal distribution with unknown mean. Lately, Strawderman (1974) method was extended to a general scale parameter by Bobotas and Kourouklis (2009). Below, we first derive Strawderman-type estimators for the ratio $\rho = \sigma_2/\sigma_1$ of two normal variances. Let X_1, X_2, S_1 and S_2 be independent statistics where X_1, X_2 have multivariate normal distributions $N_p(\mu_1, \sigma_1 I_p)$, $N_q(\mu_2, \sigma_2 I_q)$ with unknown mean vectors μ_1, μ_2 and S_1/σ_1, S_2/σ_2 have chi-square distributions χ^2_n, χ^2_m. We set $W_1 = \|X_1\|^2/S_1$ and $W_2 = \|X_2\|^2/S_2$.

Example 1 (Squared error loss) In this case, $\delta_{01} = (n-4)S_1^{-1}$ and $\delta_{02} = (m+2)^{-1}S_2$. For estimating σ_2, Maruyama and Strawderman (2006) derived the improved estimator $\delta_{\phi_2} = (1+W_2)S_2/[(m+2)(r_2+1+W_2)]$, whereas Bobotas and Kourouklis (2010) obtained the improved estimator $\delta_{\phi_1} = (n-4)(r_1+1+W_1)S_1^{-1}/(1+W_1)$ of $1/\sigma_1$, for $0 < r_i < r_{0i}$ where r_{0i} are specified constants, $i = 1, 2$. Since $L(t) = (t-1)^2$ is convex in t^k, $1 \leqslant k \leqslant 2$, Corollary 2 provides the Strawderman-type estimator of ρ

$$\delta = \frac{n-4}{m+2}\left\{\left(\frac{r_1+1+W_1}{1+W_1}\right)^k + \left(\frac{1+W_2}{r_2+1+W_2}\right)^k - 1\right\}^{1/k}\frac{S_2}{S_1}. \tag{10}$$

Noticeably, neither δ_{ϕ_1} nor δ_{ϕ_2} satisfy Kubokawa (1994a) conditions (ii), (iv) given above. The proof of this fact, for δ_{ϕ_2}, is given in Maruyama and Strawderman (2006, p. 3829) and for δ_{ϕ_1} follows in a similar way. Thus, not even in the case $k = 1$, the dominance of δ in (10) can be established by Kubokawa (1994b, Theorems 2.2 and 2.4) despite the fact that it has then the form of Kubokawa (1994b) double-adjustment improved estimators. For $k = 1$, δ in (10) is also given in Bobotas and Kourouklis (2010).

Example 2 (Entropy loss) In this case, $\delta_{01} = (n-2)S_1^{-1}$ and $\delta_{02} = (1/m)S_2$. From Maruyama and Strawderman (2006) and Bobotas and Kourouklis (2010) we use the improved estimators of σ_2 and $1/\sigma_1$, $\delta_{\phi_2} = (1+W_2)S_2/[m(r_2+1+W_2)]$ and $\delta_{\phi_1} = (n-2)(r_1+1+W_1)S_1^{-1}/(1+W_1)$, respectively, where $0 < r_i < r'_{0i}$ and r'_{0i} are specified constants, $i = 1, 2$. Since $L(t) = t - \ln t - 1$ is convex in $\ln t$ and t^k, $0 < k \leqslant 1$, by Corollaries 1, 2, we obtain the improved Strawderman-type estimators

$$\delta_{\phi_1}\delta_{\phi_2} = \frac{n-2}{m}\frac{(r_1+1+W_1)(1+W_2)}{(1+W_1)(r_2+1+W_2)}\frac{S_2}{S_1} \tag{11}$$

and

$$\begin{aligned}&\left\{(\delta_{\phi_1}\delta_{02})^k + (\delta_{01}\delta_{\phi_2})^k - (\delta_{01}\delta_{02})^k\right\}^{1/k}\\&\quad = \frac{n-2}{m}\left\{\left(\frac{r_1+1+W_1}{1+W_1}\right)^k + \left(\frac{1+W_2}{r_2+1+W_2}\right)^k - 1\right\}^{1/k}\frac{S_2}{S_1}.\end{aligned} \tag{12}$$

In this case too, the dominance of these estimators cannot be ensured by Kubokawa (1994b) or Kubokawa and Srivastava (1996). The estimator in (12) for $k = 1$ and $\delta_{\phi_1}\delta_{\phi_2}$ in (11) were also derived by Bobotas and Kourouklis (2010).

Strawderman-type estimators of ρ in general models have not as yet appeared in the literature. In the following example we obtain such estimators under the loss $t+1/t-2$. Analogous results can be derived for other losses of Table 2 as well but are not given here for the sake of brevity.

Example 3 Consider the symmetric loss $L(t) = t + 1/t - 2$ and assume the model in (5). We also suppose that (A) and (B) hold. In this case $\delta_{01} = c_{01}S_1^{-1}$ and $\delta_{02} = c_{02}S_2$, where $c_{01}^2 = \mathsf{E}(S_1)/\mathsf{E}(S_1^{-1})$ and $c_{02}^2 = \mathsf{E}(S_2^{-1})/\mathsf{E}(S_2)$. The expectations are evaluated at $\sigma_1 = \sigma_2 = 1$ and are assumed to be finite. In the Appendix it is shown that δ_{01} and δ_{02} are improved by the Strawderman-type estimators

$$\delta_{\phi_1} = \begin{cases} c_{01}\left\{1 + \dfrac{r_1}{(1+W_1)^{\epsilon_1}}\right\} S_1^{-1} & \text{if } W_1 > 0, \\ \delta_{01} & \text{otherwise} \end{cases} \tag{13}$$

and

$$\delta_{\phi_2} = \begin{cases} c_{02}\left\{1 - \dfrac{r_2}{(1+W_2)^{\epsilon_2}}\right\} S_2 & \text{if } W_2 > 0, \\ \delta_{02} & \text{otherwise,} \end{cases} \tag{14}$$

respectively, where $\epsilon_i > 0, 0 < r_i < B_i(\epsilon_i)$, and $B_i(\epsilon_i)$ is given in (16), $i = 1, 2$. Since $L(t)$ is convex in $\ln t$ and t^k, $-1 \leqslant k \leqslant 1$, $k \neq 0$, Corollaries 1 and 2 provide the improved Strawderman-type estimators of ρ, $\delta_{\phi_1}\delta_{\phi_2}$ and $\delta = \left\{(\delta_{\phi_1}\delta_{02})^k + (\delta_{01}\delta_{\phi_2})^k - (\delta_{01}\delta_{02})^k\right\}^{1/k}$, i.e.,

$$\delta_{\phi_1}\delta_{\phi_2} = \begin{cases} c_{01}c_{02}\left(1 + \dfrac{r_1}{(1+W_1)^{\epsilon_1}}\right)\left(1 - \dfrac{r_2}{(1+W_2)^{\epsilon_2}}\right)\dfrac{S_2}{S_1} & \text{if } W_1 > 0,\ W_2 > 0, \\ c_{01}c_{02}\left(1 + \dfrac{r_1}{(1+W_1)^{\epsilon_1}}\right)\dfrac{S_2}{S_1} & \text{if } W_1 > 0,\ W_2 \leqslant 0, \\ c_{01}c_{02}\left(1 - \dfrac{r_2}{(1+W_2)^{\epsilon_2}}\right)\dfrac{S_2}{S_1} & \text{if } W_1 \leqslant 0,\ W_2 > 0, \\ \delta_{01}\delta_{02} & \text{if } W_1 \leqslant 0,\ W_2 \leqslant 0 \end{cases}$$

and

$$\delta = \begin{cases} c_{01}c_{02}\left\{\left(1 + \dfrac{r_1}{(1+W_1)^{\epsilon_1}}\right)^k + \left(1 - \dfrac{r_2}{(1+W_2)^{\epsilon_2}}\right)^k - 1\right\}^{1/k}\dfrac{S_2}{S_1} & \\ & \text{if } W_1 > 0,\ W_2 > 0, \\ c_{01}c_{02}\left(1 + \dfrac{r_1}{(1+W_1)^{\epsilon_1}}\right)\dfrac{S_2}{S_1} & \text{if } W_1 > 0,\ W_2 \leqslant 0, \\ c_{01}c_{02}\left(1 - \dfrac{r_2}{(1+W_2)^{\epsilon_2}}\right)\dfrac{S_2}{S_1} & \text{if } W_1 \leqslant 0,\ W_2 > 0, \\ \delta_{01}\delta_{02} & \text{if } W_1 \leqslant 0,\ W_2 \leqslant 0. \end{cases}$$

3.3 Improved estimators of the ratio of two normal variances based on Shinozaki modifications

Let X_1, X_2, S_1, S_2 be as in the beginning of Sect. 3.2 and set $T_i = \|X_i\|^2$ for $i = 1, 2$. For estimating σ_2, Shinozaki (1995) proposed various methods for modifying classical improved scale equivariant estimators (e.g., Stein's estimator). Shinozaki's estimators are not scale equivariant, improve on the best equivariant estimator of σ_2, and are expected to offer larger improvement in a certain region of the parameter space. Here, we only discuss Shinozaki modifications of Strawderman-type estimators of $\rho = \sigma_2/\sigma_1$ under the squared error loss. For estimating σ_2, as a special case of Shinozaki (1995, Theorem 2.3) we get the improved estimator $\delta_{\psi_2} = \frac{1}{m+2}\{1 - r_2 b_2(S_2 + T_2)\frac{S_2^{\epsilon_2}}{(S_2+T_2)^{\epsilon_2}}\}S_2$, where $b_2(\cdot)$ is non-decreasing on $(0, \infty)$ and satisfies $0 \le b_2(\cdot) \le 1, \epsilon_2 > 0$ and $0 < r_2 < \frac{2q\epsilon_2}{m+q+2}\frac{\Gamma(q/2+m/2+2\epsilon_2+2)\Gamma(m/2+\epsilon_2+1)}{\Gamma(q/2+m/2+\epsilon_2+2)\Gamma(m/2+2\epsilon_2+2)}$. Following Shinozaki (1995) approach, we can modify a class of Strawderman-type estimators of a normal precision given by Bobotas and Kourouklis (2010) and obtain the improved estimator of $1/\sigma_1$, $\delta_{\psi_1} = (n-4)\{1 + r_1 b_1(S_1 + T_1)\frac{S_1^{\epsilon_1}}{(S_1+T_1)^{\epsilon_1}}\}S_1^{-1}$, where $b_1(\cdot)$ is non-decreasing on $(0, \infty)$ and satisfies $0 \le b_1(\cdot) \le 1, \epsilon_1 > 0$ and $0 < r_1 < \frac{2p\epsilon_1}{n-4}\frac{\Gamma(n/2+\epsilon_1-2)\Gamma(n/2+p/2+2\epsilon_1-2)}{\Gamma(n/2+2\epsilon_1-2)\Gamma(n/2+p/2+\epsilon_1-1)}$. The proof of this result is similar to that of Theorem 2.1 in Bobotas and Kourouklis (2010). Therefore, using δ_{ψ_1} and δ_{ψ_2}, Theorems 5 and 6 apply and give the improved Shinozaki-type estimators of ρ, $\left\{(\delta_{\psi_1}\delta_{02})^k + (\delta_{01}\delta_{\psi_2})^k - (\delta_{01}\delta_{02})^k\right\}^{1/k}$, $1 \le k \le 2$.

Appendix

We consider the model in (5) and construct Strawderman-type estimators for $1/\sigma_1$ and σ_2 under the symmetric loss $L(t) = t + 1/t - 2$. The respective best equivariant estimators are given by $\delta_{01} = c_{01}S_1^{-1}$, $\delta_{02} = c_{02}S_2$, with $c_{01}^2 = \mathsf{E}(S_1)/\mathsf{E}(S_1^{-1})$ and $c_{02}^2 = \mathsf{E}(S_2^{-1})/\mathsf{E}(S_2)$. Following Strawderman (1974), for improving on δ_{01} and δ_{02} we consider estimators of the form δ_{ϕ_1}, δ_{ϕ_2} in (13) and (14), respectively, where as before $W_i = T_i/S_i$ and r_1, r_2 are positive constants to be determined below. For convenience, we set $a_i = (-1)^i$, $i = 1, 2$, so that δ_{ϕ_1} and δ_{ϕ_2} can be written as

$$\delta_{\phi_i} = \begin{cases} c_{0i}\left\{1 - a_i\dfrac{r_i}{(1+W_i)^{\epsilon_i}}\right\}S_i^{a_i} & \text{if } W_i > 0, \\ \delta_{0i} & \text{otherwise.}\end{cases} \tag{15}$$

Recall that μ_{0i}, σ_{0i} are defined in the beginning of Sect. 2 and $V_i = S_i/\sigma_i$, $i = 1, 2$.

Theorem 7 *Assume that Conditions (A) and (B) hold. Then, for every $\epsilon_i > 0$, $i = 1, 2$, the risk of δ_{ϕ_i} is strictly smaller than that of δ_{0i} provided $0 < r_i < B_i(\epsilon_i)$, where*

$$B_i(\epsilon_i) = a_i\left\{1 - \frac{1}{c_{0i}^2}\frac{\mathsf{E}_{\mu_i=\mu_{0i},\sigma_i=\sigma_{0i}}\left\{(1+W_i)^{-\epsilon_i}V_i^{-a_i}\right\}}{\mathsf{E}_{\mu_i=\mu_{0i},\sigma_i=\sigma_{0i}}\left\{(1+W_i)^{-\epsilon_i}V_i^{a_i}\right\}}\right\}, \tag{16}$$

and a_i is as in (15).

Proof Due to the form of δ_{0i} and δ_{ϕ_i}, their risks difference is given by

$$
\begin{aligned}
&RD(\delta_{0i},\delta_{\phi_i})\\
&\quad= \mathsf{E}\left\{\delta_{0i}/\sigma_i^{a_i} + (\delta_{0i}/\sigma_i^{a_i})^{-1} - 2\right\} - \mathsf{E}\left\{\delta_{\phi_i}/\sigma_i^{a_i} + (\delta_{\phi_i}/\sigma_i^{a_i})^{-1} - 2\right\}\\
&\quad= r_i\mathsf{E}\left\{a_i\left(\frac{c_{0i}(S_i/\sigma_i)^{a_i}}{(1+W_i)^{\epsilon_i}} - \left[c_{0i}(S_i/\sigma_i)^{a_i}\left(1-\frac{a_i r_i}{(1+W_i)^{\epsilon_i}}\right)(1+W_i)^{\epsilon_i}\right]^{-1}\right)I(W_i>0)\right\}\\
&\quad= r_i a_i\int_0^\infty\int_0^\infty\left\{\frac{c_{0i}y_i^{a_i}}{(1+w_i)^{\epsilon_i}} - \left[c_{0i}y_i^{a_i}\left(1-\frac{a_i r_i}{(1+w_i)^{\epsilon_i}}\right)(1+w_i)^{\epsilon_i}\right]^{-1}\right\}\\
&\qquad\times y_i g_i(y_i)h_i(w_i y_i;\mu_i,\sigma_i)I\big(w_i y_i>\kappa_i(\mu_i,\sigma_i)\big)\mathrm{d}y_i\mathrm{d}w_i.
\end{aligned}
$$

By a change of variables $u_i = 1/(1+w_i)$, $v_i = (1+w_i)y_i$ and setting $\beta_i = \max\{\kappa_i/(1-u_i), 0\}$, $i = 1, 2$, we obtain

$$
\begin{aligned}
RD(\delta_{0i},\delta_{\phi_i}) &= r_i a_i\int_0^1\int_{\beta_i}^\infty\left\{c_{0i}u_i^{\epsilon_i+a_i}v_i^{a_i} - \frac{u_i^{\epsilon_i-a_i}}{c_{0i}v_i^{a_i}(1-a_i r_i u_i^{\epsilon_i})}\right\}\\
&\qquad\times v_i g_i(u_i v_i)h_i\big((1-u_i)v_i;\mu_i,\sigma_i\big)\mathrm{d}v_i\mathrm{d}u_i\\
&\geqslant r_i a_i\int_0^1\int_{\beta_i}^\infty\left\{c_{0i}u_i^{\epsilon_i+a_i}v_i^{a_i} - \frac{u_i^{\epsilon_i-a_i}}{c_{0i}v_i^{a_i}(1-a_i r_i)}\right\}v_i g_i(u_i v_i)\\
&\qquad\times h_i\big((1-u_i)v_i;\mu_i,\sigma_i\big)\mathrm{d}v_i\mathrm{d}u_i.
\end{aligned}
$$

A straightforward calculation for each value $i = 1, 2$ separately entails that $RD(\delta_{0i},\delta_{\phi_i}) > 0$ provided that

$$
\begin{aligned}
r_i &< a_i\left\{1 - \frac{1}{c_{0i}^2}\frac{\int_0^1 u_i^{\epsilon_i-a_i}\int_{\beta_i}^\infty v_i^{1-a_i}g_i(u_i v_i)h_i\big((1-u_i)v_i;\mu_i,\sigma_i\big)\mathrm{d}v_i\mathrm{d}u_i}{\int_0^1 u_i^{\epsilon_i+a_i}\int_{\beta_i}^\infty v_i^{1+a_i}g_i(u_i v_i)h_i\big((1-u_i)v_i;\mu_i,\sigma_i\big)\mathrm{d}v_i\mathrm{d}u_i}\right\}\\
&= a_i\left\{1 - c_{0i}^{-2}I_i(\epsilon_i;\mu_i,\sigma_i)\right\},
\end{aligned}
$$

say. Using a similar argument as in the proof of the relations (2.11) and (2.14) in Bobotas and Kourouklis (2009), it can be shown that, under (A) and (B), $I_1(\epsilon_1;\mu_1,\sigma_1) \geqslant I_1(\epsilon_1;\mu_{01},\sigma_{01}) > c_{01}^2$ and $I_2(\epsilon_2;\mu_2,\sigma_2) \leqslant I_2(\epsilon_2;\mu_{02},\sigma_{02}) < c_{02}^2$. Moreover, it is easily seen that $I_i(\epsilon_i;\mu_{0i},\sigma_{0i}) = \mathsf{E}_{\mu_i=\mu_{0i},\sigma_i=\sigma_{0i}}\{(1+W_i)^{-\epsilon_i}V_i^{-a_i}\}/\mathsf{E}_{\mu_i=\mu_{0i},\sigma_i=\sigma_{0i}}\{(1+W_i)^{-\epsilon_i}V_i^{a_i}\}$ and this completes the proof. □

Acknowledgments We wish to thank a referee whose suggestion led to treating non-scale equivariant estimators and, in particular, Shinozaki-type estimators.

References

Bobotas, P., Kourouklis, S. (2009). Strawderman-type estimators for a scale parameter with application to the exponential distribution. *Journal of Statistical Planning and Inference, 139*, 3001–3012.

Bobotas, P., Kourouklis, S. (2010). On the estimation of a normal precision and a normal variance ratio. *Statistical Methodology, 7*(4), 445–463.

Brewster, J. F., Zidek, J. V. (1974). Improving on equivariant estimators. *The Annals of Statistics, 2*, 21–38.
Brown, L. D. (1968). Inadmissibility of the usual estimators of scale parameters in problems with unknown location and scale parameters. *Annals of Mathematical Statistics, 39*, 29–48.
Gelfand, A. E., Dey, D. K. (1988). On the estimation of a variance ratio. *Journal of Statistical Planning and Inference, 19*, 121–131.
Ghosh, M., Kundu, S. (1996). Decision theoretic estimation of the variance ratio. *Statistics Decisions, 14*, 161–175.
Iliopoulos, G., Kourouklis, S. (1999). Improving on the best affine equivariant estimator of the ratio of the generalized variances. *Journal of Multivariate Analysis, 68*, 176–192.
James, W., Stein, C. (1961). Estimation with quadratic loss. In *Proceedings of the Fourth Symposium on Mathematical Statistics and Probability, 1*, (pp. 361–380). Berkeley: University of California Press.
Kubokawa, T. (1994a). A unified approach to improving equivariant estimators. *The Annals of Statistics, 22*, 290–299.
Kubokawa, T. (1994b). Double shrinkage estimation of ratio of scale parameters. *Annals of the Institute of Statistical Mathematics, 46*(1), 95–116.
Kubokawa, T. (1999). Shrinkage and modification techniques in estimation of variance and the related problems: A review. *Communications in Statistics-Theory and Methods, 28*(3–4), 613–650.
Kubokawa, T., Konno, Y. (1990). Estimating the covariance matrix and the generalized variance under a symmetric loss. *Annals of the Institute of Statistical Mathematics, 42*(2), 331–343.
Kubokawa, T., Srivastava, M. S. (1996). Double shrinkage estimators of ratio of variances. In *Multidimensional Statistical Analysis and Theory of Random Matrices* (Proceedings of the Sixth Eugene Lukacs Symposium, Bowling Green, OH, USA, Eds: A. K. Gupta and V. L. Girko), 139–154, VSP, Netherlands.
Maatta, J. M., Casella, G. (1990). Developments in decision-theoretic variance estimation. *Statistical Science, 5*(1), 90–101.
Madi, M. (1995). On the invariant estimation of a normal variance ratio. *Journal of Statistical Planning and Inference, 44*, 349–357.
Maruyama, Y., Strawderman, W. E. (2006). A new class of minimax generalized Bayes estimators of a normal variance. *Journal of Statistical Planning and Inference, 136*, 3822–3836.
Pal, N. (1988). Decision-theoretic estimation of generalized variance and generalized precision. *Communications in Statistics–Theory and Methods, 17*(12), 4221–4230.
Pal, N., Ling, C., Lin, J. (1998). Estimation of a normal variance: A critical review. *Statistical Papers, 4*, 389–404.
Shinozaki, N. (1995). Some modifications of improved estimators of a normal variance. *Annals of the Institute of Statistical Mathematics, 47*(2), 273–286.
Stein, C. (1964). Inadmissibility of the usual estimator for the variance of a normal distribution with unknown mean. *Annals of the Institute of Statistical Mathematics, 16*, 155–160.
Strawderman, W. E. (1974). Minimax estimation of powers of the variance of a normal population under squared error loss. *The Annals of Statistics, 2*, 190–198.