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Forms of four-word indicator functions with implications to two-level factorial designs

N. Balakrishnan · Po Yang

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Abstract Indicator functions are new tools to study fractional factorial designs. In this paper, we study indicator functions with four words and provide possible forms of the indicator functions and explain their implications to two-level factorial designs.

Keywords Fractional factorial designs · Two-level factorial designs · Indicator functions · Words · Replicates · Non-regular factorial designs · Aliased effects · Orthogonal effects

1 Introduction

Fractional factorial designs have many applications in industry and agriculture. Recently, a new approach, called an *indicator function approach*, has been adopted by Fontana et al. (2000) to study the two-level unreplicated factorial designs. Subsequently, Ye (2003) proposed indicator functions for replicated factorial designs. This new method allows us to discuss not only the regular factorial designs, where effects are either orthogonal or fully aliased, but also non-regular factorial designs, in which some effects are neither orthogonal nor fully aliased. This method has become a powerful tool for studying general two-level factorial designs; see, for example, Cheng et al. (2004) and Loeppky et al. (2006). Some other interesting theoretical results have also been obtained using indicator functions by Ye (2004) and Balakrishnan and Yang

N. Balakrishnan
Department of Mathematics and Statistics, McMaster University, Hamilton, ON L8S 4K1, Canada
e-mail: bala@mcmail.cis.mcmaster.ca

P. Yang (✉)
Department of Mathematical Sciences, DePaul University, Chicago, IL 60614, USA
e-mail: pyang3@depaul.edu

(2006a,b). Furthermore, Cheng and Ye (2004) and Pistone and Rogantin (2008) have shown that multi-level or mixed-level factorial designs can also be represented by indicator functions.

Let D be a 2^m full factorial design, i.e., D contains all the runs $x = \{x_1, x_2, \ldots, x_m\}$, where $x_i = 1$ or $-1, i = 1, 2, \ldots, m$. Let $\mathcal{F}$ be a design such that for any $x \in \mathcal{F}$, $x \in D$, and x can be repeated in $\mathcal{F}$. The *indicator function* of $\mathcal{F}$ is a function $f(x)$ defined on D such that

$$f(x) = \begin{cases} r_x & \text{if } x \in \mathcal{F} \\ 0 & \text{if } x \notin \mathcal{F}, \end{cases}$$

where r_x is the number of appearances of x in $\mathcal{F}$. Denote by

$$L = \{\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \mid \alpha_i = 1 \text{ or } 0 \text{ for } i = 1, 2, \ldots, m\}$$

and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$. Then, $f(x)$ can be uniquely represented by a polynomial function [see Fontana et al. (2000) and Ye (2003)]

$$f(x) = \sum_{\alpha \in L} b_\alpha x^\alpha, \tag{1}$$

where b_α can be calculated by the formula

$$b_\alpha = \frac{1}{2^m} \sum_{x \in \mathcal{F}} x^\alpha. \tag{2}$$

In particular, $b_0 = n/2^m$, where n is the total number of runs. Each term in $f(x)$ is called a *word* of the design. If a word contains even (or odd) number of letters, we refer to the word as *even-letter (or odd-letter) word.*

The forms of indicator functions with the fixed number of words, for example, four words, for replicated designs are complicated. For example, an indicator function of a 6-factor fractional factorial design is

$$f(x) = \frac{1}{2} + \frac{1}{4}x_1x_2x_3 + \frac{1}{4}x_1x_2x_4 + \frac{1}{4}x_1x_3x_5x_6 - \frac{1}{4}x_1x_4x_5x_6. \tag{3}$$

If we add a fraction with indicator function $g(x) = \frac{1}{2} + \frac{1}{2}x_1x_2x_3$, then the indicator function of the combined fraction or the replicated design is

$$f_1(x) = f(x) + g(x) = 1 + \frac{3}{4}x_1x_2x_3 + \frac{1}{4}x_1x_2x_4 + \frac{1}{4}x_1x_3x_5x_6 - \frac{1}{4}x_1x_4x_5x_6.$$

We can also add some other fractions to get indicator functions with same four words but different coefficients. For this reason, we study in this paper only the two-level unreplicated fractional factorial designs. Note that if we know the indicator function of an unreplicated design, then the indicator function of the complete replicates of the design can be obtained by multiplying the number of the replicates.

Balakrishnan and Yang (2006a) considered indicator functions of unreplicated designs with one to three words. In this paper, we study the forms of the indicator functions with four words.

For the indicator function of an unreplicated fractional factorial design, $0 < b_0 < 1$ and $|b_\alpha/b_0| \le 1$ for any $\alpha \in L$, and so $|b_\alpha| < 1$. Fontana et al. (2000) and Ye (2003) showed that $\mathcal{F}$ is a regular design if and only if $|b_\alpha| = b_0$ for any $\alpha \in L$ such that $b_\alpha \neq 0$. Thus, for a non-regular design, $|b_\alpha| < b_0$ for some $b_\alpha \neq 0$ in (1). For the design with the indicator function (3), since $|b_\alpha| = 1/4$, which is less than $b_0 = 1/2$, for all the terms in $f(x)$, the design is a non-regular design. Each word in an indicator function indicates an alias relation. For example, the word $x_1x_2x_3$ in (3) indicates that x_1 and x_2x_3 are partially aliased, since the coefficient of $x_1x_2x_3$ is less than the constant 1/2. Define the *generalized word length* of a word x^α as $m + 1 - b_\alpha/b_0$, where m is the number of the letters in x^α. The *generalized resolution* of a design is defined as the length of the shortest word of the design. In (3), both $x_1x_2x_3$ and $x_1x_2x_4$ have the shortest length 3.5, and so the generalized resolution of this design is 3.5.

In Sect. 2, we consider the possible combined signs of four words. The forms of four-word indicator functions are then presented in Sect. 3 and their implications to two-level factorial designs are explained.

2 Possible combined signs of four words

To study the forms of four-word indicator functions, we first need to study the signs of the four words. We know that a word can take values 1 and -1 since $x_i = \pm 1$, $i = 1, 2, \ldots, m$. By Remark 1 in Balakrishnan and Yang (2006a), the signs of any two words can be all the possible combined signs, as follows:

$$\begin{array}{rr} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{array}$$

However, when there are three or more words, the signs of the words may not have all the possible combined signs. In this section, we discuss the combined signs of three words first, and then find the combined signs of four words.

Lemma 1 *Any three words have either all the possible eight combined signs or the following four combined signs:*

$$\begin{array}{rrr} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{array} \qquad (4)$$

In this case, any one of the three words is the product of the other two words.

Proof We prove this result by considering the following three cases:

1. If there exists one (or more) letter such that it is in only one of the three words, then, assume that x^θ, where $\theta \in L$, is in the last word but not in the first two words. Since the first two words can have all the four combined signs and x^θ can be either 1 or -1, the three words have all the eight combined signs by using the argument in Remark 1 of Balakrishnan and Yang (2006a).
2. If any letter in the three words is in two and only two of the three words, then, the three words can be written as $x^{\theta_1}x^{\theta_2}$, $x^{\theta_2}x^{\theta_3}$ and $x^{\theta_1}x^{\theta_3}$, where $\theta_1, \theta_2, \theta_3 \in L$. Since x^{θ_i}, $i = 1, 2, 3$, can only be 1 and -1, it is easy to check that the three words have only the four possible combined signs in (4). Clearly, in this case, any one of the three words is the product of the other two words.
3. If there exists one (or more) letter which is in all the three words and no letter is in only one of the three words, then, assume that x^θ is in all the three words. Then, the three words can be written as $x^\theta x^{\theta_1}$, $x^\theta x^{\theta_2}$ and $x^\theta x^{\theta_3}$, where $\theta_1, \theta_2, \theta_3 \in L$. Consider the three words x^{θ_1}, x^{θ_2} and x^{θ_3}. Note that any letter in the three words must be in two of the three words, the three words must have the four combined signs in (4) by Part 2 above. Now, since x^θ can be 1 and -1, the three words $x^\theta x^{\theta_1}$, $x^\theta x^{\theta_2}$ and $x^\theta x^{\theta_3}$ must have the eight combined signs.

Hence, the lemma. □

Balakrishnan and Yang (2006a) have provided the forms of three-word indicator functions. The three words in all the forms can have only the four combined signs in (4). Thus, any word among the three is the product of the other two words.

Lemma 2 *Any four words have one of the following three possible combined signs:*

1.

$$\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \end{array} \tag{5}$$

In this case, the first three words are such that one is the product of the other two words;

2.

$$\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 \end{array} \tag{6}$$

In this case, any word in the four words is the product of the rest of the three words;
3. *All the possible sixteen combined signs.*

Proof By Lemma 1, we need to consider only the following two cases:

1. Case when the first three words have the four combined signs in (4): If there exists a letter in the fourth word which is not in any of the first three words, then, clearly, the four words can have the eight combined signs in (5). Otherwise, assume that the first three words are $x^{\theta_1}x^{\theta_2}$, $x^{\theta_2}x^{\theta_3}$ and $x^{\theta_1}x^{\theta_3}$. We can write the fourth word as $x^{\theta_{10}}x^{\theta_{20}}x^{\theta_{30}}$, where $\theta_{10}, \theta_{20}, \theta_{30} \in L$ and satisfy $x^{\theta_1} = x^{\theta_{10}}x^{\theta_{11}}$, $x^{\theta_2} = x^{\theta_{20}}x^{\theta_{21}}$ and $x^{\theta_3} = x^{\theta_{30}}x^{\theta_{31}}$ for some $\theta_{11}, \theta_{21}, \theta_{31} \in L$. If the fourth term is x^{θ_1}, x^{θ_2}, x^{θ_3} or $x^{\theta_1}x^{\theta_2}x^{\theta_3}$, one can easily check that the four words have the eight combined signs in (5). Otherwise, given the signs of x^{θ_1}, x^{θ_2} and x^{θ_3}, $x^{\theta_{10}}x^{\theta_{20}}x^{\theta_{30}}$ can take on two values 1 and -1 by changing the signs of $x^{\theta_{11}}$, $x^{\theta_{21}}$ or $x^{\theta_{31}}$. This implies that the four words have the eight combined signs in (5).
2. Case when the first three words have all the possible eight combined signs: if there exists one letter in the fourth word which is not in any of the first three words, then the four words have all the possible sixteen combined signs. Otherwise, if the fourth word is a product of any two words from the first three words, then the four words have the signs in (5) if we change the order of the four words; if the fourth word is the product of the first three words, then one can check that the four words have the combined signs in (6), also, in this case, one can see that any one of the four words is the product of the other three words; if the fourth word is formed by the parts of the first three words, then, by Lemma 1, either there exists one (or more) letter which is in only one of the first three words or which is in all the first three words and no letter is in only one word: in the first case, the first three words can be written as $x^{\theta_1}x^{\theta_3}x^{\theta_4}$, $x^{\theta_1}x^{\theta_2}x^{\theta_5}$, $x^{\theta_2}x^{\theta_3}x^{\theta_6}$, where $\theta_i \in L, i = 1, \ldots, 6$, and the fourth word can be represented by $x^{\theta_{11}}x^{\theta_{21}}x^{\theta_{31}}x^{\theta_{41}}x^{\theta_{51}}x^{\theta_{61}}$, where $\theta_{ik} \in L, k = 1$, and $x^{\theta_{ik}}$ is a part of x^{θ_i}; then by using an argument similar to the one in part 1 above, the four words have all the sixteen combined signs; in the second case, the four words can be written as $x^{\theta}x^{\theta_1}$, $x^{\theta}x^{\theta_2}$, $x^{\theta}x^{\theta_3}$, and $x^{\theta_0}x^{\theta_{00}}$, where $\theta_l, \theta_0, \theta_{00} \in L, l = 1, 2, 3$, x^{θ_1}, x^{θ_2}, and x^{θ_3} are such that any letter is in two of the three words, x^{θ_0} is a part of x^{θ}, $x^{\theta_{00}}$ includes some letters in x^{θ_1}, x^{θ_2}, and x^{θ_3}, and by part 1, the four words x^{θ_1}, x^{θ_2}, x^{θ_3}, and $x^{\theta_{00}}$ have the eight combined signs in (5), and by changing the sign of one letter in x^{θ_0}, we can get the sixteen combined signs (if $x^{\theta_0} = 1$, changing the sign of x^{θ}).

Hence, the lemma. □

3 Four-word indicator functions

Balakrishnan and Yang (2006b) showed that there is no three-word indicator function which has only one odd-letter word. In this section, we show that this is true in general. We then consider the four-word indicator functions with one even-letter word. Finally, we discuss the possible forms of indicator functions with four words.

Theorem 1 *There is no factorial design whose indicator function has more than two words but only one odd-letter word.*

Proof Denote $||\alpha|| = \sum_{i=1}^{m} \alpha_i$, i.e., $||\alpha||$ is the number of letters of x^α, and

$$\Omega = \{\alpha \in L \mid b_\alpha \neq 0 \quad \text{and} \quad ||\alpha|| \text{ is 0 or even }\}.$$

Then, for any $\alpha \in \Omega$, if $\alpha \neq 0$, then x^α is an even-letter word, and the indicator function in (1) can be written as

$$f(x) = \sum_{\alpha \in \Omega} b_\alpha x^\alpha + b_\theta x^\theta,$$

where x^θ is an odd-letter word. Note that for any $x = \{x_1, x_2, \ldots, x_m\} \in D$, $-x = \{-x_1, -x_2, \ldots, -x_m\} \in D$. Since $f(x)$ can only be 1 or 0 for any $x \in D$, we have

$$f(x) = \sum_{\alpha \in \Omega} b_\alpha x^\alpha + b_\theta x^\theta = 0 \text{ or } 1$$

and

$$f(-x) = \sum_{\alpha \in \Omega} b_\alpha x^\alpha - b_\theta x^\theta = 0 \text{ or } 1.$$

Since $b_\theta \neq 0$, $b_\theta x^\theta \neq 0$ for any $x \in D$. Thus, the two equations imply that $\sum_{\alpha \in \Omega} b_\alpha x^\alpha \equiv \frac{1}{2}$ for any $x \in D$, which is impossible, and so an indicator function cannot contain only one odd-letter word. □

Balakrishnan and Yang (2006a) mentioned that if an indicator function contains only one odd-letter word, then the coefficient of the odd-letter word must be $\pm\frac{1}{2}$. Here, we have shown that these indicator functions actually do not exist.

Lemma 3 *There is no four-word indicator function which contains only one even-letter word.*

Proof Assume that $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + b_\gamma x^\gamma + b_\sigma x^\sigma$ is the indicator function of $\mathcal{F}$. When there is one even-letter word, say x^α, the four words cannot have the combined signs in (6), since an even-letter word cannot be a product of three odd-letter words. Thus, they also cannot have the sixteen combined signs, since the sixteen combined signs include the eight combined signs in (6). Note that for any three odd-letter words, one of them cannot be the product of the other two, and so we only need to consider the case when the four words have the combined signs in (5).

By Proposition 3.2 of Balakrishnan and Yang (2006a), b_0 and b_α can have the following possible values:

1. $b_0 = b_\alpha = \frac{1}{4}$ or $b_0 = b_\alpha = \frac{1}{2}$: In this case, since $f(x)$ can only equal 0 or 1 for any $x \in \mathcal{F}$, the last four combined signs of x^α, x^β, x^γ and x^σ in (5) imply

$$\begin{aligned} b_\beta - b_\gamma + b_\sigma &= 0 \text{ or } 1, \\ b_\beta - b_\gamma - b_\sigma &= 0 \text{ or } 1, \\ -b_\beta + b_\gamma + b_\sigma &= 0 \text{ or } 1, \\ -b_\beta + b_\gamma - b_\sigma &= 0 \text{ or } 1. \end{aligned}$$

The first and last equations yield $b_\beta - b_\gamma + b_\sigma = 0$, while the middle two equations yield $b_\beta - b_\gamma - b_\sigma = 0$. It then follows that $b_\sigma = 0$, which is impossible.
2. $b_0 = -3b_\alpha = \frac{3}{4}$: Note that the last four combined signs of x^α, x^β, x^γ and x^σ in (5) imply

$$\begin{aligned} b_\beta - b_\gamma + b_\sigma &= 0 \text{ or } -1, \\ b_\beta - b_\gamma - b_\sigma &= 0 \text{ or } -1, \\ -b_\beta + b_\gamma + b_\sigma &= 0 \text{ or } -1, \\ -b_\beta + b_\gamma - b_\sigma &= 0 \text{ or } -1, \end{aligned}$$

which again, similarly, give $b_\sigma = 0$, a contradiction.
3. $b_0 = \frac{1}{3}b_\alpha = \frac{1}{4}$ or $b_0 = 3b_\alpha = \frac{3}{4}$: Since $(1, 1, \ldots, 1) \in \mathcal{F}$ by Proposition 3.2 of Balakrishnan and Yang (2006a), the first three combined signs of x^α, x^β, x^γ and x^σ in (5) imply

$$\begin{aligned} b_\beta + b_\gamma + b_\sigma &= 0, \\ b_\beta + b_\gamma - b_\sigma &= 0 \text{ or } -1, \\ -b_\beta - b_\gamma + b_\sigma &= 0 \text{ or } -1. \end{aligned}$$

Now, the last two equations yield $b_\beta + b_\gamma - b_\sigma = 0$. Combining this with the first equation, we get $b_\sigma = 0$, which is a contradiction.
4. $b_0 = -b_\alpha = \frac{1}{4}, \frac{1}{2}$ or $\frac{3}{4}$: Since $(1, 1, \ldots, 1) \notin \mathcal{F}$ by Proposition 3.2 of Balakrishnan and Yang (2006a), the first three combined signs of x^α, x^β, x^γ and x^σ in (5) imply

$$\begin{aligned} b_\beta + b_\gamma + b_\sigma &= 0, \\ b_\beta + b_\gamma - b_\sigma &= 0 \text{ or } 1, \\ -b_\beta - b_\gamma + b_\sigma &= 0 \text{ or } 1, \end{aligned}$$

which, similarly, give $b_\sigma = 0$, that is impossible.

Hence, $f(x)$ cannot contain only one even-letter word. □

Theorem 2 *Assume that $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + b_\gamma x^\gamma + b_\sigma x^\sigma$ is the indicator function of a fractional factorial design $\mathcal{F}$. Then, one of the four words must be a product of the other three words and $f(x)$ must be of one of the following forms:*

1. *When there are two even-letter words, say* x^α *and* x^β,

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma - \frac{1}{4}x^\sigma & \textit{if } (1,\ldots,1), (-1,\ldots,-1) \in \mathcal{F} \\ \frac{1}{2} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma + \frac{1}{4}x^\sigma & \textit{if } (1,\ldots,1), (-1,\ldots,-1) \notin \mathcal{F} \\ \frac{1}{2} - \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma + \frac{1}{4}x^\sigma & \textit{if } (1,\ldots,1) \in \mathcal{F}, (-1,\ldots,-1) \notin \mathcal{F} \\ \frac{1}{2} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma - \frac{1}{4}x^\sigma & \textit{if } (1,\ldots,1) \notin \mathcal{F}, (-1,\ldots,-1) \in \mathcal{F}; \end{cases} \tag{7}$$

2. *When all the words are even-letter words,*

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma - \frac{1}{4}x^\sigma & \textit{if } (1,\ldots,1), (-1,\ldots,-1) \in \mathcal{F} \\ \frac{1}{2} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma + \frac{1}{4}x^\sigma & \textit{if } (1,\ldots,1), (-1,\ldots,-1) \notin \mathcal{F}; \end{cases} \tag{8}$$

3. *When all the words are odd-letter words,*

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma - \frac{1}{4}x^\sigma & \textit{if } (1,\ldots,1) \in \mathcal{F}, (-1,\ldots,-1) \notin \mathcal{F} \\ \frac{1}{2} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma + \frac{1}{4}x^\sigma & \textit{if } (1,\ldots,1) \notin \mathcal{F}, (-1,\ldots,-1) \in \mathcal{F}. \end{cases} \tag{9}$$

Proof By Theorem 1 and Lemma 3, we only need to consider the following three cases:

Case 1 When there are two even-letter words, say x^α and x^β, and two odd-letter words. By Proposition 3.1 and Theorem 4.2 of Balakrishnan and Yang (2006a), we have

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + b_\gamma x^\gamma - b_\gamma x^\sigma & \text{if } (1,\ldots,1), (-1,\ldots,-1) \in \mathcal{F} \\ \frac{1}{2} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + b_\gamma x^\gamma - b_\gamma x^\sigma & \text{if } (1,\ldots,1), (-1,\ldots,-1) \notin \mathcal{F} \\ \frac{1}{2} - \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + b_\gamma x^\gamma + (\frac{1}{2} - b_\gamma)x^\sigma & \text{if } (1,\ldots,1) \in \mathcal{F}, (-1,\ldots,-1) \notin \mathcal{F} \\ \frac{1}{2} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + b_\gamma x^\gamma + (-\frac{1}{2} - b_\gamma)x^\sigma & \text{if } (1,\ldots,1) \notin \mathcal{F}, (-1,\ldots,-1) \in \mathcal{F}. \end{cases} \tag{10}$$

1. Case when the four words have the combined signs in (5): Note that x^α and x^β are even-letter words, and x^γ is an odd-letter word, one of x^α, x^β and x^γ cannot be the product of the other two words. Similarly, one of x^α, x^β and x^σ cannot be the product of the other two words. Thus, we only need to consider the following two cases:
   (a) When one of x^α, x^γ and x^σ is the product of the other two words, then x^α, x^β, x^γ and x^σ can have the signs 1, −1, 1, and 1. However, in this case, $f(x) = \frac{1}{2}$ for the first two and the last forms in (10), and $f(x) = \frac{3}{2}$ for the third form in (10). Thus, the indicator function cannot be of any form in (10).

(b) Similarly, when one of x^β, x^γ and x^σ is the product of the other two words, then if $x^\alpha, x^\beta, x^\gamma$ and x^σ have the signs -1, 1, 1, and 1, $f(x) \neq 1$ or 0 for all the forms in (10). Thus, $f(x)$ cannot have any form in (10).

2. Case when the four words have the combined signs in (6):
If $f(x) = \frac{1}{2} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + b_\gamma x^\gamma - b_\gamma x^\sigma$, when the signs of $x^\alpha, x^\beta, x^\gamma$ and x^σ are $1, -1, 1, -1$, $b_\gamma = \pm\frac{1}{4}$ since $f(x)$ can only be 0 or 1. One can check that $f(x)$ =0 or 1 for all the other signs in (6) if $b_\gamma = \pm\frac{1}{4}$. Thus, $f(x)$ has the first form in (7).
If $f(x) = \frac{1}{2} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + b_\gamma x^\gamma - b_\gamma x^\sigma$, we similarly get $b_\gamma = \pm\frac{1}{4}$, which gives the second form in (7).
If $f(x) = \frac{1}{2} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + b_\gamma x^\gamma + (\frac{1}{2} - b_\gamma)x^\sigma$, the signs $1, -1, 1, -1$ of $x^\alpha, x^\beta, x^\gamma$ and x^σ indicates that $b_\gamma = \pm\frac{1}{4}$. However, $f(x)$ =0 or 1 for all the other signs in (6) only when $b_\gamma = \frac{1}{4}$. This gives the third form of (7).
If $f(x) = \frac{1}{2} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + b_\gamma x^\gamma + (-\frac{1}{2} - b_\gamma)x^\sigma$, and $x^\alpha, x^\beta, x^\gamma$ and x^σ have the signs $1, -1, 1, -1$, we obtain $b_\gamma = -\frac{1}{4}$ or $-\frac{3}{4}$. But only $b_\gamma = -\frac{1}{4}$ satisfies $f(x)$ =0 or 1 for all the other signs in (6). Thus, $f(x)$ has the fourth form in (7).

Case 2 When all the four words are even-letter words.
Then, we have

$$b_0 + b_\alpha + b_\beta + b_\gamma + b_\sigma = \begin{cases} 1 & \text{if } (1, \ldots, 1) \in \mathcal{F}, (-1, \ldots, -1) \in \mathcal{F} \\ 0 & \text{if } (1, \ldots, 1) \notin \mathcal{F}, (-1, \ldots, -1) \notin \mathcal{F}. \end{cases}$$

1. When the four words have the combined signs in (5), note that $f(x)$ can only be 0 or 1, the signs of $x^\alpha, x^\beta, x^\gamma$ and x^σ in (5) imply

$$\begin{aligned}
b_0 + b_\alpha + b_\beta + b_\gamma + b_\sigma &= 0 \text{ or } 1 \\
b_0 + b_\alpha + b_\beta + b_\gamma - b_\sigma &= 0 \text{ or } 1, \\
b_0 + b_\alpha - b_\beta - b_\gamma + b_\sigma &= 0 \text{ or } 1, \\
b_0 + b_\alpha - b_\beta - b_\gamma - b_\sigma &= 0 \text{ or } 1, \\
b_0 - b_\alpha + b_\beta - b_\gamma + b_\sigma &= 0 \text{ or } 1, \\
b_0 - b_\alpha + b_\beta - b_\gamma - b_\sigma &= 0 \text{ or } 1, \\
b_0 - b_\alpha - b_\beta + b_\gamma + b_\sigma &= 0 \text{ or } 1, \\
b_0 - b_\alpha - b_\beta + b_\gamma - b_\sigma &= 0 \text{ or } 1.
\end{aligned}$$

Note that $b_\alpha, b_\beta, b_\gamma, b_\sigma$ cannot be 1 or 0, and so one can check that the equations have no solutions for the two cases $b_0 + b_\alpha + b_\beta + b_\gamma + b_\sigma = 1$ or $b_0 + b_\alpha + b_\beta + b_\gamma + b_\sigma = 0$.

2. When the four words have the combined signs in (6), we obtain the following equations:

$$\begin{aligned}
b_0 + b_\alpha + b_\beta + b_\gamma + b_\sigma &= 1, \\
b_0 + b_\alpha + b_\beta - b_\gamma - b_\sigma &= 0 \text{ or } 1,
\end{aligned}$$

$$
\begin{aligned}
b_0 + b_\alpha - b_\beta + b_\gamma - b_\sigma &= 0 \text{ or } 1,\\
b_0 - b_\alpha + b_\beta + b_\gamma - b_\sigma &= 0 \text{ or } 1,\\
b_0 + b_\alpha - b_\beta - b_\gamma + b_\sigma &= 0 \text{ or } 1,\\
b_0 - b_\alpha - b_\beta + b_\gamma + b_\sigma &= 0 \text{ or } 1,\\
b_0 - b_\alpha + b_\beta - b_\gamma + b_\sigma &= 0 \text{ or } 1,\\
b_0 - b_\alpha - b_\beta - b_\gamma - b_\sigma &= 0.
\end{aligned}
$$

(a) If $b_0 + b_\alpha + b_\beta + b_\gamma + b_\sigma = 1$, then $b_0 - b_\alpha - b_\beta - b_\gamma - b_\sigma$ must equal 0, since b_0 cannot be 1. Solving the equations, we obtain the following four solutions:

$$
\begin{aligned}
-b_\alpha = b_\beta = b_\gamma = b_\sigma &= \frac{1}{4},\\
b_\alpha = -b_\beta = b_\gamma = b_\sigma &= \frac{1}{4},\\
b_\alpha = b_\beta = -b_\gamma = b_\sigma &= \frac{1}{4},\\
b_\alpha = b_\beta = b_\gamma = -b_\sigma &= \frac{1}{4}.
\end{aligned}
$$

Since x^α, x^β, x^γ and x^σ are symmetric, the indicator functions have the first form in (8).

(b) If $b_0 + b_\alpha + b_\beta + b_\gamma + b_\sigma = 0$, then $b_0 - b_\alpha - b_\beta - b_\gamma - b_\sigma$ must equal 1. Similarly, one can show that the indicator function have the second form in (8).

Case 3 When all the four words are odd-letter words.

In this case, the four words cannot have the eight combined signs in (5), since the product of any two odd-letter words is an even-letter word.

By Proposition 3.1 of Balakrishnan and Yang (2006a), $b_0 = \frac{1}{2}$ and

$$
b_\alpha + b_\beta + b_\gamma + b_\sigma = \begin{cases} \frac{1}{2} & \text{if } (1, \ldots, 1) \in \mathcal{F}, (-1, \ldots, -1) \notin \mathcal{F} \\ -\frac{1}{2} & \text{if } (1, \ldots, 1) \notin \mathcal{F}, (-1, \ldots, -1) \in \mathcal{F}. \end{cases}
$$

Case when the four words have the combined signs in (6): If $b_\alpha + b_\beta + b_\gamma + b_\sigma = \frac{1}{2}$, then $b_0 + b_\alpha + b_\beta + b_\gamma + b_\sigma = 1$ and $b_0 - b_\alpha - b_\beta - b_\gamma - b_\sigma = 0$. This is the same as Part (*a*) of *Case 2*, which gives the first form of (9). If $b_\alpha + b_\beta + b_\gamma + b_\sigma = -\frac{1}{2}$, then $b_0 + b_\alpha + b_\beta + b_\gamma + b_\sigma = 0$ and $b_0 - b_\alpha - b_\beta - b_\gamma - b_\sigma = 1$. This is the same as Part (*b*) of *Case 2*, and consequently the indicator function has the second form in (9).

From the above three cases, we see that the four words actually cannot have the combined signs in (5). Thus, they also cannot have the sixteen combined signs. Therefore, the four words can only have the signs in (6), which implies that one of the four words must be a product of the other three words. This completes the proof of the theorem. □

Remark Corollary 3.5 of Fontana et al. (2000) implies that if $f(x)$ and $g(x)$ are indicator functions of two complementary factorial designs, then $f(x) + g(x) = 1$. Conversely, if $f(x) + g(x) = 1$, we can see that the two designs must be complementary factorial designs, since the indicator function of a full factorial design D is $h(x) \equiv 1$, for any $x \in D$. Thus, when two indicator functions contain the same words and have the first and second forms in (7), respectively, the corresponding two factorial designs are complementary. Similarly, the designs with indicator functions of the third and fourth forms in (7), the first and second forms in (8), and the first and second forms in (9), are complementary, respectively.

From Theorem 2, one can see that there exist non-regular two-level factorial designs with only four words in their indicator functions. Since $|b_\alpha/b_0| = 1/2$ for any word in each indicator function, the resolution of each design is r.5, where r is the number of the shortest word in each indicator function.

By Theorem 2, if an indicator function contains four words, its constant must be 1/2 and the coefficient of each word must be 1/4 or $-1/4$. From the above Remark, if we know that a design has four words in its indicator function, then, its complementary design must also have four words. Also, Theorem 2 provides a way to construct a design with four words: since the four words have the relation that one is the product of the other three, we just need to set up three words, then let the fourth word be the product of the three words, and use Theorem 2 to get a design with four words. For example, to obtain a 6-factor design with resolution 4.5, let the three words be $x_1x_2x_3x_4$, $x_1x_2x_3x_5$, and $x_1x_2x_3x_6$, then, the fourth word is $x_1x_2x_3x_4x_5x_6$; now, since the four words are even-letter words, by the first form of (8), we obtain a design with indicator function

$$f(x) = \frac{1}{2} + \frac{1}{4}x_1x_2x_3x_4 + \frac{1}{4}x_1x_2x_3x_5 + \frac{1}{4}x_1x_2x_3x_6 - \frac{1}{4}x_1x_2x_3x_4x_5x_6.$$

We can also change the order of the four words to get other designs. For instance, by switching the third and the fourth words, we obtain a design with the indicator function

$$f(x) = \frac{1}{2} + \frac{1}{4}x_1x_2x_3x_4 + \frac{1}{4}x_1x_2x_3x_5 + \frac{1}{4}x_1x_2x_3x_4x_5x_6 - \frac{1}{4}x_1x_2x_3x_6.$$

Since $b_0 = \frac{1}{2}$, the run size is $n = 2^m b_0 = 2^{m-1}$, which is half runs comparing to the full m-factor factorial designs. Note that at least five factors are needed to obtain a resolution 3.5 design using the forms in (7) and (9), while we need at least 16 runs to obtain a resolution 3.5 design with four words. Similarly, at least 32 runs are needed to obtain a resolution 4.5 design with six factors and four words. If we compare run sizes and alias structures, clearly, the regular designs with one or three words are better than the designs with four words.

4 Conclusion

In this paper, we have showed that indicator functions of two-level unreplicated factorial designs with only one odd-letter word do not exist. For the indicator functions with four words, we have proved that the four words cannot contain only one even-letter word and provided the forms of indicator functions; we have also pointed out the relation among the four words that one must be the product of the other three words, and the complementary relations between the corresponding factorial designs. The results indicate that there is no valuable factorial design with four words in its indicator function in terms of run size and alias structure.

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