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Bivariate Fibonacci polynomials of order k with statistical applications

Kiyoshi Inoue · Sigeo Aki

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Abstract In the present article, we investigate the properties of bivariate Fibonacci polynomials of order k in terms of the generating functions. For k and ℓ $(1 \leq \ell \leq k-1)$, the relationship between the bivariate Fibonacci polynomials of order k and the bivariate Fibonacci polynomials of order ℓ is elucidated. Lucas polynomials of order k are considered. We also reveal the relationship between Lucas polynomials of order k and Lucas polynomials of order ℓ. The present work extends several properties of Fibonacci and Lucas polynomials of order k, which will lead us a new type of geneses of these polynomials. We point out that Fibonacci and Lucas polynomials of order k are closely related to distributions of order k and show that the distributions possess properties analogous to the bivariate Fibonacci and Lucas polynomials of order k.

Keywords Fibonacci polynomials · Lucas polynomials · Success runs · Waiting time · Distributions of order k · Probability generating function · Continued fraction

1 Introduction

The classical sequences of polynomials, for example, Hermite, Legendre, Chebyshev and Laguerre have received considerable attention in the literature (see Lebedev 1965).

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K. Inoue (✉)
Faculty of Economics, Seikei University, 3-3-1 Kichijoji-Kitamachi,
Musasino-shi, Tokyo 180-8633, Japan
e-mail: kinoue@econ.seikei.ac.jp

S. Aki
Faculty of Engineering Science, Kansai University, 3-3-35 Yamate-cho,
Suita-shi, Osaka 564-8680, Japan

We can mention Fibonacci polynomials of order k (in symbols: $F^{(k)}$) as the interesting class of polynomials, which have a deep relationship with the famous Fibonacci (see Koshy 2001). The theory and applications of the polynomials are still popular subjects of study for researchers in a wide range of areas such as statistics, probability theory, reliability, algebra, geometry and number theory (see Philippou and Makri 1985; Philippou et al. 1985; Philippou 1986). Numerous properties of Fibonacci polynomials of order k have been studied, which play a central role in these studies (see Antzoulakos and Koutras 2005; Djordjevic 2001). However, for k and ℓ ($1 \le \ell \le k-1$), the relationships between $F^{(k)}$ and $F^{(\ell)}$ have never been examined so far.

In the present paper, we investigate the properties of bivariate Fibonacci polynomials of order k (in symbols: $BF^{(k)}$) in terms of the generating functions. In Sect. 2, the relationship between $BF^{(k)}$ and $BF^{(\ell)}$ is elucidated, which reveals the relationship between $F^{(k)}$ and $F^{(\ell)}$. We also study Lucas polynomials of order k (in symbols: $L^{(k)}$) and investigate the relationship between $L^{(k)}$ and $L^{(\ell)}$. In Sect. 3, we point out that the bivariate Fibonacci and Lucas polynomials of order k are closely related to run-related distributions, which have been termed as *distributions of order k* (see Koutras 1997a; Fu and Koutras 1994; Inoue and Aki 2003) and show that the distributions possess properties analogous to the bivariate Fibonacci polynomials of order k. With the exception of Aki and Hirano (2007) study, up to now, very little work has appeared in the statistical literature on the relationships between the distributions of different orders (see Aki and Hirano 1994, 1995). We elucidate the relationships between the distributions of different orders through the bivariate Fibonacci and Lucas polynomials of order k, which will provide further insights into the distributions of order k. For example, we obtain in Proposition 3 a new relationship between the Markov geometric distributions of order k and ℓ.

2 General results

We will discuss bivariate type of Fibonacci polynomials of order k, which are the special case of multivariate Fibonacci polynomials of order k introduced by Philippou and Antzoulakos (1990) (see Philippou and Antzoulakos 1991).

2.1 Fibonacci polynomials of order k

We consider the following bivariate Fibonacci polynomials of order k:

$$F_n^{(k)}(\boldsymbol{x}) = x\, F_{n-1}^{(k)}(\boldsymbol{x}) + y \sum_{i=2}^{\min(n,k)} F_{n-i}^{(k)}(\boldsymbol{x}), \quad n \ge 2, \tag{1}$$

$$F_0^{(k)}(\boldsymbol{x}) = 0, \qquad F_1^{(k)}(\boldsymbol{x}) = 1, \tag{2}$$

where $\boldsymbol{x} = (x, y)$. As already announced, Philippou and Antzoulakos (1990) consider the multivariate Fibonacci polynomials of order k, $H_n^{(k)}(x_1, x_2, \ldots, x_k)$. We should notice that the above polynomials $F_n^{(k)}(\boldsymbol{x})$ are the special case of multivariate Fibonacci polynomials of order k, i.e. $F_n^{(k)}(\boldsymbol{x}) = H_n^{(k)}(x, y, \ldots, y)$. The generating function of the sequence $\{F_n^{(k)}(\boldsymbol{x})\}_{n\geq 0}$ will be denoted by $G^{(k)}(z; \boldsymbol{x})$, that is,

$$G^{(k)}(z; \boldsymbol{x}) = \sum_{n=0}^{\infty} F_n^{(k)}(\boldsymbol{x}) z^n.$$

Through the simple algebraic calculations on (1) and (2), the generating function $G^{(k)}(z; x)$ is given by

$$\begin{aligned} G^{(k)}(z; \boldsymbol{x}) &= \sum_{n=0}^{\infty} F_n^{(k)}(\boldsymbol{x}) z^n = \frac{z}{1 - xz - yz^2 \sum_{i=0}^{k-2} z^i} \\ &= \frac{z}{1 - xz - yz^2 P_{k-2}(z)}, \end{aligned}$$

where $P_k(z) = 1 + z + \cdots + z^k$. Details of the derivation of the above formula can be found in Philippou and Antzoulakos (1990) (see Philippou and Antzoulakos 1991).

For k and ℓ $(1 \leq \ell \leq k-1)$, we can elucidate the relationship between $BF^{(k)}$ and $BF^{(\ell)}$ through the generating functions. This is illustrated in the next theorem.

Theorem 1 *The generating function $G^{(k)}(z; \boldsymbol{x})$ can be expressed in terms of the generating function $G^{(\ell)}(z; \boldsymbol{x})$, $1 \leq \ell \leq k-1$ as*

$$G^{(k)}(z; \boldsymbol{x}) = \frac{G^{(\ell)}(z; \boldsymbol{x})}{1 - yz^{\ell} P_{k-\ell-1}(z) G^{(\ell)}(z; \boldsymbol{x})}, \tag{3}$$

which takes on the more appealing form for $\ell = k-1$

$$G^{(k)}(z; \boldsymbol{x}) = \frac{G^{(k-1)}(z; \boldsymbol{x})}{1 - yz^{k-1} G^{(k-1)}(z; \boldsymbol{x})}. \tag{4}$$

Proof The formula (3) is easily established by observing that

$$\begin{aligned} G^{(k)}(z; \boldsymbol{x}) &= \frac{z}{1 - xz - yz^2 P_{\ell-2}(z) - yz^{\ell+1} P_{k-\ell-1}(z)} \\ &= \frac{\dfrac{z}{1 - xz - yz^2 P_{\ell-2}(z)}}{1 - yz^{\ell} P_{k-\ell-1}(z) \cdot \dfrac{z}{1 - xz - yz^2 P_{\ell-2}(z)}}. \end{aligned}$$

The formula (4) is the immediate consequence of the formula (3). The proof is completed. ☐

In the case of $\boldsymbol{x} = (x, x)$, the generating function $G^{(k)}(z; \boldsymbol{x})$ reduces to the generating function of Fibonacci polynomials of order k. Therefore, we can investigate the relationship between $F^{(k)}$ and $F^{(\ell)}$ by making use of Theorem 1.

From the Eq. 4, we have

$$G^{(k)}(z;\boldsymbol{x}) = \frac{1}{yz^k} - \frac{\dfrac{1}{yz^k}}{1+yz^k G^{(k+1)}(z;\boldsymbol{x})}, \tag{5}$$

$$G^{(k+1)}(z;\boldsymbol{x}) = \frac{1}{yz^{k+1}} - \frac{\dfrac{1}{yz^{k+1}}}{1+yz^{k+1} G^{(k+2)}(z;\boldsymbol{x})}. \tag{6}$$

Substituting the Eq. 6 into the right-hand side of the Eq. 5, we obtain

$$G^{(k)}(z;\boldsymbol{x}) = \frac{1}{yz^k} - \frac{\dfrac{1}{yz^k}}{1+\dfrac{1}{z} - \dfrac{\dfrac{1}{z}}{1+yz^{k+1} G^{(k+2)}(z;\boldsymbol{x})}}.$$

Iterating the process of the substitution, we finally obtain the expression for the generating function $G^{(k)}(z;\boldsymbol{x})$ in the form of a continued fraction.

Proposition 1 *The expression for the generating function $G^{(k)}(z;\boldsymbol{x})$ in the form of a continued fraction is given by*

$$G^{(k)}(z;\boldsymbol{x}) = \frac{1}{yz^k} - \frac{\dfrac{1}{yz^k}}{1+\dfrac{1}{z} - \dfrac{\dfrac{1}{z}}{1+\dfrac{1}{z} - \dfrac{\dfrac{1}{z}}{1+\dfrac{1}{z} - \dfrac{\dfrac{1}{z}}{1+\dfrac{1}{z} - \cdots}}}}.$$

We will mention *Lucas polynomials of order k* (see Charalambides 1991), which is defined by the sequences of polynomials $\{L_n^{(k)}(x)\}_{n\geq 0}$ as follows:

$$L_n^{(k)}(x) = x\left\{n + \sum_{i=1}^{n-1} L_{n-i}^{(k)}(x)\right\}, \quad n = 2, 3, \ldots, k, \tag{7}$$

$$L_n^{(k)}(x) = x\sum_{i=1}^{k} L_{n-i}^{(k)}(x), \quad n \geq k+1, \tag{8}$$

$$L_0^{(k)}(x) = 0, \qquad L_1^{(k)}(x) = x. \tag{9}$$

The generating function of the sequence $\{L_n^{(k)}(x)\}_{n\geq 0}$ will be denoted by $L^{(k)}(z; x)$, that is,

$$L^{(k)}(z; x) = \sum_{n=0}^{\infty} L_n^{(k)}(x) z^n.$$

After some simple algebraic calculations on (7), (8) and (9), we get

$$L^{(k)}(z; x) = \sum_{n=0}^{\infty} L_n^{(k)}(x) z^n = \frac{xz \sum_{i=0}^{k-1}(i+1)\, z^i}{1 - xz \sum_{i=0}^{k-1} z^i} = \frac{xzR_{k-1}(z)}{1 - xzP_{k-1}(z)},$$

where $R_k(z) = 1 + 2z + \cdots + (k+1)z^k$.

Expanding the generating function $L^{(k)}(z; x)$ in a power series of z and picking out the coefficient of z^n, we may easily obtain the expression for Lucas polynomials of order k as

$$L_n^{(k)}(x) = \sum_{i=1}^{n} \sum_{n_1+2n_2+\cdots+kn_k=n-i} i \binom{n_1+\cdots+n_k}{n_1, n_2, \ldots, n_k} x^{n_1+n_2+\cdots+n_k+1}.$$

Note that $L_n^{(k)}(1)$ represents the Lucas numbers of order k (see Balakrishnan and Koutras 2002). The expression for $L_n^{(k)}(1)$ can be obtained as

$$L_n^{(k)}(1) = \sum_{i=1}^{n} \sum_{n_1+2n_2+\cdots+kn_k=n-i} i \binom{n_1+\cdots+n_k}{n_1, n_2, \ldots, n_k}.$$

The Lucas numbers of order k has a combinatorial interpretation (see Balakrishnan and Koutras 2002). We mention that $L_n^{(k)}(1)$ enumerates the number of different ways in which n (in total) symbols $\{S, F\}$ can be arranged on a circle in such a way that k or more consecutive S's do not appear.

We can capture the relationship between $L^{(k)}$ and $L^{(\ell)}$ for k and ℓ $(1 \leq \ell \leq k-1)$ through the generating function. The next theorem provides the details.

Theorem 2 *The generating function $L^{(k)}(z; x)$ can be expressed in terms of the generating function $L^{(\ell)}(z; x)$, $1 \leq \ell \leq k-1$ as*

$$L^{(k)}(z; x) = \frac{R_{k-1}(z) L^{(\ell)}(z; x)}{R_{\ell-1}(z) - z^{\ell} P_{k-\ell-1}(z) L^{(\ell)}(z; x)},$$

which takes on the more appealing form for $\ell = k-1$

$$L^{(k)}(z; x) = \frac{R_{k-1}(z) L^{(k-1)}(z; x)}{R_{k-2}(z) - z^{k-1} L^{(k-1)}(z; x)},$$

where $R_k(z) = 1 + 2z + \cdots + (k+1)z^k$.

Proof The proof is completed by recalling the Eqs. 3, 4 in Theorem 1 and observing that

$$L^{(k)}(z;x) = xR_{k-1}(z)\cdot\frac{z}{1-xzP_{k-1}(z)} = xR_{k-1}(z)\,G^{(k)}(z;x,x).$$

□

Working in the same fashion as we did in establishing the result presented in Proposition 1, we obtain the next proposition.

Proposition 2 *The expression for the generating function $L^{(k)}(z;x)$ in the form of a continued fraction is given by*

$$L^{(k)}(z;x) = \frac{R_{k-1}(z)}{z^k} - \cfrac{\frac{R_k(z)R_{k-1}(z)}{z^k}}{R_k(z)\left(1+\frac{1}{z}\right) - \cfrac{\frac{R_{k+1}(z)R_k(z)}{z}}{R_{k+1}(z)\left(1+\frac{1}{z}\right) - \cfrac{\frac{R_{k+2}(z)R_{k+1}(z)}{z}}{R_{k+2}(z)\left(1+\frac{1}{z}\right) - \cfrac{\frac{R_{k+3}(z)R_{k+2}(z)}{z}}{R_{k+3}(z)\left(1+\frac{1}{z}\right) - \cdots}}}}.$$

3 Applications

Let $X_0, X_1, X_2, \ldots$ be a time homogeneous $\{0, 1\}$-valued Markov chain with transition probabilities,

$$p_{ij} = P(X_t = j | X_{t-1} = i), \tag{10}$$

for $t \geq 1$, $i, j = 0, 1$ and initial probabilities

$$P(X_0 = 0) = p_0, \quad P(X_0 = 1) = p_1 \tag{11}$$

(we say success and failure for the outcomes "1" and "0", respectively). Clearly, for $p_0 = 1$, $p_1 = 0$, $p_{01} = p_{11} = p$, $p_{10} = p_{00} = q$, $p + q = 1$ in Eqs. (10), (11), the underlying sequence reduces to the i.i.d. Bernoulli trials.

3.1 Geometric distribution of order k

In the literature, there are different ways of counting runs (see Fu and Koutras 1994; Balakrishnan and Koutras 2002). The important and frequently used ways of counting runs are the "non-overlapping", the "at least" and the "overlapping" scheme, which are called the Type I, II and III counting scheme, respectively (see Balakrishnan and Koutras 2002; Inoue and Aki 2005).

We denote the waiting time for the r-th occurrence of success run of length k by $T_r^{(k,\alpha)}$ ($r \geq 1$), where the α represents the type of counting scheme employed for the success run of length k; $\alpha = \text{I}$ will indicate the non-overlapping counting, $\alpha = \text{II}$ the at least scheme and $\alpha = \text{III}$ overlapping one.

For $\alpha = \text{I, II, III}$, the probability generating function and the double generating function of $T_r^{(k,\alpha)}$ are denoted by $H_r^{(k,\alpha)}(z)$ and $H^{(k,\alpha)}(z, w)$, respectively;

$$H_r^{(k,\alpha)}(z) = E[z^{T_r^{(k,\alpha)}}] = \sum_{n=0}^{\infty} \Pr[T_r^{(k,\alpha)} = n]z^n,$$

$$H^{(k,\alpha)}(z,t) = \sum_{r=0}^{\infty} H_r^{(k,\alpha)}(z)t^r = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \Pr[T_r^{(k,\alpha)} = n]z^n t^r,$$

(convention: $H_0^{(k,\alpha)}(z) = 1$). In the special case where $r = 1$, the distribution of the waiting time $T_1^{(k,\alpha)}$ for the first occurrence of a success run of length k is referred as *Markov geometric distribution of order k* (in symbols: $MG^{(k)}$). The formula for the probability generating function $H_1^{(k,\alpha)}(z)$ of the waiting time $T_1^{(k,\alpha)}$ is given by

$$\begin{aligned} H_1^{(k,\alpha)}(z) &= \frac{[p_1 + (p_0 p_{01} - p_1 p_{00})z](p_{11}z)^{k-1}}{1 - p_{00}z - p_{01}p_{10}z^2 \sum_{i=0}^{k-2}(p_{11}z)^i} \\ &= \frac{B(z)(p_{11}z)^{k-1}}{1 - p_{00}z - p_{01}p_{10}z^2 P_{k-2}(p_{11}z)}, \end{aligned}$$

where $B(z) = p_1 + (p_0 p_{01} - p_1 p_{00})z$ (see Koutras 1997a,b; Fu and Lou 2003). Balakrishnan and Koutras (2002) established expressions for $H_r^{(k,\alpha)}(z)$ and $H^{(k,\alpha)}(z,t)$ as

$$H_r^{(k,\alpha)}(z) = H_1^{(k,\alpha)}(z)\left[A^{(k,\alpha)}(z)\right]^{r-1},$$

$$H^{(k,\alpha)}(z,t) = 1 + \frac{t\,H_1^{(k,\alpha)}(z)}{1 - t\,A^{(k,\alpha)}(z)},$$

where

$$A^{(k,\alpha)}(z) = \begin{cases} \dfrac{C(z)(p_{11}z)^{k-1}}{1 - p_{00}z - p_{01}p_{10}z^2 P_{k-2}(p_{11}z)} & \alpha = I, \\[2ex] \dfrac{p_{10}z}{1 - p_{11}z} \cdot \dfrac{(p_{01}z)(p_{11}z)^{k-1}}{1 - p_{00}z - p_{01}p_{10}z^2 P_{k-2}(p_{11}z)} & \alpha = II, \\[2ex] p_{11}z + \dfrac{(p_{01}z)(p_{10}z)(p_{11}z)^{k-1}}{1 - p_{00}z - p_{01}p_{10}z^2 P_{k-2}(p_{11}z)} & \alpha = III, \end{cases}$$

with $C(z) = p_{11}z + (p_{01}p_{10} - p_{11}p_{00})z^2$. It is noteworthy that the probability generating function $H_1^{(k)}(z)$ can be expressed in terms of bivariate Fibonacci polynomials of order k as

$$\begin{aligned} H_1^{(k,\alpha)}(z) &= B(z)(p_{11}z)^{k-2} \cdot \frac{p_{11}z}{1 - p_{00}z - p_{01}p_{10}z^2 P_{k-2}(p_{11}z)} \\ &= B(z)(p_{11}z)^{k-2} \cdot G^{(k)}\left(p_{11}z; \frac{p_{00}}{p_{11}}, \frac{p_{01}p_{10}}{p_{11}^2}\right). \end{aligned}$$

By virtue of Theorem 1, we can therefore reveal the relationship between $MG^{(k)}$ and $MG^{(\ell)}$ for k and ℓ $(1 \le \ell \le k-1)$.

Proposition 3 *The probability generating function $H_1^{(k,\alpha)}(z)$ can be expressed in terms of the probability generating function $H_1^{(\ell,\alpha)}(z)$, $1 \le \ell \le k-1$ as*

$$H_1^{(k,\alpha)}(z) = \frac{B(z)(p_{11}z)^{k-\ell}\, H_1^{(\ell,\alpha)}(z)}{B(z) - p_{01}p_{10}z^2 P_{k-\ell-1}(p_{11}z) H_1^{(\ell,\alpha)}(z)}.$$

Working in a similar fashion as we did in establishing the result presented in Proposition 1, we arrive at the following proposition.

Proposition 4 *The expression for the probability generating function $H_1^{(k)}(z)$ in the form of a continued fraction is given by*

$$H_1^{(k,\alpha)}(z) = \frac{B(z)}{p_{01}p_{10}z^2} - \cfrac{\cfrac{B(z)}{p_{01}p_{10}z^2}}{1 + \cfrac{1}{p_{11}z} - \cfrac{\cfrac{1}{p_{11}z}}{1 + \cfrac{1}{p_{11}z} - \cfrac{\cfrac{1}{p_{11}z}}{1 + \cfrac{1}{p_{11}z} - \cfrac{\cfrac{1}{p_{11}z}}{1 + \cfrac{1}{p_{11}z} - \cdots}}}}.$$

3.2 Negative binomial distribution of order k

The distributions of the three waiting times $T_r^{(k,\alpha)}$ will be called *TypeI, II, III Markov negative binomial distributions of order k* (in symbols: $MNG_r^{(k,I)}$, $MNG_r^{(k,II)}$,

$MNG_r^{(k,III)}$). It is interesting to note that the double generating function $H^{(k,\alpha)}(z,t)$ can be expressed in terms of bivariate Fibonacci polynomials of order k as

$$H^{(k,\alpha)}(z,t) = \begin{cases} 1+\dfrac{tB(z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\,\frac{p_{00}}{p_{11}},\,\frac{p_{01}p_{10}}{p_{11}^2}\right)}{1-t\,C(z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\,\frac{p_{00}}{p_{11}},\,\frac{p_{01}p_{10}}{p_{11}^2}\right)} & \alpha=I,\\[2ex] 1+\dfrac{tB(z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\,\frac{p_{00}}{p_{11}},\,\frac{p_{01}p_{10}}{p_{11}^2}\right)}{1-t\,\frac{p_{10}z}{1-p_{11}z}\cdot(p_{01}z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\,\frac{p_{00}}{p_{11}},\,\frac{p_{01}p_{10}}{p_{11}^2}\right)} & \alpha=II,\\[2ex] 1+\dfrac{tB(z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\,\frac{p_{00}}{p_{11}},\,\frac{p_{01}p_{10}}{p_{11}^2}\right)}{1-t\left(p_{11}z+(p_{01}z)(p_{10}z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\,\frac{p_{00}}{p_{11}},\,\frac{p_{01}p_{10}}{p_{11}^2}\right)\right)} & \alpha=III. \end{cases}$$

By making use of Theorem 1, we can therefore reveal the relationship between $MNG_r^{(k,\alpha)}$ and $MNG_r^{(\ell,\alpha)}$ for k and ℓ $(1\le\ell\le k-1)$, $\alpha=I,II,III$.

Proposition 5 *The double generating function $H^{(k,\alpha)}(z,t)$ can be expressed in terms of the double generating function $H^{(\ell,\alpha)}(z,t)$, $1\le\ell\le k-1$ as*

$$H^{(k,\alpha)}(z,t) = \begin{cases} 1+\dfrac{tB(z)(p_{11}z)^{k-\ell}(H^{(\ell,\alpha)}(z,t)-1)}{tB(z)+[t\,C(z)(1-(p_{11}z)^{k-\ell})-p_{01}p_{10}z^2P_{k-\ell-1}(p_{11}z)](H^{(\ell,\alpha)}(z,t)-1)} & \alpha=I,\\[2ex] 1+\dfrac{tB(z)(p_{11}z)^{k-\ell}(H^{(\ell,\alpha)}(z,t)-1)}{tB(z)-(1-t)p_{01}p_{10}z^2P_{k-\ell-1}(p_{11}z)(H^{(\ell,\alpha)}(z,t)-1)} & \alpha=II,\\[2ex] 1+\dfrac{(p_{11}z)^{k-\ell}tB(z)(H^{(\ell,\alpha)}(z,t)-1)}{tB(z)+p_{01}p_{10}z^2[t(1-(p_{11}z)^{k-\ell})-P_{k-\ell-1}(p_{11}z)(1-tp_{11}z)](H^{(\ell,\alpha)}(z,t)-1)} & \alpha=III. \end{cases}$$

3.3 Binomial distribution of order k

Let $N_n^{(k,\alpha)}$ denote the number of success runs of length k in a sequence of Markov dependent trials $X_0, X_1, \ldots, X_n$ defined by Eqs. 10, 11 under Type α(=I, II, III) enumeration scheme and

$$\Phi^{(k,\alpha)}(z,t)=\sum_{n=0}^{\infty}\sum_{x=0}^{\infty}P(N_n^{(k,\alpha)}=x)\,t^x\,z^n$$

be the double generating function of $N_n^{(k,\alpha)}$ (see Koutras and Alexandrou 1995). Koutras (1997b) showed the following relationship between the generating functions $\Phi^{(k,\alpha)}(z,t)$ and $H^{(k,\alpha)}(z,t)$

$$\Phi^{(k,\alpha)}(z,t) = \frac{1}{(1-z)}\left[1 - \frac{(1-t)H_1^{(k,\alpha)}(z)}{1-tA^{(k,\alpha)}(z)}\right]$$

(see Inoue and Aki 2007a; Inoue and Aki 2008). The distributions of the three run-enumerated variables $N_n^{(k,\alpha)}$ will be called *TypeI, II, III Markov binomial distributions of order k* (in symbols: $MB^{(k,I)}$, $MB^{(k,II)}$, $MB^{(k,III)}$). It should be noted that $\Phi^{(k,\alpha)}(z,t)$ can be expressed in terms of bivariate Fibonacci polynomials of order k as

$$\Phi^{(k,\alpha)}(z,t)$$
$$= \begin{cases} \dfrac{1}{1-z}\left[1 - \dfrac{(1-t)B(z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\, \frac{p_{00}}{p_{11}},\, \frac{p_{01}p_{10}}{p_{11}^2}\right)}{1-t\,C(z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\, \frac{p_{00}}{p_{11}},\, \frac{p_{01}p_{10}}{p_{11}^2}\right)}\right] & \alpha = I, \\ \dfrac{1}{1-z}\left[1 - \dfrac{(1-t)B(z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\, \frac{p_{00}}{p_{11}},\, \frac{p_{01}p_{10}}{p_{11}^2}\right)}{1-t\,\frac{p_{10}z}{1-p_{11}z}\cdot(p_{01}z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\, \frac{p_{00}}{p_{11}},\, \frac{p_{01}p_{10}}{p_{11}^2}\right)}\right] & \alpha = II, \\ \dfrac{1}{1-z}\left[1 - \dfrac{(1-t)B(z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\, \frac{p_{00}}{p_{11}},\, \frac{p_{01}p_{10}}{p_{11}^2}\right)}{1-t\left(p_{11}z+(p_{01}z)(p_{10}z)(p_{11}z)^{k-2}G^{(k)}\left(p_{11}z;\, \frac{p_{00}}{p_{11}},\, \frac{p_{01}p_{10}}{p_{11}^2}\right)\right)}\right] & \alpha = III. \end{cases}$$

By making use of Theorem 1, we can reveal the relationship between $MB^{(k,\alpha)}$ and $MB^{(\ell,\alpha)}$ for k and ℓ $(1 \le \ell \le k-1)$, $\alpha =$I, II, III. More specifically, we have the next results.

Proposition 6 *The double generating function $\Phi^{(k,\alpha)}(z,t)$ can be expressed in terms of the double generating function $\Phi^{(\ell,\alpha)}(z,t)$, $1 \le \ell \le k-1$ as*

$$\Phi^{(k,\alpha)}(z,t)$$
$$= \begin{cases} \frac{1}{1-z}\left[1 - \frac{(1-t)B(z)(p_{11}z)^{k-\ell}[1-(1-z)\Phi^{(\ell,\alpha)}(z,t)]}{(1-t)B(z)+[tC(z)(1-(p_{11}z)^{k-\ell})-p_{01}p_{10}z^2P_{k-\ell-1}(p_{11}z)][1-(1-z)\Phi^{(\ell,\alpha)}(z,t)]}\right] & \alpha = I, \\ \frac{1}{1-z}\left[1 - \frac{B(z)(p_{11}z)^{k-\ell}[1-(1-z)\Phi^{(\ell,\alpha)}(z,t)]}{B(z)-p_{01}p_{10}z^2P_{k-\ell-1}(p_{11}z)[1-(1-z)\Phi^{(\ell,\alpha)}(z,t)]}\right] & \alpha = II, \\ \frac{1}{1-z}\left[1 - \frac{(1-t)B(z)(p_{11}z)^{k-\ell}[1-(1-z)\Phi^{(\ell,\alpha)}(z,t)]}{(1-t)B(z)+p_{01}p_{10}z^2[t(1-(p_{11}z)^{k-\ell})-(1-tp_{11}z)P_{k-\ell-1}(p_{11}z)][1-(1-z)\Phi^{(\ell,\alpha)}(z,t)]}\right] & \alpha = III. \end{cases}$$

3.4 The longest success run

Let L_n be the longest success run in the fixed number of Markov dependent trials $X_0, X_1, \ldots, X_n$ and

$$\Psi^{(k)}(z) = \sum_{n=0}^{\infty} P\left(L_n \le k\right) z^n$$

be the generating function of L_n. The generating function $\Psi^{(k)}(z)$ is given by

$$\Psi^{(k)}(z) = \sum_{n=0}^{\infty} P\left(N_n^{(k+1,I)} = 0\right) z^n = \frac{p_0 + (p_1 p_{10} z + B(z)) P_{k-1}(p_{11} z)}{1 - p_{00} z - p_{01} p_{10} z^2 P_{k-1}(p_{11} z)}.$$

Details of the derivation of the above formula can be found in Inoue and Aki (2007b) (see Balakrishnan and Koutras 2002). It is worth mentioning here that the generating function $\Psi^{(k)}(z)$ can be expressed in terms of bivariate Fibonacci polynomials of order k as

$$\Psi^{(k)}(z) = \frac{p_0 + (p_1 p_{10} z + B(z)) P_{k-1}(p_{11} z)}{p_{11} z} \cdot G^{(k+1)}\left(p_{11} z; \frac{p_{00}}{p_{11}}, \frac{p_{01} p_{10}}{p_{11}^2}\right).$$

By making use of Theorem 1, we have the following results.

Proposition 7 *The generating function $\Psi^{(k)}(z)$ can be expressed in terms of the generating function $\Psi^{(\ell)}(z)$, $1 \le \ell \le k-1$ as*

$$\Psi^{(k)}(z) = \frac{[p_0 + (p_1 p_{10} z + B(z)) P_{k-1}(p_{11} z)] \Psi^{(\ell)}(z)}{p_0 + (p_1 p_{10} z + B(z)) P_{\ell-1}(p_{11} z) - p_{01} p_{10} z^2 (p_{11} z)^{\ell} P_{k-\ell-1}(p_{11} z) \Psi^{(\ell)}(z)}.$$

Next, we consider a fixed number of i.i.d. Bernoulli trials $X_1, X_2, \ldots, X_n$ ($p_0 = 1$, $p_1 = 0$, $p_{01} = p_{11} = p$, $p_{10} = p_{00} = q$, $p + q = 1$ in Eqs. 10, 11), where the n Bernoulli trials are arranged on a circle (circular sequence). Here, we assume that the outcomes of the n Bernoulli trials are bent into a circle so that additional success runs may be formed by combining successes at the beginning and end of the sequence. Let L_n^c be the length of the longest run of successes in a circular sequence of n independent Bernoulli trials and

$$\Psi_c^{(k)}(z) = \sum_{n=0}^{\infty} P\left(L_n^c \le k\right) z^n$$

be the generating function of L_n. The generating function $\Psi_c^{(k)}(z)$ is given by

$$\Psi_c^{(k)}(z) = \frac{qz \sum_{i=0}^{k} (i+1)(pz)^i}{1 - qz \sum_{i=0}^{k} (pz)^i} + \frac{1 - (pz)^{k+1}}{1 - pz} = \frac{qz R_k(pz)}{1 - qz P_k(pz)} + P_k(pz).$$

Details of the derivation of the above formula can be found in Charalambides (1991) (see Makri and Philippou 1994; Koutras et al. 1994, 1995; Charalambides 1994; Inoue and Aki 2005). It needs to be mentioned here that the generating function $\Psi_c^{(k)}(z)$ can be expressed in terms of Lucas or bivariate Fibonacci polynomials of order k as

$$\Psi_c^{(k)}(z) = L^{(k+1)}\left(pz; \frac{q}{p}\right) + P_k(pz)$$
$$= \frac{qz}{pz} R_k(pz) \cdot G^{(k+1)}\left(pz; \frac{q}{p}, \frac{q}{p}\right) + P_k(pz).$$

By making use of Theorems 1 and 2, we have the following proposition.

Proposition 8 *The generating function $\Psi_c^{(k)}(z)$ can be expressed in terms of the generating function $\Psi_c^{(\ell)}(z)$, $1 \le \ell \le k-1$ as*

$$\Psi_c^{(k)}(z) = P_k(pz) + \frac{R_k(pz)\left[\Psi_c^{(\ell)}(z) - P_\ell(pz)\right]}{R_\ell(pz) - (pz)^{\ell+1} P_{k-\ell-1}(pz)\left[\Psi_c^{(\ell)}(z) - P_\ell(pz)\right]}.$$

3.5 Waiting time for success run of length k in a marked point process

Let $((T_n), (Y_n))$ be the following marked point process in continuous time with mark space $\{0, 1\}$, where $\{T_n\}_{n=0}^{\infty}$ is a sequence of increasing random timepoints such that $T_0 = 0$ and $T_1, T_2 - T_1, \ldots, T_n - T_{n-1}, \ldots$ follow independently a common distribution whose moment generating function (mgf) $M(t)$, i.e., $\{Tn\}$ is a 0-delayed renewal process (see Jacobsen 2006). Here, $\{Y_n\}_{n=0}^{\infty}$ is a sequence of $\{0, 1\}$-valued random variables. Further, we assume $\{T_n\}$ and $\{Y_n\}$ are independent. The marginal distribution of $\{Y_n\}$ is the same as that of $\{X_n\}$. That is, $\{Y_n\}$ is the time homogeneous Markov chain. Let N be the number of trials for obtaining the first success run of length k in $\{Y_n\}$ and let $\tau_k = T_N$ be the waiting time for the first success run in the mark sequence. Then we obtain the mgf $\nu_k(t)$ of τ_k as follows:

$$\begin{aligned}
\nu_k(t) &= E[\, e^{t\,\tau_k}\,] \\
&= E[\, E[\, e^{t\,T_N} \mid \{Y_n\}\,]\,] \\
&= \sum_{n=0}^{\infty} P(N=n) E[\, e^{t\,T_N} \mid N=n\,] \\
&= \sum_{n=0}^{\infty} P(N=n) E[\, e^{t\,T_n}\,] \\
&= \sum_{n=0}^{\infty} P(N=n) E[\, e^{t\,(T_1+(T_2-T_1)+\cdots+(T_n-T_{n-1}))}\,] \\
&= \sum_{n=0}^{\infty} P(N=n)(M(t))^n = H_1^{(k,\alpha)}(M(t)).
\end{aligned}$$

Noting that

$$H_1^{(k,\alpha)}(M(t)) = B(M(t))(p_{11}M(t))^{k-2}G^{(k)}\left(p_{11}M(t); \frac{p_{00}}{p_{11}}, \frac{p_{01}p_{10}}{p_{11}^2}\right)$$

and Theorem 1, we obtain the following relation between the mgf's $\nu_k(t)$ and $\nu_\ell(t)$.

Proposition 9 *The moment generating function $\nu_k(t)$ can be expressed in terms of the generating function $\nu_\ell(t)$, $1 \le \ell \le k-1$ as*

$$\nu_k(t) = (p_{11}M(t))^{k-\ell} \frac{\nu_\ell(t)}{1 - \frac{p_{01}p_{10}}{p_{11}^2} \frac{(p_{11}M(t))^2}{B(M(t))} P_{k-\ell-1}(p_{11}M(t)\nu_\ell(t))}.$$

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