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The geometry of the Wilks's Λ random field

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Abstract The statistical problem addressed in this paper is to approximate the P value of the maximum of a smooth random field of Wilks's Λ statistics. So far results are only available for the usual univariate statistics (Z, t , χ^2, F) and a few multivariate statistics (Hotelling's T^2, maximum canonical correlation, Roy's maximum root). We derive results for any differentiable scalar function of two independent Wishart random fields, such as Wilks's Λ random field. We apply our results to a problem in brain shape analysis.

Keywords Multivariate random fields · Excursion sets · Euler characteristic · Derivatives of matrix functions

1 Introduction

The motivation for this work came from practical applications in brain imaging. Changes in brain shape can be inferred from the 3D non-linear deformations required to warp a patient's MRI scan to an atlas standard. Using these vector deformations as a dependent variable, we can use a multivariate multiple regression model to detect regions of the brain whose shape is correlated with external regressors, such as disease

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state. The statistic of choice is Wilks's Λ, the likelihood ratio statistic (Anderson 1984), evaluated at each voxel in the brain. The challenging statistical problem is how to assess the P value of local maxima of a smooth 3D random field (RF) of Wilks's Λ statistics.

The problem of assessing the P value of local maxima was studied in the seminal works of Adler and Hasofer (1976) and Hasofer (1978) for the case of Gaussian RFs. The key idea consists on approximating the P value that the local maximum exceeds a high threshold value via the geometry of the excursion set, the set of points where the RF exceeds the threshold value. For high thresholds, the excursion sets consists of isolated regions, each containing a local maxima. The Euler characteristic (EC) of the excursion sets counts the number of such connected components. For high thresholds near the maximum of the RF, the EC takes the value one if the maximum is above the threshold, and zero otherwise. Thus, for high thresholds, the expected EC approximates the P value of the maximum of the RF.

During the last decade this approach has became an important tool in different areas of applied probability. This is due to the many applications that have emerged principally in medical imaging and astrophysics (Gott et al. 1990; Beaky et al. 1992; Vogeley et al. 1994; Worsley 1995a,b; Worsley et al. 1998; Cao and Worsley 2001). So far, results are available for some fields of standard univariate statistics such as Z, t, χ^2 and F RFs (Worsley 1994). Some other useful extensions has been obtained for Hotelling's T^2 (Cao and Worsley 1999a), cross-correlation (Cao and Worsley 1999b), maximum canonical correlation (Taylor and Worsley 2008) and Roy's maximum root RFs (Taylor and Worsley 2008). However, the extension of such results to other RFs of classical multivariate statistics has not so far been fully studied (Worsley et al. 2004). RFs of multivariate statistics appear when dealing with multivariate RF data and multiple contrasts, where the classical example is the Wilks's Λ statistic mentioned above.

The purpose of this work is to extend Adler's results to a wide class of RFs of multivariate statistics, which the Wilks's Λ RF belongs to. We build these random fields from Gaussian RFs in the same way as we build the statistics themselves. First let $Z = Z(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^N$, be a smooth isotropic Gaussian RF with zero mean, unit variance, and with

$$\mathrm{Var}(\dot{Z}) = \mathbf{I}_N, \tag{1}$$

where dot represents derivative with respect to $\mathbf{t}$, and $\mathbf{I}_N$ is the $N \times N$ identity matrix. Note that if we have another isotropic Gaussian RF Z^* with $\mathrm{Var}(\dot{Z}^*) = v^2\mathbf{I}_N$ and $v \neq 1$, then we can always re-scale the parameter $\mathbf{t}$ by dividing by v to define $Z(\mathbf{t}) = Z^*(\mathbf{t}/v)$ so that (1) is satisfied.

Now suppose that $\mathbf{Z}(\mathbf{t})$ is an $\nu \times m$ matrix independent Gaussian RFs each with the same distribution as $Z(\mathbf{t})$. Then the Wishart RF with ν degrees of freedom is defined as

$$W(\mathbf{t}) = \mathbf{Z}(\mathbf{t})'\mathbf{Z}(\mathbf{t}) \sim \mathrm{Wishart}_m(\mathbf{I}_m, \nu) \tag{2}$$

at each fixed $\mathbf{t}$ (Cao and Worsley 1999b). Let $\mathbf{U}(\mathbf{t})$ and $\mathbf{V}(\mathbf{t})$ be independent Wishart RFs with ν and η degrees of freedom, respectively, whose component Gaussian RFs all have the same distribution as $Z(\mathbf{t})$ above. We shall be concerned with the class of RFs $\Lambda(\mathbf{t})$ defined as

$$\Lambda(\mathbf{t}) = \Lambda\left(\mathbf{U}(\mathbf{t}), \mathbf{V}(\mathbf{t})\right), \tag{3}$$

where Λ is any differentiable function of two matrix arguments. Our aim is to provide approximate values for the expected EC of the excursion sets of $\Lambda(\mathbf{t})$.

The plan of the paper is the following. In Sect. 2 we shall state the notation as well as some preliminaries facts about RF theory and matrix differential theory. In Sect. 3 we obtain expressions for the derivatives up to second order of the random field $\Lambda(\mathbf{t})$, while, in Sects. 4 and 5 we get the EC densities of $\Lambda(\mathbf{t})$. Finally, we applied these results to detecting brain damage of a group of 17 non-missile trauma patients compared to a group of 19 age- and sex-matched controls in the Sect. 6.

2 Preliminaries

In this section we shall settle the notation to be used throughout the paper, as well as some basic facts about RFs and matrix differentiation.

2.1 Basic notations

We shall use the notations $\mathbf{I}_m$ and $\mathbf{K}_m$ to represent the identity matrix and the permutation matrix of order m, respectively. By $\text{Normal}_{m\times n}(\mu, \mathbf{\Sigma})$ we denote the multivariate normal distribution of a matrix $\mathbf{X} \in \mathbb{R}^{m\times n}$ with mean μ and covariance matrix $\mathbf{\Sigma}$, and write $\mathbf{\Sigma} = \mathbf{M}(\mathbf{I}_{mn})$ when $\text{cov}(\mathbf{X}_{ij}, \mathbf{X}_{kl}) = \varepsilon(i, j, k, l) - \delta_{ij}\delta_{kl}$ with $\varepsilon(i, j, k, l)$ symmetric in its arguments and with $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Let $\text{Wishart}_m(\nu, \mathbf{\Sigma})$ be represent the Wishart distribution of a $m \times m$ matrix with expectation $\nu\mathbf{\Sigma}$ and degrees of freedom ν, and use $\text{Beta}_m\left(\frac{1}{2}\nu, \frac{1}{2}\eta\right)$ for the multivariate beta distribution of a $m\times m$ matrix with degrees of freedom ν and η. For any vector we shall use subscripts j and $|j$ to denote the jth component and the first j components, respectively. In a similar way, for any matrix, the subscripts $|j$ and $|jj$ shall denote the submatrix of the first j rows and the first j rows and columns, respectively. For any scalar b, let $b^+ = b$ if $b > 0$ and zero otherwise. For any symmetric matrix $\mathbf{B}$, let $\mathbf{B}^- = \mathbf{B}$ if $\mathbf{B}$ is negative definite and zero otherwise and let $\text{detr}_j(\mathbf{B})$ be the sum of the determinants of all $j \times j$ principal minors of $\mathbf{B}$ and define $\text{detr}_0(\mathbf{B}) = 1$. We shall denote by $\mathbf{B}^{1/2}$ the square root of the matrix $\mathbf{B}$, which is defined by $\mathbf{B}^{1/2}(\mathbf{B}^{1/2})' = \mathbf{B}$.

2.2 Random field theory

Let $Y = Y(\mathbf{t}), \mathbf{t} \in S \subset \mathbb{R}^N$ be a real valued isotropic RF and let S be a compact subset of $\mathbb{R}^N$ with a twice differentiable boundary ∂S. Denote the first derivative of Y by $\dot{Y}$ and the $N \times N$ matrix of second-order partial derivatives of Y by $\ddot{Y}$. For $j > 0$ define the j-dimensional *Euler Characteristic* (*EC*) *density* as

$$\rho_j(y) = \mathbb{E}\left(\dot{Y}_j^+ \det(-\ddot{Y}_{\mid j-1,j-1}) \mid \dot{Y}_{|j-1} = 0, Y = y\right) \theta_{|j-1}(0, y) \tag{4}$$

where $\mathbb{E}(\cdot)$ denotes mathematical expectation and $\theta_{|j-1}(\cdot,\cdot)$ is the joint probability density function of $\dot{Y}_{|j-1}$ and Y. For $j = 0$ define $\rho_0(y) = \mathbb{P}(Y \geq y)$. Let $a_j = 2\pi^{j/2}/\Gamma(j/2)$ be the surface area of a unit $(j-1)$-sphere in $\mathbb{R}^N$. Define $\mu_N(S)$ to be the Lebesgue measure of S and $\mu_j(S)$, $j = 0, \ldots, N-1$ to be proportional to the j-dimensional *Minkowski functional* or *intrinsic volume*

$$\mu_j(S) = \frac{1}{a_{N-j}} \int_{\partial S} \operatorname{detr}_{N-1-j}(C)\, \mathrm{d}t$$

with C denoting the inside curvature matrix of ∂S at the point t. The expected EC χ of the excursion set of Y above the threshold y inside S $(A_y(Y, S))$ is given by

$$\mathbb{E}(\chi(A_y(Y, S))) = \sum_{j=0}^{N} \mu_j(S)\, \rho_j(y).$$

Then, the P value of the maximum of Z inside S is well approximated by (see Adler 2000; Taylor et al. 2005)

$$\mathbb{P}\left(\max_{\mathbf{t}\in S} Y(\mathbf{t}) \geq y\right) \approx \mathbb{E}(\chi(A_y(Y, S))). \tag{5}$$

In this paper Y is a function of independent RFs each with the same distribution as the standard Gaussian RF Z, and the EC densities $\rho_j(y)$ are calculated assuming unit variance of the derivative (1). If instead

$$\operatorname{Var}(\dot{Z}) = v^2 \mathbf{I}_m \tag{6}$$

then $\mathbf{t}$ can be re-scaled by dividing $\mathbf{t}$ by v to satisfy (1). This is equivalent to multiplying $\mu_j(S)$ by v^j, to give

$$\mathbb{P}\left(\max_{\mathbf{t}\in S} Y(\mathbf{t}) \geq y\right) \approx \mathbb{E}(\chi(A_y(Y, S))) = \sum_{j=0}^{N} \mu_j(S) v^j \rho_j(y). \tag{7}$$

For this reason all the subsequent theory for the EC densities will assume $v = 1$.

2.3 Matrix functions

Throughout the next section we shall need the following result, which follows easily from the properties of the operators $\operatorname{vec}(\cdot)$ (vectorization) and $\otimes$ (Kronecker tensor product). If the $p \times q$ matrix $\mathbf{X}$ is such that $\mathbf{X} \sim \text{Normal}_{p\times q}(\mathbf{0}, \mathbf{I}_{pq})$ then for any $s \times p$ matrix $\mathbf{A}$ and $q \times r$ matrix $\mathbf{B}$,

$$\mathbf{AXB} \sim \text{Normal}_{s\times r}\left(\mathbf{0}, \left(\mathbf{B}'\mathbf{B}\right)\otimes\left(\mathbf{AA}'\right)\right). \tag{8}$$

We shall also make intensive use the differential theory of matrix argument functions proposed in Magnus and Neudecker (1988). It is based on expressing the derivative of a matrix-valued function of a matrix argument $\mathbf{F}(\mathbf{X}) : \mathbb{R}^{p\times q} \longrightarrow \mathbb{R}^{m\times n}$ by the $mn \times pq$ matrix

$$D_{\mathbf{X}}\mathbf{F}(\mathbf{X}) = \frac{d\,(\text{vec}\,(\mathbf{F}(\mathbf{X})))}{d\,(\,\text{vec}\,(\mathbf{X}))}.$$

3 Representations of derivatives

In this section we obtain expressions for the first and second derivatives of the random field $\Lambda\,(\mathbf{t})$ with respect to $\mathbf{t}$, denoted by $\dot{\Lambda}$ and $\ddot{\Lambda}$ respectively. These expressions generalize, for instance, the formulas for the derivatives of the F field obtained in Worsley (1994).

Theorem 1 *The two first derivatives of the random field* Λ *can be expressed as*

$$\begin{aligned}
\dot{\Lambda} &= \mathbf{A}_1\text{vec}(\mathbf{M}_0(\mathbf{W},\mathbf{B})),\\
\ddot{\Lambda} &= \text{tr}(\mathbf{M}_1(\mathbf{W},\mathbf{B}))\mathbf{I}_N + \text{tr}(\mathbf{M}_2(\mathbf{W},\mathbf{B}))^{1/2}\mathbf{H} + \text{tr}(\mathbf{M}_3(\mathbf{W},\mathbf{B}))\mathbf{P} + \text{tr}(\mathbf{M}_4(\,\mathbf{W},\mathbf{B}))\mathbf{Q}\\
&\quad + \mathbf{A}_1\mathbf{M}_5(\mathbf{W},\mathbf{B})\mathbf{A}_1' + \mathbf{A}_1\mathbf{M}_6(\mathbf{W},\mathbf{B})\mathbf{A}_2' + \mathbf{A}_2\mathbf{M}_7(\mathbf{W},\mathbf{B})\mathbf{A}_1' + \mathbf{A}_2\mathbf{M}_8(\mathbf{W},\mathbf{B})\mathbf{A}_2',
\end{aligned}$$

where equality means equality in law,

$$\begin{aligned}
\mathbf{A}_1, \mathbf{A}_2 &\sim \text{Normal}_{N\times m^2}(\mathbf{0}, \mathbf{I}_{Nm^2}), \quad \mathbf{H} \sim \text{Normal}_{N\times N}\left(0, M\left(\mathbf{I}_{N^2}\right)\right),\\
\mathbf{P} &\sim \text{Wishart}_N\left(\nu - m, \mathbf{I}_N\right), \quad \mathbf{Q} \sim \text{Wishart}_N(\eta - m, \mathbf{I}_N),\\
\mathbf{W} &\sim \text{Wishart}_m(\nu+\eta, \mathbf{I}_m), \quad \mathbf{B} \sim \text{Beta}_m\left(\tfrac{1}{2}\nu, \tfrac{1}{2}\eta\right),
\end{aligned}$$

independently, and $\mathbf{M}_i(\mathbf{W},\mathbf{B})$, $i = 0,\ldots,8$ *are matrices that depends on* $\mathbf{B}$ *and* $\mathbf{W}$.

Proof For each $j = 1,\ldots,N$ we have

$$\dot{\Lambda}_j = D_{\mathbf{U}}\Lambda\,(\mathbf{U},\mathbf{V})\,D_{t_j}\mathbf{U} + D_{\mathbf{V}}\Lambda\,(\mathbf{U},\mathbf{V})\,D_{t_j}\,\mathbf{V}. \tag{9}$$

According to Neudecker (1969) one is able to find two matrix functions $\mathbf{G}^1(\mathbf{U},\mathbf{V})$, $\mathbf{G}^2(\mathbf{U},\mathbf{V})$,

$$\mathbf{G}^1, \mathbf{G}^2 : \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times m} \longrightarrow \mathbb{R}^{m\times m}$$

such that

$$\begin{aligned}
D_{\mathbf{U}}\Lambda\,(\mathbf{U},\mathbf{V}) &= \text{vec}(\mathbf{G}^1)',\\
D_{\mathbf{V}}\Lambda\,(\mathbf{U},\mathbf{V}) &= \text{vec}(\mathbf{G}^2)'.
\end{aligned}$$

From Lemma 3.2 in Cao and Worsley (1999a) we make the substitutions

$$D_{t_j}\mathbf{U} = \text{vec}\left(\mathbf{U}^{1/2}\mathbf{A}_j^{\mathbf{U}} + (\mathbf{U}^{1/2}\mathbf{A}_j^{\mathbf{U}})'\right), \quad D_{t_j}\mathbf{V} = \text{vec}\left(\mathbf{V}^{1/2}\mathbf{A}_j^{\mathbf{V}} + (\mathbf{V}^{1/2}\mathbf{A}_j^{\mathbf{V}})'\right), \tag{10}$$

where

$$\left(\mathbf{A}_1^{\mathbf{U}}, \ldots, \mathbf{A}_N^{\mathbf{U}}\right), \left(\mathbf{A}_1^{\mathbf{V}}, \ldots, \mathbf{A}_N^{\mathbf{V}}\right) \sim \text{Normal}_{m\times mN}\left(\mathbf{0}, \mathbf{I}_{m^2N}\right),$$

independently. Thus,

$$\begin{aligned}
\dot{\Lambda}_j &= \text{vec}(\mathbf{G}^1)'\text{vec}\left(\mathbf{U}^{1/2}\mathbf{A}_j^{\mathbf{U}} + (\mathbf{U}^{1/2}\mathbf{A}_j^{\mathbf{U}})'\right) + \text{vec}(\mathbf{G}^2)'\text{vec}\left(\mathbf{V}^{1/2}\mathbf{A}_j^{\mathbf{V}} + (\mathbf{V}^{1/2}\mathbf{A})'\right) \\
&= \text{tr}\left(((\mathbf{G}^1)' + \mathbf{G}^1)\mathbf{U}^{1/2}\mathbf{A}_j^{\mathbf{U}}\right) + \text{tr}\left(((\mathbf{G}^2)' + \mathbf{G}^2)\mathbf{V}^{1/2}\mathbf{A}_j^{\mathbf{V}}\right) \\
&= \text{tr}\left(\mathbf{G}\mathbf{U}^{1/2}\mathbf{A}_j^{\mathbf{U}} + \mathbf{F}\mathbf{V}^{1/2}\mathbf{A}_j^{\mathbf{V}}\right) \\
&= \text{tr}\left(\mathbf{C}_1^{-1}\mathbf{A}_j^1\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{G} &= (\mathbf{G}^1)' + \mathbf{G}^1, \\
\mathbf{F} &= (\mathbf{G}^2)' + \mathbf{G}^2, \\
\mathbf{C}_1 &= (\mathbf{GUG} + \mathbf{FVF})^{-1/2}, \\
\mathbf{A}_j^1 &= \mathbf{C}_1(\mathbf{G}\mathbf{U}^{1/2}\mathbf{A}_j^{\mathbf{U}} + \mathbf{F}\mathbf{V}^{1/2}\mathbf{A}_j^{\mathbf{V}}),
\end{aligned}$$

$j = 1, \ldots, N$. It is easily verified from (8) that, conditional on $\mathbf{U}, \mathbf{V}$, $\mathbf{A}_j^1 \sim \text{Normal}_{m\times m}\left(\mathbf{0}, \mathbf{I}_{m^2}\right)$, which yields

$$\dot{\Lambda} = \mathbf{A}_1\text{vec}(\mathbf{C}_1^{-1}),$$

where

$$\mathbf{A}_1 = \left(\text{vec}(\mathbf{A}_1^1), \ldots, \text{vec}(\mathbf{A}_N^1)\right)' \sim \text{Normal}_{N\times m^2}(\mathbf{0}, \mathbf{I}_{m^2N}).$$

Using the fact that

$$\mathbf{W} = \mathbf{U} + \mathbf{V} \sim \text{Wishart}_m(\nu + \eta, \mathbf{I}_m),$$

$$\mathbf{B} = (\mathbf{U} + \mathbf{V})^{-1/2}\mathbf{U}(\mathbf{U} + \mathbf{V})^{-1/2} \sim \text{Beta}_m\left(\tfrac{1}{2}\nu, \tfrac{1}{2}\eta\right),$$

independently (Olkin and Rubin 1962), we obtain the first identity by rewriting the matrix $\mathbf{C}_1^{-1}$ in terms of $\mathbf{W}$ and $\mathbf{B}$ by means of the substitutions

$$\mathbf{U} = \mathbf{W}^{1/2}\mathbf{B}\mathbf{W}^{1/2}, \quad \mathbf{V} = \mathbf{W}^{1/2}(\mathbf{I}_m - \mathbf{B})\mathbf{W}^{1/2}. \tag{11}$$

For the second identity we obtain from (9)

$$\begin{aligned}\ddot{\Lambda}_{kj} &= (D_{t_j}\mathbf{U})' D^2_{\mathbf{UU}}\Lambda\,(\mathbf{U},\mathbf{V})\, D_{t_k}\mathbf{U} + (D_{t_j}\mathbf{U})' D^2_{\mathbf{VU}}\Lambda\,(\mathbf{U},\mathbf{V})\, D_{t_k}\mathbf{V} \\ &\quad + (D_{t_j}\mathbf{V})' D^2_{\mathbf{UV}}\Lambda\,(\mathbf{U},\mathbf{V})\, D_{t_k}\mathbf{U} + (D_{t_j}\mathbf{V})' D^2_{\mathbf{VV}}\Lambda\,(\mathbf{U},\mathbf{V})\, D_{t_k}\mathbf{V} \\ &\quad + (D_{\mathbf{U}}\Lambda\,(\mathbf{U},\mathbf{V}))\, D^2_{t_k t_j}\mathbf{U} + (D_{\mathbf{V}}\Lambda\,(\mathbf{U},\mathbf{V}))\, D^2_{t_k t_j}\mathbf{V},\end{aligned} \tag{12}$$

From Lemma 3.2 in Cao and Worsley (1999a) we make the substitutions

$$\begin{aligned}D^2_{t_k t_j}\mathbf{U} &= \text{vec}\left(\widetilde{\mathbf{P}}_{jk} + \widetilde{\mathbf{P}}_{kj} + \mathbf{U}^{1/2}\mathbf{H}^{\mathbf{U}}_{kj} + (\mathbf{U}^{1/2}\mathbf{H}^{\mathbf{U}}_{kj})' + (\mathbf{A}^{\mathbf{U}}_j)'\mathbf{A}^{\mathbf{U}}_k + (\mathbf{A}^{\mathbf{U}}_k)'\mathbf{A}^{\mathbf{U}}_j - 2\delta_{kj}\mathbf{U}\right), \\ D^2_{t_k t_j}\mathbf{V} &= \text{vec}\left(\widetilde{\mathbf{Q}}_{jk} + \widetilde{\mathbf{Q}}_{kj} + \mathbf{V}^{1/2}\mathbf{H}^{\mathbf{V}}_{kj} + (\mathbf{V}^{1/2}\mathbf{H}^{\mathbf{V}}_{kj})' + (\mathbf{A}^{\mathbf{V}}_j)'\mathbf{A}^{\mathbf{V}}_k + (\mathbf{A}^{\mathbf{V}}_k)'\mathbf{A}^{\mathbf{V}}_j - 2\delta_{kj} V\right),\end{aligned} \tag{13}$$

where

$$\begin{aligned}\widetilde{\mathbf{P}} &= (\widetilde{\mathbf{P}}_{jk}) \sim \text{Wishart}_{mN}(\nu - m, \mathbf{I}_{mN}), \\ \widetilde{\mathbf{Q}} &= (\widetilde{\mathbf{Q}}_{jk}) \sim \text{Wishart}_{mN}(\eta - m, \mathbf{I}_{mN}), \\ \mathbf{H}^{\mathbf{U}}_{kj}, \mathbf{H}^{\mathbf{V}}_{kj} &\sim \text{Normal}_{m\times m}\left(\mathbf{0}, M\left(\mathbf{I}_{m^2}\right)\right).\end{aligned}$$

The expressions (10) and (13) yield

$$\begin{aligned}(D_{t_j}\mathbf{U})'(D^2_{\mathbf{UU}}\Lambda)D_{t_k}\,\mathbf{U} &= \text{vec}(\mathbf{U}^{1/2}\mathbf{A}^{\mathbf{U}}_j)'(\mathbf{I}_{m^2} + \mathbf{K}'_m)(D^2_{\mathbf{UU}}\Lambda)\left(\mathbf{I}_{m^2} + \mathbf{K}_m\right)\text{vec}(\mathbf{U}^{1/2}\mathbf{A}^{\mathbf{U}}_k), \\ (D_{t_j}\mathbf{U})'(D^2_{\mathbf{VU}}\Lambda)D_{t_k}\,\mathbf{V} &= \text{vec}(\mathbf{U}^{1/2}\mathbf{A}^{\mathbf{U}}_j)'(\mathbf{I}_{m^2} + \mathbf{K}'_m)(D^2_{\mathbf{VU}}\Lambda)\left(\mathbf{I}_{m^2} + \mathbf{K}_m\right)\text{vec}(\mathbf{V}^{1/2}\mathbf{A}^{\mathbf{V}}_k), \\ (D_{t_j}\mathbf{V})'(D^2_{\mathbf{UV}}\Lambda)D_{t_k}\,\mathbf{U} &= \text{vec}(\mathbf{V}^{1/2}\mathbf{A}^{\mathbf{V}}_j)'(\mathbf{I}_{m^2} + \mathbf{K}'_m)(D^2_{\mathbf{UV}}\Lambda)\left(\mathbf{I}_{m^2} + \mathbf{K}_m\right)\text{vec}(\mathbf{U}^{1/2}\mathbf{A}^{\mathbf{U}}_k), \\ (D_{t_j}\mathbf{V})'(D^2_{\mathbf{VV}}\Lambda)D_{t_k}\,\mathbf{V} &= \text{vec}(\mathbf{V}^{1/2}\mathbf{A}^{\mathbf{V}}_j)'(\mathbf{I}_{m^2} + \mathbf{K}'_m)(D^2_{\mathbf{VV}}\Lambda)\left(\mathbf{I}_{m^2} + \mathbf{K}_m\right)\text{vec}(\mathbf{V}^{1/2}\mathbf{A}^{\mathbf{V}}_k),\end{aligned} \tag{14}$$

and

$$\begin{aligned}D_{\mathbf{U}}\Lambda(D^2_{t_k t_j}\mathbf{U}) &= -2\delta_{kj}\text{vec}(\mathbf{G}^1)'\text{vec}\,(\mathbf{U}) + \text{vec}(\mathbf{G})'\text{vec}(\widetilde{\mathbf{P}}_{jk}) + \text{vec}(\mathbf{G})'\text{vec}(\mathbf{U}^{1/2}\mathbf{H}^{\mathbf{U}}_{jk}) \\ &\quad + \text{vec}(\mathbf{G})'\text{vec}((\mathbf{A}^{\mathbf{U}}_j)'\mathbf{A}^{\mathbf{U}}_k), \\ D_{\mathbf{V}}\Lambda(D^2_{t_k t_j}\mathbf{V}) &= -2\delta_{kj}\text{vec}(\mathbf{G}^2)'\text{vec}\,(\mathbf{V}) + \text{vec}(\mathbf{F})'\text{vec}(\widetilde{\mathbf{Q}}_{jk}) + \text{vec}(\mathbf{F})'\text{vec}(\mathbf{V}^{1/2}\mathbf{H}^{\mathbf{V}}_{jk}) \\ &\quad + \text{vec}(\mathbf{F})'\text{vec}((\mathbf{A}^{\mathbf{V}}_j)'\mathbf{A}^{\mathbf{V}}_k).\end{aligned} \tag{15}$$

We can find a linear combination of $\mathbf{A}^U_j$ and $\mathbf{A}^{\mathbf{V}}_j$, $j = 1, \ldots, N$ that is independent of $\mathbf{A}^1_j$, namely,

$$\mathbf{A}^2_j = \mathbf{C}_2\left(\mathbf{G}^{-1}\mathbf{U}^{-1/2}\mathbf{A}^{\mathbf{U}}_j - \mathbf{F}^{-1}\mathbf{V}^{-1/2}\mathbf{A}^{\mathbf{V}}_j\right), \mathbf{C}_2 = \left((\mathbf{GUG})^{-1} + (\mathbf{FVF})^{-1}\right)^{-1/2}.$$

This implies that the matrices $\mathbf{A}_j^{\mathbf{U}}, \mathbf{A}_j^{\mathbf{V}}$ can be written in terms of $\mathbf{A}_j^1$ and $\mathbf{A}_j^2$ by suitable expressions (unequivocally determined), which when substituted in each of the right members of (14) and (15) give

$$\begin{aligned}\ddot{\Lambda}_{kj} &= -2\delta_{kj}\mathrm{tr}(\mathbf{G}^1\mathbf{U}+\mathbf{G}^2\mathbf{V}) + \mathrm{tr}(\mathbf{C}_1^{-1}\mathbf{H}_{kj}) + \mathrm{tr}(\mathbf{G}\widetilde{\mathbf{P}}_{jk}) + \mathrm{tr}(\mathbf{F}\widetilde{\mathbf{Q}}_{jk}) \\ &\quad +(\mathrm{vec}(\mathbf{A}_j^1))'\mathbf{S}_1\mathrm{vec}(\mathbf{A}_k^1) + (\mathrm{vec}(\mathbf{A}_j^1))'\mathbf{S}_2\mathrm{vec}(\mathbf{A}_k^2) \\ &\quad +(\mathrm{vec}(\mathbf{A}_j^2))'\mathbf{S}_3\mathrm{vec}(\mathbf{A}_k^1) + (\mathrm{vec}(\mathbf{A}_j^2))'\mathbf{S}_4\mathrm{vec}(\mathbf{A}_k^2), \end{aligned} \tag{16}$$

for certain matrices $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4$ that only depends on $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{H}_{kj}$ given by

$$\mathbf{H}_{kj} = \mathbf{C}_1\left(\mathbf{G}\mathbf{U}^{1/2}\mathbf{H}_{jk}^{\mathbf{U}} + \mathbf{F}\mathbf{V}^{1/2}\mathbf{H}_{kj}^{\mathbf{V}}\right).$$

It can be easily shown that the matrix $(\mathrm{tr}(\mathbf{C}_1^{-1}\mathbf{H}_{kj}))_{kj}$ can be rewritten as

$$\left(\mathrm{tr}(\mathbf{C}_1^{-1}\mathbf{H}_{kj})\right)_{kj} = \mathrm{tr}(\mathbf{C}_1^{-2})^{1/2}\mathbf{H},$$

with

$$\mathbf{H} = \mathrm{tr}(\mathbf{C}_1^{-2})^{-1/2}\left(\mathbf{I}_N \otimes \left(\mathrm{vec}(\mathbf{C}_1^{-1})\right)'\right)(\mathrm{vec}(\mathbf{H}_{kj}))_{kj}, \tag{17}$$

which, conditional on $\mathbf{U}$, $\mathbf{V}$, is distributed as $Normal_{N\times N}\left(0, M\left(\mathbf{I}_{N^2}\right)\right)$. On the other hand, noting that

$$\begin{aligned}(\mathrm{tr}(\mathbf{G}\widetilde{\mathbf{P}}_{kj}))_{kj} &= (\mathbf{I}_N \otimes \mathrm{vec}(\mathbf{G}^{1/2})')(\mathbf{I}_m \otimes \widetilde{\mathbf{P}}_{kj})_{kj}(\mathbf{I}_N \otimes \mathrm{vec}(\mathbf{G}^{1/2})), \\ (\mathrm{tr}(\mathbf{F}\widetilde{\mathbf{Q}}_{kj}))_{kj} &= (\mathbf{I}_N \otimes \mathrm{vec}(\mathbf{F}^{1/2})')(\mathbf{I}_m \otimes \widetilde{\mathbf{Q}}_{kj})_{kj}(\mathbf{I}_N \otimes \mathrm{vec}(\mathbf{F}^{1/2})), \\ (\mathbf{I}_m \otimes \widetilde{\mathbf{P}}_{kj})_{kj} &\sim \mathrm{Wishart}_{m^2N}\left(\nu - m, \mathbf{I}_{m^2N}\right), \\ (\mathbf{I}_m \otimes \widetilde{\mathbf{Q}}_{kj})_{kj} &\sim \mathrm{Wishart}_{m^2N}(\eta - m, \mathbf{I}_{m^2N}), \end{aligned}$$

we define the matrices

$$\begin{aligned}\mathbf{P} &= (\mathrm{tr}(\mathbf{G}^{-1}))^{-1}(\mathbf{I}_N \otimes \mathrm{vec}(\mathbf{G}^{1/2})')(\mathbf{I}_m \otimes \widetilde{\mathbf{P}}_{kj})_{kj}(\mathbf{I}_N \otimes \mathrm{vec}(\mathbf{G}^{1/2})), \\ \mathbf{Q} &= (\mathrm{tr}(\mathbf{F}^{-1}))^{-1}(\mathbf{I}_N \otimes \mathrm{vec}(\mathbf{F}^{1/2})')(\mathbf{I}_m \otimes \widetilde{\mathbf{Q}}_{kj})_{kj}(\mathbf{I}_N \otimes \mathrm{vec}(\mathbf{F}^{1/2})), \end{aligned} \tag{18}$$

which distribute independently as $\mathrm{Wishart}_N(\nu - m, \mathbf{I}_N)$ and $\mathrm{Wishart}_N(\eta - m, \mathbf{I}_N)$, respectively. It is also verified that

$$\left(\mathrm{vec}(\mathbf{A}_j^1)'\mathbf{S}_1\mathrm{vec}(\mathbf{A}_k^1)\right)_{kj} = \mathbf{A}_1\mathbf{S}_1\mathbf{A}_1',$$
$$\left(\mathrm{vec}(\mathbf{A}_j^2)'\mathbf{S}_2\mathrm{vec}(\mathbf{A}_k^2)\right)_{kj} = \mathbf{A}_2\mathbf{S}_2\mathbf{A}_2',$$
$$\left(\mathrm{vec}(\mathbf{A}_j^1)'\mathbf{S}_3\mathrm{vec}(\mathbf{A}_k^2)\right)_{kj} = \mathbf{A}_1\mathbf{S}_3\mathbf{A}_2',$$
$$\left(\mathrm{vec}(\mathbf{A}_j^2)'\mathbf{S}_4\mathrm{vec}(\mathbf{A}_k^1)\right)_{kj} = \mathbf{A}_2\mathbf{S}_4\mathbf{A}_1',$$

with $\mathbf{A}_2 = (\mathrm{vec}(\mathbf{A}_1^2), \ldots, \mathrm{vec}(\mathbf{A}_N^2))' \sim \mathrm{Normal}_{N\times m^2}\left(0, \mathbf{I}_{m^2N}\right)$ independently of $\mathbf{A}^1$. Combining these last expressions with (17) and (18) and substituting in (16) we obtain

$$\ddot{\Lambda} = -2\mathrm{tr}(\mathbf{G}^{\mathbf{U}}\mathbf{U} + \mathbf{G}^{\mathbf{V}}\mathbf{V})\mathbf{I}_N + \mathrm{tr}(\mathbf{C}_1^{-2})^{1/2}\mathbf{H} + \mathrm{tr}\left(\mathbf{G}^{-1}\right)\mathbf{P} + \mathrm{tr}\left(\mathbf{F}^{-1}\right)\mathbf{Q}$$
$$+\mathbf{A}_1\mathbf{S}_1\mathbf{A}_1' + \mathbf{A}_1\mathbf{S}_2\mathbf{A}_2' + \mathbf{A}_2\mathbf{S}_3\mathbf{A}_1' + \mathbf{A}_2\mathbf{S}_4\mathbf{A}_2'.$$

The substitutions (11) conclude the proof. □

4 Euler characteristic densities

In this section, we obtain expressions for the EC densities of the random field Λ. We begin with the following lemma.

Lemma 2 *The following equalities hold*

$$\dot{\Lambda}_{|j} = \mathrm{tr}\,(\mathbf{M}_2)^{1/2}\,\mathbf{z}$$
$$\ddot{\Lambda}_{|jj} \mid (\dot{\Lambda}_{|j} = \mathbf{0}) = \mathrm{tr}(\mathbf{M}_1)\mathbf{I}_j + \mathrm{tr}\,(\mathbf{M}_2)^{1/2}\,\widetilde{\mathbf{H}} + \mathrm{tr}\,(\mathbf{M}_3)\,\widetilde{\mathbf{P}} + \mathrm{tr}\,(\mathbf{M}_4)\,\widetilde{\mathbf{Q}} + \mathbf{X}_1\mathbf{M}_5\mathbf{X}_1'$$
$$+\mathbf{X}_1\mathbf{M}_6\mathbf{X}_2' + \mathbf{X}_2\mathbf{M}_7\mathbf{X}_1' + \mathbf{X}_2\mathbf{M}_8\mathbf{X}_2',$$

where

$$\mathbf{X}_1 \sim \mathrm{Normal}_{j\times m^2}(\mathbf{0}, \mathbf{\Sigma}\otimes\mathbf{I}_j),\quad \mathbf{X}_2 \sim \mathrm{Normal}_{j\times m^2}(\mathbf{0}, \mathbf{I}_{jm^2}),$$
$$\mathbf{z} \sim \mathrm{Normal}_j(\mathbf{0}, \mathbf{I}_j),\quad \widetilde{\mathbf{H}} \sim \mathrm{Normal}_{j\times j}(0, M(\mathbf{I}_{j^2})),$$
$$\widetilde{\mathbf{P}} \sim \mathrm{Wishart}_j(\nu - m, \mathbf{I}_j),\quad \widetilde{\mathbf{Q}} \sim \mathrm{Wishart}_j(\eta - m, \mathbf{I}_j)$$
$$\mathbf{W} \sim \mathrm{Wishart}_m(\nu + \eta, \mathbf{I}_m),\quad \mathbf{B} \sim \mathrm{Beta}_m\left(\tfrac{1}{2}\nu, \tfrac{1}{2}\eta\right)$$

independently, with $\mathbf{\Sigma}$ *given by*

$$\mathbf{\Sigma} = \mathbf{I}_{m^2\times m^2} - \mathrm{tr}(\mathbf{M}_2)^{-1}\mathrm{vec}(\mathbf{M}_0)\mathrm{vec}(\mathbf{M}_0)'.$$

Proof Define the vector **b** and the matrix **C** by

$$\mathbf{b} = (\mathbf{A}_1)_{|j}\mathrm{vec}\,(\mathbf{M}_0) \quad \text{and} \quad \mathbf{C} = [(\mathbf{A}_1)_{|j}, \mathbf{b}],$$

respectively. It is easily seen from the definition of $\mathbf{M}_0$ and $\mathbf{M}_2$ in the theorem above that

$$\mathrm{vec}\,(\mathbf{M}_0)'\,\mathrm{vec}\,(\mathbf{M}_0) = \mathrm{tr}(\mathbf{M}_2),$$

which implies

$$\mathbf{b} \sim \mathrm{Normal}_j(\mathbf{0}, \mathrm{tr}(\mathbf{M}_2)\,\mathbf{I}_j).$$

The first equality follows by noting that $\dot{\Lambda}_{|j} = \mathbf{b}$. Moreover it is very easy to check that

$$\mathbf{C} \sim \mathrm{Normal}_{j\times(m^2+1)}(\mathbf{0}, \ \widetilde{\boldsymbol{\Sigma}}),$$

where

$$\widetilde{\boldsymbol{\Sigma}} = \begin{pmatrix} \mathbf{I}_{jm^2\times jm^2} & \mathrm{vec}(\mathbf{M}_0)\otimes\mathbf{I}_j \\ \mathrm{vec}(\mathbf{M}_0)'\otimes\mathbf{I}_j & \mathrm{tr}(\mathbf{M}_2)\mathbf{I}_j \end{pmatrix}.$$

From Theorem 3.2.3 and Theorem 3.2.4 in Mardia and Bibby (1979) we obtain

$$(\mathbf{A}_1)_{|j} \mid (\mathbf{b} = \mathbf{0}) \sim \mathrm{Normal}_{j\times m^2}(\mathbf{0}, \boldsymbol{\Sigma}_1), \tag{19}$$

independently of $\mathbf{b}$, where

$$\begin{aligned} \boldsymbol{\Sigma}_1 &= \mathbf{I}_{jm^2\times jm^2} - (\mathrm{vec}(\mathbf{M}_0)\otimes\mathbf{I}_j)(\mathrm{tr}(\mathbf{M}_2)\mathbf{I}_j)^{-1}(\mathrm{vec}(\mathbf{M}_0)'\otimes\mathbf{I}_j) \\ &= \mathbf{I}_{jm^2\times jm^2} - \mathrm{tr}(\mathbf{M}_2)^{-1}\mathrm{vec}(\mathbf{M}_0)\mathrm{vec}(\mathbf{M}_0)'\otimes\mathbf{I}_j \\ &= (\mathbf{I}_{m^2\times m^2} - \mathrm{tr}(\mathbf{M}_2)^{-1}\mathrm{vec}(\mathbf{M}_0)\mathrm{vec}(\mathbf{M}_0)')\otimes\mathbf{I}_j. \end{aligned}$$

According to Theorem 1 the condition $(\dot{\Lambda}_{|j} = \mathbf{0})$ is equivalent to $\mathbf{b} = \mathbf{0}$. This implies, by (19),

$$\begin{aligned} \ddot{\Lambda}_{|jj} \mid (\dot{\Lambda}_{|j} = \mathbf{0}) = \ & \mathrm{tr}(\mathbf{M}_1)\mathbf{I}_j + \mathrm{tr}\,(\mathbf{M}_2)^{1/2}\,\widetilde{\mathbf{H}} + \mathrm{tr}\,(\mathbf{M}_3)\,\widetilde{\mathbf{P}} + \mathrm{tr}\,(\mathbf{M}_4)\,\widetilde{\mathbf{Q}} \\ & + \mathbf{X}_1\mathbf{M}_5\mathbf{X}_1' + \mathbf{X}_1\mathbf{M}_6\mathbf{X}_2' + \mathbf{X}_2\mathbf{M}_7\mathbf{X}_1' + \mathbf{X}_2\mathbf{M}_8\mathbf{X}_2', \end{aligned}$$

where

$$\mathbf{X}_1 \sim \mathrm{Normal}_{j\times m^2}(\mathbf{0}, \ \boldsymbol{\Sigma}), \mathbf{X}_2 \sim \mathrm{Normal}_{j\times m^2}(\mathbf{0}, \mathbf{I}_{jm^2}), \widetilde{\mathbf{H}} \sim \mathrm{Normal}_{j\times j}(0, M(\mathbf{I}_{j^2})),$$
$$\widetilde{\mathbf{P}} \sim \mathrm{Wishart}_j(\nu - m, \mathbf{I}_j), \widetilde{\mathbf{Q}} \sim \mathrm{Wishart}_j(\eta - m, \mathbf{I}_j),$$

independently, and

$$\boldsymbol{\Sigma} = \mathbf{I}_{m^2\times m^2} - \mathrm{tr}(\mathbf{M}_2)^{-1}\mathrm{vec}(\mathbf{M}_0)\mathrm{vec}(\mathbf{M}_0)', \tag{20}$$

which concludes the proof. □

The next theorem is the main result of this section. The EC densities are obtained only for $j = 1, 2, 3$, which are the most important cases in practical applications.

Theorem 3 *The EC densities* $\rho_j(\lambda)$, $j = 1, 2, 3$ *for the random field* Λ *are given by*

$$\rho_1(\lambda) = (2\pi)^{-1/2}\theta_0(\lambda)\varphi_1(\lambda),$$
$$\rho_2(\lambda) = (2\pi)^{-1}\theta_0(\lambda)\varphi_2(\lambda)$$
$$\rho_3(\lambda) = (2\pi)^{-3/2}\theta_0(\lambda)\varphi_3(\lambda),$$

where $\varphi_j(\lambda)$ *are expressions of the type*

$$\varphi_j(\lambda) = \mathbb{E}(f_j(\mathbf{B}, \mathbf{W}) \mid \Lambda(\mathbf{B}, \mathbf{W}) = \lambda),$$

for certain scalar values $f_j(\mathbf{B}, \mathbf{W})$, *and* $\theta_0(\lambda)$ *denotes the probability density function of* Λ.

Proof We shall evaluate the expectations in (4) by first conditioning on $\Lambda = \lambda$, $\mathbf{W}$ and $\mathbf{B}$, and then taking expectations over $\mathbf{W}$, $\mathbf{B}$ conditional on $\Lambda(\mathbf{B}, \mathbf{W}) = \lambda$. This is,

$$\rho_j(\lambda) = \mathbb{E}\Big(\mathbb{E}(\dot{\Lambda}_j^+ \mid \dot{\Lambda}_{|j-1} = \mathbf{0}, \Lambda = \lambda; \mathbf{B}, \mathbf{W})\mathbb{E}(\det(-\ddot{\Lambda}_{|j-1,j-1}) \mid \dot{\Lambda}_{|j-1} = \mathbf{0}, \Lambda = \lambda; \mathbf{B}, \mathbf{W}) \times \theta_{|j-1}(0; \lambda, \mathbf{B}, \mathbf{W}) \mid \Lambda(\mathbf{B}, \mathbf{W}) = \lambda\Big)\ \theta_0(\lambda), \quad (21)$$

where the corresponding derivatives should be rewritten according to Lemma 2. From Lemma 5.3.3 in Adler (1981) and Theorem2 we have

$$\mathbb{E}(\dot{\Lambda}_j^+ \mid \dot{\Lambda}_{|j-1} = \mathbf{0}, \Lambda = \lambda; \mathbf{B}, \mathbf{W}) = (2\pi)^{-1/2}\mathrm{tr}(\mathbf{M}_2)^{1/2}, \quad (22)$$

and the density of $\dot{\Lambda}_{|j-1}$ at zero conditional on $\Lambda = \lambda$ and $\mathbf{B}$, $\mathbf{W}$ given by

$$\theta_{|j-1}(0; \lambda, \mathbf{B}, \mathbf{W}) = (2\pi\,\mathrm{tr}(\mathbf{M}_2))^{-\frac{1}{2}(j-1)}, \quad (23)$$

According to Lemmas 2, 7, and 8 we obtain

$$\mathbb{E}(\det(-\ddot{\Lambda}_{|00}) \mid \dot{\Lambda}_{|0} = 0, \Lambda = \lambda; \mathbf{B}, \mathbf{W}) = 1, \quad (24)$$

$$\mathbb{E}(\det(-\ddot{\Lambda}_{|11}) \mid \dot{\Lambda}_{|1} = 0, \Lambda = \lambda; \mathbf{B}, \mathbf{W}) = -((\nu - m)\mathrm{tr}(\mathbf{M}_3) + \mathrm{tr}(\mathbf{M}_1 + \mathbf{M}_5\mathbf{\Sigma} + \mathbf{M}_8) + (\eta - m)\mathrm{tr}(\mathbf{M}_4)), \quad (25)$$

$$\begin{aligned}\mathbb{E}(\det(-\ddot{\Lambda}_{|22}) \mid \dot{\Lambda}_{|2} = \mathbf{0}, \Lambda = \lambda; \mathbf{B}, \mathbf{W}) &= \tbinom{\nu-m}{2}\mathrm{tr}(\mathbf{M}_3)^2 + \tbinom{\eta-m}{2}\mathrm{tr}(\mathbf{M}_4)^2 + \mathrm{tr}(\mathbf{M}_1)^2 \\ &\quad + (\nu - m)(\eta - m)\mathrm{tr}(\mathbf{M}_3)\,\mathrm{tr}(\mathbf{M}_4) + [(\nu - m)\mathrm{tr}(\mathbf{M}_3) + (\eta - m)\mathrm{tr}(\mathbf{M}_4)]\,\mathrm{tr}(\mathbf{M}_1) \\ &\quad + \mathrm{tr}(\mathbf{M}_5\mathbf{\Sigma} + \mathbf{M}_8)^2 \\ &\quad + [(\nu - m)\mathrm{tr}(\mathbf{M}_3) + (\eta - m)\mathrm{tr}(\mathbf{M}_4) + \mathrm{tr}(\mathbf{M}_1)]\,\mathrm{tr}(\mathbf{M}_5\mathbf{\Sigma} + \mathbf{M}_8) \\ &\quad - \mathrm{tr}(\mathbf{M}_5\mathbf{\Sigma}\mathbf{M}_5\mathbf{\Sigma} + \mathbf{M}_8^2 + \mathbf{M}_6\mathbf{M}_7\mathbf{\Sigma} + \mathbf{M}_7\mathbf{M}_6\mathbf{\Sigma} + \mathbf{M}_2),\end{aligned} \quad (26)$$

respectively, with $\mathbf{\Sigma}$ given by (20).

By substituting (22), (23) and (24) in (21) we have

$$\rho_1(\lambda) = (2\pi)^{-1/2}\theta_0(\lambda)\,\varphi_1(\lambda),$$

where $\varphi_1(\lambda) := \mathbb{E}(\mathrm{tr}(\mathbf{M}_2)^{1/2})$. In a similar way, by substituting (22), (23), (25) and (22), (23), (26) in (21) we respectively obtain

$$\rho_2(\lambda) = (2\pi)^{-1}\theta_0(\lambda)\varphi_2(\lambda)$$

and

$$\rho_3(\lambda) = (2\pi)^{-3/2}\theta_0(\lambda)\,\varphi_3(\lambda),$$

with

$$\varphi_2(\lambda) := -\mathbb{E}((\nu - m)\mathrm{tr}\,(\mathbf{M}_3) + (\eta - m)\mathrm{tr}\,(\mathbf{M}_4) + \mathrm{tr}(\mathbf{M}_1) + \mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma} + \mathbf{M}_8)),$$

and

$$\begin{aligned}\varphi_3(\lambda) := \mathbb{E}(&\mathrm{tr}(\mathbf{M}_2)^{-1/2}(2\tbinom{\nu-m}{2}\mathrm{tr}\,(\mathbf{M}_3)^2\\ &+2\tbinom{\eta-m}{2}\mathrm{tr}\,(\mathbf{M}_4)^2 + 2(\nu-m)(\eta-m)\mathrm{tr}\,(\mathbf{M}_3)\,\mathrm{tr}\,(\mathbf{M}_4)\\ &+\mathrm{tr}(\mathbf{M}_1)^2 + \mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma} + \mathbf{M}_8)^2 + ((\nu-m)\mathrm{tr}\,(\mathbf{M}_3) + (\eta-m)\mathrm{tr}(\mathbf{M}_4))\mathrm{tr}(\mathbf{M}_1)\\ &+((\nu-m)\mathrm{tr}\,(\mathbf{M}_3) + (\eta-m)\mathrm{tr}\,(\mathbf{M}_4)\\ &+\mathrm{tr}(\mathbf{M}_1))\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma} + \mathbf{M}_8) - \mathrm{tr}(\mathbf{M}_5\,\boldsymbol{\Sigma}\mathbf{M}_5\boldsymbol{\Sigma}\\ &+\mathbf{M}_8{}^2 + \mathbf{M}_6\mathbf{M}_7\boldsymbol{\Sigma} + \mathbf{M}_7\mathbf{M}_6\boldsymbol{\Sigma} + \mathbf{M}_2))),\end{aligned}$$

where all expectations above must be taken by conditioning on $\Lambda(\mathbf{B}, \mathbf{W}) = \lambda$. □

5 Wilks's Λ random field

In this section we shall apply the previous results for obtaining the EC densities of the Wilks's Λ RF.

Definition 4 Let $\mathbf{U}(\mathbf{t})$, $\mathbf{V}(\mathbf{t})$ be two independent m-dimensional Wishart RFs with ν and η degrees of freedom, respectively. The Wilks's Λ RF is then defined as

$$\Lambda(\mathbf{t}) = \frac{\det(\mathbf{U}(\mathbf{t}))}{\det(\mathbf{U}(\mathbf{t}) + \mathbf{V}(\mathbf{t}))}.$$

Note that although the Wilks's Λ statistic is well defined at a point $\mathbf{t}$ provided $\nu + \eta \geq m$, this may not be true for all points inside a compact subset of $\mathbb{R}^N$. The reason is that both the numerator and denominator if Wilks's Λ could be zero. For the particular case of $m = 1$ it was shown in Worsley (1994) that $\nu + \eta \geq N$ to avoid this, with probability one. For general m, following the same ideas in Cao and Worsley

(1999a), we obtain $\nu + \eta \geq m + N - 1$ as a necessary and sufficient condition for having Λ well defined. When used in hypothesis testing problems, one rejects null hypotheses based on small values of the Wilks's Λ statistics. So, in what follows, we shall work with the RF $Y(\mathbf{t}) = -\log(\Lambda(\mathbf{t}))$.

It is well known that for any symmetric matrix function $\mathbf{U}(\mathbf{t})$ the identities

$$\begin{aligned}\dot{\det}(\mathbf{U})_j &= \det(\mathbf{U})\mathrm{tr}(\mathbf{U}^{-1}\dot{\mathbf{U}}_j),\\ \ddot{\det}(\mathbf{U})_{kj} &= \det(\mathbf{U})^{-1}\dot{\det}(\mathbf{U})_k\dot{\det}(\mathbf{U})_j - \det(\mathbf{U})\mathrm{tr}(\mathbf{U}^{-1}\dot{\mathbf{U}}_k\mathbf{U}^{-1}\dot{\mathbf{U}}_j + \mathbf{U}^{-1}\ddot{\mathbf{U}}_{kj})\end{aligned}$$

hold, so that Theorem 1 gives

$$\begin{aligned}\dot{Y} &= 2\Lambda\mathbf{A}_1\mathrm{vec}(\mathbf{M}_0^{1/2})\\ \ddot{Y} &= 2\Lambda\left(\mathrm{tr}(\mathbf{M}_0)^{1/2}\mathbf{H} - \mathrm{tr}(\mathbf{M}_0)\mathbf{P} + \mathrm{tr}(\mathbf{M}_4)\mathbf{Q} + \mathbf{A}_1\mathbf{M}_5\mathbf{A}_1' - \mathbf{A}_1\mathbf{M}_6\mathbf{A}_2' - \mathbf{A}_2\mathbf{M}_7\mathbf{A}_1'\right),\end{aligned}$$

where

$$\begin{aligned}\mathbf{M}_0 &= (\mathbf{B}^{-1} - \mathbf{I}_m)\mathbf{W}^{-1},\\ \mathbf{M}_4 &= \mathbf{W}^{-1},\\ \mathbf{M}_5 &= (\mathbf{M}_0^{1/2} \otimes \mathbf{M}_0^{1/2}),\\ \mathbf{M}_6 &= (\mathbf{M}_0^{1/2} \otimes \mathbf{W}^{-1/2})',\\ \mathbf{M}_7 &= \mathbf{M}_6'.\end{aligned}$$

Theorem 3 yields

$$\rho_1(y) = 2(2\pi)^{-1/2}e^{-y}\theta_0(y)\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{1/2} \mid \det(\mathbf{B}) = e^{-y}), \tag{27}$$

$$\begin{aligned}\rho_2(y) = {} & 2(2\pi)^{-1}e^{-y}\theta_0(y)\mathbb{E}((\nu - m)\mathrm{tr}(\mathbf{M}_0) - (\eta - m)\mathrm{tr}\,(\mathbf{M}_4)\\ & -\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma}) \mid \det(\mathbf{B}) = e^{-y}),\end{aligned} \tag{28}$$

$$\begin{aligned}\rho_3(y) = {} & 2(2\pi)^{-3/2}e^{-y}\theta_0(y)\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-1/2}(2\tbinom{\nu-m}{2}\mathrm{tr}\,(\mathbf{M}_0)^2 + 2\tbinom{\eta-m}{2}\mathrm{tr}\,(\mathbf{M}_4)^2\\ & +\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma})^2 - 2(\nu - m)(\eta - m)\mathrm{tr}(\mathbf{M}_0)\mathrm{tr}(\mathbf{M}_4) + ((\eta - m)\mathrm{tr}(\mathbf{M}_4)\\ & -(\nu - m)\mathrm{tr}(\mathbf{M}_0))\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma})\\ & -\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma}\mathbf{M}_5\,\boldsymbol{\Sigma} + 2\mathbf{M}_6\mathbf{M}_7\boldsymbol{\Sigma}) - \mathrm{tr}(\mathbf{M}_0))) \mid \det(\mathbf{B}) = e^{-y}),\end{aligned} \tag{29}$$

where

$$\boldsymbol{\Sigma} = \mathbf{I}_{m^2\times m^2} - \mathrm{tr}(\mathbf{M}_0)^{-1}\mathrm{vec}(\mathbf{M}_0^{1/2})\mathrm{vec}(\mathbf{M}_0^{1/2})'.$$

There is no closed form expression for $\theta_0(y)$, the density of $Y = -\log(\Lambda)$. A host of approximations are available (Anderson 1984) , but it is easier numerically to use Fourier methods as follows. We can write

$$Y = \sum_{i=1}^{m} Y_i,$$

where

$$\exp(-Y_i) \sim \mathrm{Beta}_1\left(\tfrac{1}{2}(\nu + 1 - i), \tfrac{1}{2}\eta\right)$$

independently, $i = 1, \ldots, m$ (Anderson 1984). Then the product of the Fourier transforms of the densities of Y_i is the Fourier transform of the density of Y. Inverting gives the desired density $\theta_0(y)$ of Y.

5.1 EC densities

Providing closed expressions for the expectations that appear in the formulas (27)–(29) is a very difficult task. In this subsection we propose a practical way to approximate such expectations. Specifically, we shall use Taylor expansion series for the moments of functions of random variables (see Shapiro and Gross 1981). In details,

$$\mathbb{E}(f(X)) \approx f(\mathbb{E}(X)) + \frac{1}{2} f''(\mathbb{E}(X))\mathbb{E}((X-\mathbb{E}(X))^2)$$

and

$$\begin{aligned}\mathbb{E}(f(X, Y)) \approx f(\mathbb{E}(X), \mathbb{E}(Y)) &+ \frac{1}{2} f''_{xx}(\mathbb{E}(X), \mathbb{E}(Y))\mathbb{E}((X-\mathbb{E}(X))^2)\\ &+ \frac{1}{2} f''_{yy}(\mathbb{E}(X), \mathbb{E}(Y))\mathbb{E}((Y-\mathbb{E}(Y))^2)\\ &+ \frac{1}{2} f''_{xy}(\mathbb{E}(X), \mathbb{E}(Y))\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))),\end{aligned} \tag{30}$$

provided that f is sufficiently differentiable and the moments of X and Y are finite. In particular we shall deal with the functions $f(x) = x^r$ and $f(x, y) = \frac{y}{x^r}$ for the random variables $X = \mathrm{tr}(\mathbf{M}_0)$ and $Y = \mathrm{tr}(\mathbf{M}_0^k)$, with $k \geq 2$ and $r > 0$.

Thus for the EC density $\rho_1(y)$ we have

$$\begin{aligned}\rho_1(y) &= 2(2\pi)^{-1/2} e^{-y} \theta_0(y)\\ &\quad\times \left[\frac{9}{8}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0) \mid \mathcal{A})^{1/2} - \frac{1}{8}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0) \mid \mathcal{A})^{-3/2}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^2 \mid \mathcal{A})\right],\end{aligned}$$

where $\mathcal{A}$ is the event $\det(\mathbf{B}) = e^{-y}$. According to Letac and Massam (2001),

$$\begin{aligned}k_1(y) := \mathbb{E}(\mathrm{tr}(\mathbf{M}_0) \mid \mathcal{A}) &= \frac{\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m) \mid \mathcal{A})}{q}\\ &= \frac{c_1(y) - m}{q},\end{aligned}$$

and

$$k_2(y) := \mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^2 \mid \mathcal{A}) = \frac{\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m)^2 \mid \mathcal{A})}{q^2 - 1} + \frac{\mathbb{E}(\mathrm{tr}((\mathbf{B}^{-1} - \mathbf{I}_m)^2) \mid \mathcal{A})}{q^3 - q}$$
$$= \frac{c_3(y) - 2mc_1(y) + m^2}{q^2 - 1} + \frac{c_2(y) - 2c_1(y) + m}{q^3 - q},$$

where $q = \nu + \eta - m - 1$, $u = \nu - m - 1$, and $c_j(y) = \mathbb{E}(\mathrm{tr}(\mathbf{B}^{-j}) \mid \mathcal{A})$, $j = 1, 2$, $c_3(y) = \mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1})^2 \mid \mathcal{A})$ are calculated according to Appendix with second order approximations (by expressing $\mathrm{tr}(\mathbf{B}^{-1})$, $\mathrm{tr}(\mathbf{B}^{-2})$ and $\mathrm{tr}(\mathbf{B}^{-1})^2$ in terms of suitable zonal polynomials of $\mathbf{B}^{-1}$). In a similar way, we use the approximation

$$\mathbb{E}(\mathrm{tr}(\mathbf{M}_5\mathbf{\Sigma})) = \mathbb{E}(\mathrm{tr}(\mathbf{M}_0^{1/2})^2) - \mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-1}\mathrm{tr}(\mathbf{M}_0^2))$$
$$\approx \mathbb{E}(\mathrm{tr}(\mathbf{M}_0)) - \frac{\mathbb{E}(\mathrm{tr}(\mathbf{M}_0^2))}{\mathbb{E}(\mathrm{tr}(\mathbf{M}_0))} + \frac{\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mathrm{tr}(\mathbf{M}_0^2))}{\mathbb{E}(\mathrm{tr}(\mathbf{M}_0))^2}$$
$$- \frac{\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^2)\mathbb{E}(\mathrm{tr}(\mathbf{M}_0^2))}{\mathbb{E}(\mathrm{tr}(\mathbf{M}_0))^3}$$

to obtain

$$\rho_2(y) = 2^{\frac{3}{2}}(2\pi)^{-1}e^{-\frac{3y}{2}}\theta_0(y)\Bigg[(\nu - m - 1)k_1(y) - \frac{m(\eta - m)}{q}$$
$$+ \frac{k_3(y)}{k_1(y)} - \frac{k_4(y)}{k_1(y)^2} + \frac{k_2(y)k_3(y)}{k_1(y)^3}\Bigg],$$

where, according to Letac and Massam (2001),

$$k_3(y) := \mathbb{E}(\mathrm{tr}(\mathbf{M}_0^2) \mid \mathcal{A}) = \frac{\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m)^2 \mid \mathcal{A})}{q^3 - q} + \frac{\mathbb{E}(\mathrm{tr}((\mathbf{B}^{-1} - \mathbf{I}_m)^2) \mid \mathcal{A})}{q^2 - 1}$$
$$= \frac{c_3(y) - 2mc_1(y) + m^2}{q^3 - q} + \frac{c_2(y) - 2c_1(y) + m}{q^2 - 1},$$
$$k_4(y) := \mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mathrm{tr}(\mathbf{M}_0^2) \mid \mathcal{A}) = \frac{1}{q(q^2 - 1)(q^2 - 4)}[q\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m)^3 \mid \mathcal{A})$$
$$+(q^2 + 2)\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m)\mathrm{tr}((\mathbf{B}^{-1} - \mathbf{I}_m)^2) \mid \mathcal{A})]$$
$$+2q\mathbb{E}(\mathrm{tr}((\mathbf{B}^{-1} - \mathbf{I}_m)^3) \mid \mathcal{A})$$
$$= \frac{c_5(y) - 3mc_3(y) + 3m^2c_1(y) - m^3}{(q^2 - 1)(q^2 - 4)} + \frac{2(c_4(y) - 3c_2(y) + 3c_1(y) - m)}{(q^2 - 1)(q^2 - 4)}$$
$$+ \frac{(q^2 + 2)(c_6(y) - 2c_3(y) - mc_2(y) + 3mc_1(y) - m^2)}{q(q^2 - 1)(q^2 - 4)},$$

with $c_4(y) = \mathbb{E}(\mathrm{tr}(\mathbf{B}^{-3}) \mid \mathcal{A})$, $c_5(y) = \mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1})^3 \mid \mathcal{A})$, $c_6(y) = \mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1})\mathrm{tr}(\mathbf{B}^{-2}) \mid \mathcal{A})$, which are all according to the explanation given in Appendix.

Finally, for the density $\rho_3(y)$ we have,

$$\rho_3(y) = 4(2\pi)^{-\frac{3}{2}} e^{-2y}\theta_0(y)\mathbb{E}\Big(\tbinom{\nu-m}{2}\mathrm{tr}\,(\mathbf{M}_0)^{\frac{3}{2}} + \tbinom{\eta-m}{2}\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}}\mathrm{tr}\,(\mathbf{M}_4)^2$$
$$+\,\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}}\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma})^2 - (\nu-m)(\eta-m)\mathrm{tr}(\mathbf{M}_0)^{\frac{1}{2}}\mathrm{tr}(\mathbf{M}_4)$$
$$+((\eta-m)\mathrm{tr}(\mathbf{M}_4) - (\nu-m)\mathrm{tr}(\mathbf{M}_0))\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}}\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma})$$
$$-\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma}\mathbf{M}_5\,\boldsymbol{\Sigma} + 2\mathbf{M}_6\mathbf{M}_7\boldsymbol{\Sigma})\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}} - \mathrm{tr}(\mathbf{M}_0)^{\frac{1}{2}} \mid \mathcal{A}\Big).$$

Finally, in order to evaluate these expectations we shall use the following approximations

$$\mathbb{E}(\mathrm{tr}\,(\mathbf{M}_0)^{\frac{3}{2}} \mid \mathcal{A}) \approx \frac{1}{4}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{\frac{3}{2}} + \frac{3}{4}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{1}{2}}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^2\mid\mathcal{A})$$
$$= \frac{5}{8}k_1(y)^{\frac{3}{2}} + \frac{3}{8}k_1(y)^{-\frac{1}{2}}k_2(y),$$

$$\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}}\mathrm{tr}\,(\mathbf{M}_4)^2 \mid \mathcal{A})$$
$$\approx \mathbb{E}(\mathrm{tr}\,(\mathbf{M}_4)^2)\left(\frac{7}{8}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{1}{2}} + \frac{3}{8}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{5}{2}}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^2\mid\mathcal{A})\right)$$
$$-\frac{1}{4}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{3}{2}}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mathrm{tr}\,(\mathbf{M}_4)^2\mid\mathcal{A})$$
$$= \frac{m(mq+1)}{q(q^2-1)}\left(\frac{7}{8}k_1(y)^{-\frac{1}{2}} + \frac{3}{8}k_1(y)^{-\frac{5}{2}}k_2(y)\right) - \frac{1}{4}k_1(y)^{-\frac{3}{2}}k_5(y),$$

$$\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{\frac{1}{2}}\mathrm{tr}\,(\mathbf{M}_4) \mid \mathcal{A})$$
$$\approx \mathbb{E}(\mathrm{tr}\,(\mathbf{M}_4))\left(\frac{7}{8}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{\frac{1}{2}} - \frac{1}{8}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{3}{2}}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^2\mid\mathcal{A})\right)$$
$$+\frac{1}{4}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{1}{2}}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mathrm{tr}\,(\mathbf{M}_4)\mid\mathcal{A})$$
$$= \frac{m}{q}\left(\frac{7}{8}k_1(y)^{\frac{1}{2}} - \frac{1}{8}k_1(y)^{-\frac{3}{2}}k_2(y)\right) + \frac{1}{4}k_1(y)^{-\frac{1}{2}}k_6(y),$$

$$\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}}(\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma})^2 - \mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma}\mathbf{M}_5\boldsymbol{\Sigma}))\mid\mathcal{A})$$
$$\approx 2\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-\frac{3}{2}}\mathrm{tr}(\mathbf{M}_0^3)\mid\mathcal{A}) - 2\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}}\mathrm{tr}(\mathbf{M}_0^2)\mid\mathcal{A})$$
$$\approx 2\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{3}{2}}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0^3)\mid\mathcal{A}) + \frac{1}{2}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{3}{2}}\mathbb{E}\left(\mathrm{tr}(\mathbf{M}_0)\mathrm{tr}\left(\mathbf{M}_0^2\right)\mid\mathcal{A}\right)$$
$$-\mathbb{E}\left(\mathrm{tr}\left(\mathbf{M}_0^2\right)\right)\left(\frac{7}{4}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{1}{2}} + \frac{3}{4}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)\mid\mathcal{A})^{-\frac{5}{2}}\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^2\mid\mathcal{A})\right)$$
$$= k_1(y)^{-\frac{3}{2}}(2k_7(y) + \frac{1}{2}k_4(y)) - k_3(y)\left(\frac{7}{4}k_1(y)^{-\frac{1}{2}} + \frac{3}{4}k_1(y)^{-\frac{5}{2}}k_2(y)\right),$$

$$\begin{aligned}
&\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}}\mathrm{tr}(\mathbf{M}_6\mathbf{M}_7\boldsymbol{\Sigma}) \mid \mathcal{A}) \\
&\quad = \mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-\frac{3}{2}}\left(\mathrm{tr}(\mathbf{M}_0)^2\mathrm{tr}(\mathbf{M}_4) - \mathrm{tr}(\mathbf{M}_0^2\mathbf{M}_4) | \mathcal{A}\right) \\
&\quad \approx \mathbb{E}(\mathrm{tr}(\mathbf{M}_0) \mid \mathcal{A})^{-\frac{3}{2}}(\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^2\mathrm{tr}(\mathbf{M}_4) \mid \mathcal{A}) - \mathbb{E}\left(\mathrm{tr}(\mathbf{M}_0^2\mathbf{M}_4) \mid \mathcal{A}\right) \\
&\quad = k_1(y)^{-\frac{3}{2}}(k_9(y) - k_8(y)), \\
&\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{\frac{1}{2}}\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma}) \mid \mathcal{A}) \\
&\quad \approx \mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{\frac{3}{2}} | \mathcal{A}) - \mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}}\mathrm{tr}(\mathbf{M}_0^2) | \mathcal{A}) \\
&\quad \approx \frac{5}{8}k_1(y)^{\frac{3}{2}} + \frac{3}{8}k_1(y)^{-\frac{1}{2}}k_2(y) + \frac{1}{4}k_1(y)^{-\frac{3}{2}}k_4(y) \\
&\qquad -k_3(y)\left(\frac{7}{8}k_1(y)^{-\frac{1}{2}} + \frac{3}{8}k_1(y)^{-\frac{5}{2}}k_2(y)\right), \\
&\mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-\frac{1}{2}}\mathrm{tr}(\mathbf{M}_4)\mathrm{tr}(\mathbf{M}_5\boldsymbol{\Sigma}) \mid \mathcal{A}) \\
&\quad \approx \mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^{-\frac{3}{2}}(\mathrm{tr}(\mathbf{M}_0)^2\mathrm{tr}(\mathbf{M}_4) - \mathrm{tr}(\mathbf{M}_0^2)\mathrm{tr}(\mathbf{M}_4) | \mathcal{A}) \\
&\quad \approx k_1(y)^{-\frac{3}{2}}(k_9(y) - k_{10}(y)),
\end{aligned}$$

where

$$\begin{aligned}
k_5(y) &:= \mathbb{E}(\mathrm{tr}\,(\mathbf{M}_0)\,\mathrm{tr}\,(\mathbf{M}_4)^2 \mid \mathcal{A}) = \frac{m^2(q^2-2)+3qm+4}{q(q^2-1)(q^2-4)}\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m) \mid \mathcal{A}) \\
&= \frac{(m^2(q^2-2)+3qm+4)(c_1(y)-m)}{q(q^2-1)(q^2-4)}, \\
k_6(y) &:= \mathbb{E}(\mathrm{tr}\,(\mathbf{M}_0)\,\mathrm{tr}\,(\mathbf{M}_4) \mid \mathcal{A}) = \frac{mq+1}{q^3-q}\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m) \mid \mathcal{A}) \\
&= \frac{(mq+1)(c_1(y)-m)}{q^3-q}, \\
k_7(y) &:= \mathbb{E}(\mathrm{tr}(\mathbf{M}_0^3) \mid \mathcal{A}) \\
&= \frac{1}{q(q^2-1)(q^2-4)}[2\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m)^3 \mid \mathcal{A}) + q^2\mathbb{E}(\mathrm{tr}((\mathbf{B}^{-1} - \mathbf{I}_m)^3) \mid \mathcal{A}) \\
&\quad +3q\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m)\mathrm{tr}((\mathbf{B}^{-1} - \mathbf{I}_m)^2) \mid \mathcal{A})] \\
&= \frac{2(c_5(y)-3mc_3(y)+3m^2c_1(y)-m^3)+q^2(c_4(y)-3c_2(y)+3c_1(y)-m)}{q(q^2-1)(q^2-4)} \\
&\quad + \frac{3(c_6(y)-2c_3(y)-mc_2(y)+3mc_1(y)-m^2)}{(q^2-1)(q^2-4)}, \\
k_8(y) &:= \mathbb{E}(\mathrm{tr}(\mathbf{M}_0^2\mathbf{M}_4) \mid \mathcal{A}) \\
&= \frac{2(m+q)\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1} - \mathbf{I}_m)^2 \mid \mathcal{A}) + (q^2+mq)\mathbb{E}(\mathrm{tr}((\mathbf{B}^{-1} - \mathbf{I}_m)^2) \mid \mathcal{A})}{q(q^2-1)(q^2-4)} \\
&= \frac{2(m+q)(c_3(y)-2mc_1(y)+m^2)+(q^2+mq)(c_2(y)-2c_1(y)+m)}{q(q^2-1)(q^2-4)}.
\end{aligned}$$

$$
\begin{aligned}
k_9(y) &:= \mathbb{E}(\mathrm{tr}(\mathbf{M}_0)^2\mathrm{tr}(\mathbf{M}_4)\mid\mathcal{A})\\
&= \frac{(m(q^2-2)+2q)\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1}-\mathbf{I}_m)^2\mid\mathcal{A})}{q(q^2-1)(q^2-4)}\\
&\quad+\frac{(mq+4)\mathbb{E}(\mathrm{tr}((\mathbf{B}^{-1}-\mathbf{I}_m)^2)\mid\mathcal{A})}{q(q^2-1)(q^2-4)}\\
&= \frac{(m(q^2-2)+2q)(c_3(y)-2mc_1(y)+m^2)+(mq+4)(c_2(y)-2c_1(y)+m)}{q(q^2-1)(q^2-4)}.\\
k_{10}(y) &:= \mathbb{E}(\mathrm{tr}(\mathbf{M}_0^2)\mathrm{tr}(\mathbf{M}_4)\mid\mathcal{A})\\
&= \frac{(mq+4)\mathbb{E}(\mathrm{tr}(\mathbf{B}^{-1}-\mathbf{I}_m)^2\mid\mathcal{A})}{q(q^2-1)(q^2-4)}\\
&\quad+\frac{(m(q^2-2)+2q)\mathbb{E}(\mathrm{tr}((\mathbf{B}^{-1}-\mathbf{I}_m)^2)\mid\mathcal{A})}{q(q^2-1)(q^2-4)}\\
&= \frac{(mq+4)(c_3(y)-2mc_1(y)+m^2)+(m(q^2-2)+2q)(c_2(y)-2c_1(y)+m)}{q(q^2-1)(q^2-4)}.
\end{aligned}
$$

5.2 Simulation results

In this subsection we shall show some simulation results for validating the previous formulae. We generated 200 $Y = -\log(\Lambda)$ RFs on a 32^3 voxel rectilinear lattice as follows. We first generated lattices of independent standard Gaussian random variables then smoothed them with a Gaussian shaped filter of standard deviation $\sigma = 3.2$ voxels, which gives $v = 1/(\sqrt{2}\sigma) = 0.22$ as the standard deviation of the spatial derivative (Worsley et al. 1996). The Wilks's Λ RF was generated by

$$
\Lambda = \frac{\det(\mathbf{Z}_1'\mathbf{Z}_1)}{\det(\mathbf{Z}_1'\mathbf{Z}_1+\mathbf{Z}_2'\mathbf{Z}_2)}, \tag{31}
$$

where $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are $\nu \times m$ and $\eta \times m$ matrices of independent smoothed Gaussian random fields as above. The EC $\chi(A_y)$ of the excursion sets A_y was calculated for each of the 200 Y RFs at $y = 0, 0.1, 0.2, \ldots, 6.0$. Then, the average of the measurements $\chi(A_y)$ was used as an estimator of $\mathbb{E}(\chi(A_y))$. This value and the theoretical approximation obtained in the subsection above are plotted in Fig. 1 for $\nu = 31$, $\eta = 10$, and $m = 3$. Note that for thresholds greater that 1, both the theoretical approximation and the simulated values are very close.

The P value approximation (5) based on the expected EC, the (uncorrected) P value at a single voxel, and the Bonferroni P value are plotted in Fig. 2. The parameter values were chosen to match the example in Sect. 6: $m = 3$, $\nu = 31$ and $\eta = 1, 2, 3, 4$. The parameter set S was a 3D ball of radius with radius 68 mm sampled on a 2 mm voxel lattice. The data was smoothed by a Gaussian filter with standard deviation 5.65 mm to give $v = 0.125$. In the first plot with $\eta = 1$, Wilks's Λ is equivalent to Hotelling's $T^2 = \nu(\exp(Y) - 1)$. Exact results for the expected EC of Hotelling's T^2 from Cao and Worsley (1999a) are added to the plot. We can see that these are in reasonable

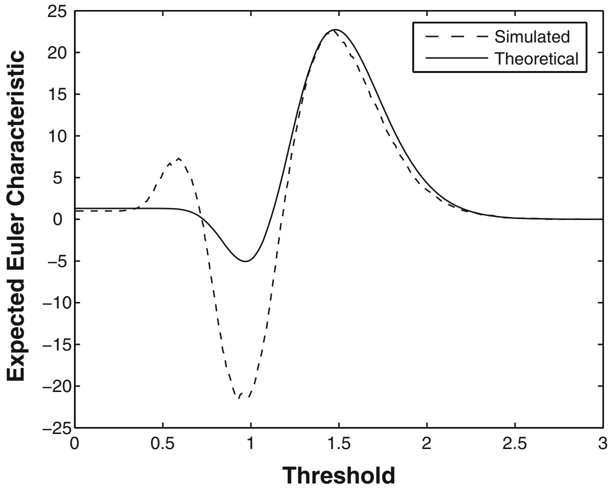

Fig. 1 Euler characteristic of the excursion set of $Y = -\log(\Lambda)$ for $\nu = 7$, $\eta = 4$ and $m = 2$ sampled on a 32^3 lattice smoothed by Gaussian filter with standard deviation 6.37 voxels

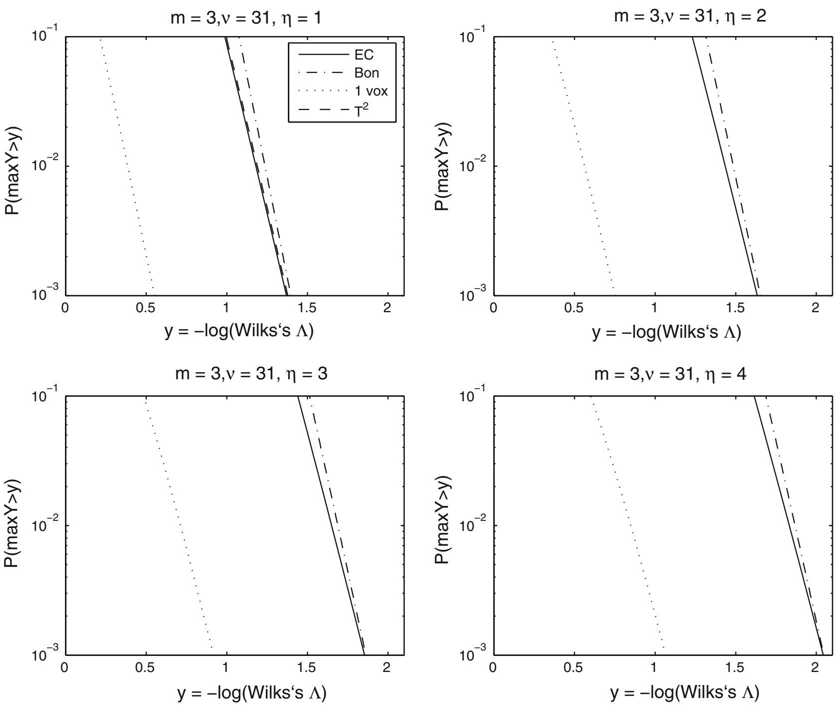

Fig. 2 *P* values of the maximum of $Y = -\log(\Lambda)$ over the brain, approximated by a 3D ball with radius 68 mm sampled on a 2 mm voxel lattice. The data was smoothed by a Gaussian filter with standard deviation 5.65 mm. *EC* Euler Characteristic approximation (5), *Bon* Bonferroni upper bound, *1 vox* uncorrected *P* value at a single voxel (lower bound), T^2 exact expected EC for equivalent Hotelling's T^2 ($\eta = 1$ only)

agreement with our Wilks's Λ approximation, which appears to be too liberal by a factor of 3 to 4.

6 Application

6.1 Multivariate linear models

We apply our results to a multivariate linear model. Suppose we have n independent RFs of multivariate observations $\mathbf{y}_i(\mathbf{t}) \in \mathbb{R}^m$, $i = 1, \ldots, n$, where $\mathbf{t} \in S \subset \mathbb{R}^N$, and a multivariate linear model (Cao and Worsley 1999a):

$$\mathbf{y}_i(\mathbf{t})' = \mathbf{x}_i'\beta(\mathbf{t}) + \epsilon_i(\mathbf{t})'\boldsymbol{\Sigma}(\mathbf{t})^{1/2}, \tag{32}$$

where $\mathbf{x}_i$ is a p-vector of known regressors, and $\beta(\mathbf{t})$ is an unknown $p \times m$ coefficient matrix. The error $\epsilon_i(\mathbf{t})$ is a m-vector of independent zero mean, unit variance isotropic Gaussian components with the same spatial correlation structure, and $\mathrm{Var}(\mathbf{y}_i(\mathbf{t})) = \boldsymbol{\Sigma}(\mathbf{t})$ is an unknown $m \times m$ matrix. We can now detect how the regressors are related to the multivariate data at point $\mathbf{t}$ by testing contrasts in the rows of $\beta(\mathbf{t})$. Classical multivariate test statistics evaluated at each point $\mathbf{t}$ then form a random field.

Let $\widehat{\beta}(\mathbf{t})$ be the usual least-squares estimate of $\beta(\mathbf{t})$. The Wilks's Λ random field is defined in terms of two independent Wishart random fields. The first is the error sum of squares matrix

$$\mathbf{U}(\mathbf{t}) = \sum_{i=1}^{n} \left(\mathbf{y}_i(\mathbf{t})' - \mathbf{x}_i'\widehat{\beta}(\mathbf{t})\right)' \left(\mathbf{y}_i(\mathbf{t})' - \mathbf{x}_i'\widehat{\beta}(\mathbf{t})\right) \sim \mathrm{Wishart}_m(\boldsymbol{\Sigma}(\mathbf{t}), \nu)$$

where $\nu = n - p$. Let $\mathbf{X} = (\mathbf{x}_1', \ldots, \mathbf{x}_n')'$ be the design matrix. The second is the regression sum of squares matrix for a $\eta \times p$ matrix of contrasts $\mathbf{C}$

$$\mathbf{V}(\mathbf{t}) = (\mathbf{C}\widehat{\beta}(\mathbf{t}))'(\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1}(\mathbf{C}\widehat{\beta}(\mathbf{t})) \sim \mathrm{Wishart}_m(\boldsymbol{\Sigma}(\mathbf{t}), \eta)$$

under the null hypothesis of no effect, $\mathbf{C}\beta = \mathbf{0}$. In Wilks's Λ (31), $\boldsymbol{\Sigma}(\mathbf{t})$ will cancel so under the null hypothesis, (31) will be a Wilks's Λ random field.

6.2 Brain shape analysis

As an illustration of the methods, we apply the P value approximations for Wilks's Λ to a data set on non-missile trauma (Tomaiuolo et al. 2005) that was analyzed in a similar way in Worsley et al. (2004). The subjects were 17 patients with non-missile brain trauma who were in a coma for 3–14 days. MRI images were taken after the trauma, and the multivariate data were the $N = 3$ component vector deformations needed to warp the MRI images to an atlas standard sampled on a 2 mm voxel lattice. The same data were also collected on a group of 19 age- and sex-matched controls.

Damage is expected in white mater areas, so the search region S was defined as the voxels where smoothed average control subject white matter density exceeded 5%. For calculating the intrinsic volumes, this was approximated by a sphere with the same volume, 1.31cc, which is slightly liberal for a non-spherical search region. The effective v from (6), averaged over the search region, was 0.125.

The first analysis was to look for brain damage by comparing the deformations of the 17 trauma patients with the 19 controls, so the sample size is $n = 36$. We are looking at a single contrast, the difference between trauma and controls, so $\eta = 1$ and the residual degrees of freedom is $\nu = 34$. In this case Wilks's Λ is Hotelling's T^2. The $P = 0.05$ threshold, found using the EC densities in Cao and Worsley (1999a) and by equating (7) to 0.05 and solving for y, was $y = 54.0$. The thresholded data, together with the estimated contrast (mean trauma–control deformations) is shown in Fig. 3a. A large region near the corpus callosum seems to be damaged. The nature of the damage, judged by the direction of the arrows, is away from the center (see Fig. 3b). This can be interpreted as expansion of the ventricles, or more likely, atrophy of the surrounding white matter, which causes the ventricle/white matter boundary to move outwards.

We might also be interested in functional anatomical connectivity: are there any regions of the brain whose shape (as measured by the deformations) is correlated with shape at a reference point? In other words, the explanatory variables are the deformations at a pre-selected reference point, and the test statistic is Wilks's Λ. We chose as the reference the point with maximum Hotelling's T^2 for damage, marked by axis lines in Fig. 3. Figure 3c shows the resulting $-\log$ Wilks's Λ field above the $P = 0.05$ threshold of 1.56 for the combined trauma and control data sets removing separate means for both groups ($\nu = 31, \eta = 3$). Obviously there is strong correlation with points near the reference, due to the smoothness of the data. The main feature is the strong correlation with contralateral points, indicating that brain anatomy tends to be symmetric.

A more interesting question is whether the correlations observed in the control subjects are modified by the trauma (Friston et al. 1997). In other words, is there any evidence for an interaction between group and reference vector deformations? To do this, we simply add another three covariates to the linear model whose values are the reference vector deformations for the trauma patients, and the negative of the reference vector deformations for the control subjects. The resulting $-\log$ Wilks's Λ field for testing for these three extra covariates, thresholded at 1.75 ($P = 0.05, \eta = 3, \nu = 28$) is shown in Fig. 3d. Apart from changes in the neighborhood of the reference point, there is some evidence of a change in correlation at a location in the contralateral side, slightly anterior. Looking at the maximum canonical correlations in the two groups separately, we find that correlation has increased at this location from 0.719 to 0.927, perhaps indicating that the damage is strongly bilateral. These results are in close agreement with those in Worsley et al. (2004) which used Roy's maximum root rather than Wilks's Λ.

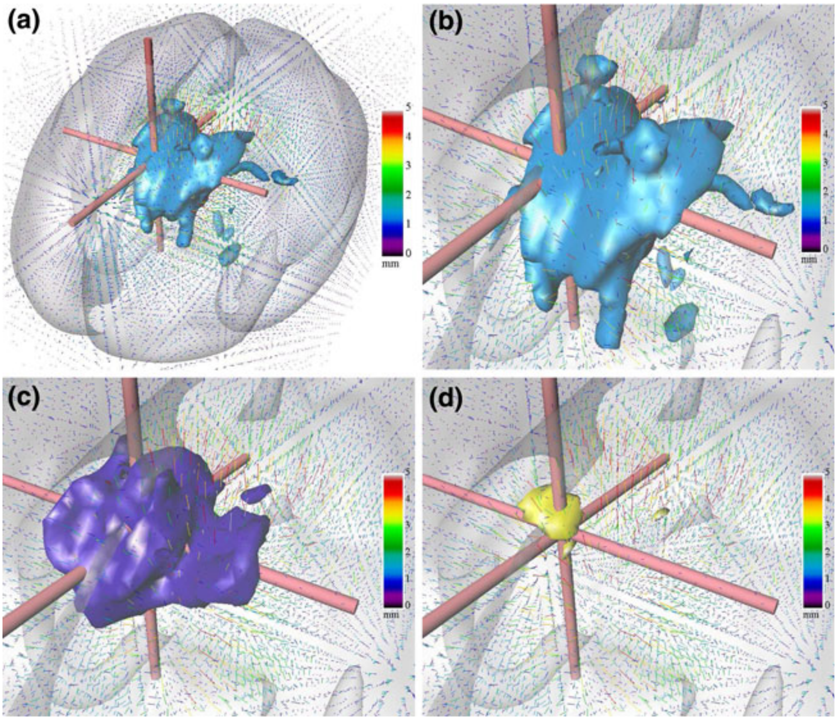

Fig. 3 Shape analysis of non-missile brain trauma data. **a** Trauma minus control average deformations (*arrows* and *color bar*), sampled every 6 mm inside the brain, with Hotelling's T^2 field for significant group differences (threshold $y = 54.0$, $P = 0.05$). The reference point of maximum Hotelling's T^2 is marked by the intersection of the three axes. **b** Closeup of **a** showing that the damage is an outward movement of the anatomy, either due to swelling of the ventricles or atrophy of the surrounding white matter. **c** Regions of effective anatomical connectivity with the reference point, assessed by the Wilks's Λ field (threshold $y = 1.56$, $P = 0.05$). The reference point is connected with its neighbors (due to smoothness) and with contralateral regions (due to symmetry). **d** Regions where the connectivity is different between trauma and control groups, assessed by the Wilks's Λ field (threshold $y = 1.75$, $P = 0.05$). The small region in the contralateral hemisphere (*right*) is more correlated in the trauma group than the control group

A Appendix

Lemma 5 (Lemma A.2 in Worsley (1994)) *Let* $\mathbf{H} \sim Normal_{j\times j}(\mathbf{0}, M(\mathbf{I}_j))$, *let* h *be a fixed scalar, and let* $\mathbf{A}$ *be a fixed symmetric* $j \times j$ *matrix. Then*

$$\mathbb{E}(\det(\mathbf{A} + h\mathbf{H})) = \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^i (2i)!}{2^i i!} h^{2i} \operatorname{detr}_{j-2i}(\mathbf{A}).$$

Lemma 6 *Let*

$$\mathbf{P} \sim \text{Wishart}_j(\nu, \mathbf{I}_j), \quad \mathbf{Q} \sim \text{Wishart}_j(\eta, \mathbf{I}_j),$$
$$\mathbf{X}_1 \sim \text{Normal}_{j\times m}(\mathbf{0}, \boldsymbol{\Sigma} \otimes \mathbf{I}_j), \quad \mathbf{X}_2 \sim \text{Normal}_{j\times m}(\mathbf{0}, \mathbf{I}_{jm})$$

independently, a, b, c be fixed scalars, and $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$ be fixed $m \times m$ matrices. Then

$$\begin{aligned}
&\mathbb{E}(\mathrm{detr}_i(a\mathbf{P} + b\mathbf{Q} + c\mathbf{I}_j + \mathbf{X}_1\mathbf{C}_1\mathbf{X}_1' + \mathbf{X}_1\mathbf{C}_2\mathbf{X}_2' + \mathbf{X}_2\mathbf{C}_3\mathbf{X}_1' + \mathbf{X}_2\mathbf{C}_4\mathbf{X}_2')) \\
&= \sum_{k=0}^{i} \frac{j!}{(j-i+k)!} \sum_{r=0}^{i-k} \binom{\nu}{i-k-r} \binom{\eta}{r} a^{i-k-r} b^r \sum_{l=0}^{k} \binom{j-l}{k-l} c^{k-l} \\
&\times \mathbb{E}(\mathrm{detr}_l(\mathbf{X}_1\mathbf{C}_1{\mathbf{X}_1}' + \mathbf{X}_1\mathbf{C}_2{\mathbf{X}_2}' + \mathbf{X}_2\mathbf{C}_3{\mathbf{X}_1}' + \mathbf{X}_2\mathbf{C}_4{\mathbf{X}_2}')).
\end{aligned}$$

Proof Holding $\mathbf{X}_1$ and $\mathbf{X}_2$ fixed and using Lemma in Worsley (1994) we obtain

$$\begin{aligned}
&\mathbb{E}(\mathrm{detr}_i(a\mathbf{P} + b\mathbf{Q} + c\mathbf{I}_j + \mathbf{X}_1\mathbf{C}_1\mathbf{X}_1' + \mathbf{X}_1\mathbf{C}_2\mathbf{X}_2' + \mathbf{X}_2\mathbf{C}_3\mathbf{X}_1' + \mathbf{X}_2\mathbf{C}_4\mathbf{X}_2')) \\
&= \sum_{k=0}^{i} \mathbb{E}(\mathrm{detr}_{i-k}(a\mathbf{P} + b\mathbf{Q}))\mathrm{detr}_k(c\mathbf{I}_j + \mathbf{X}_1\mathbf{C}_1\mathbf{X}_1' + \mathbf{X}_1\mathbf{C}_2\mathbf{X}_2' + \mathbf{X}_2\mathbf{C}_3\mathbf{X}_1' \\
&+\mathbf{X}_2\mathbf{C}_4\mathbf{X}_2') = \sum_{k=0}^{i} \frac{j!}{(j-i+k)!} \sum_{r=0}^{i-k} \binom{\nu}{i-k-r} \binom{\eta}{r} a^{i-k-r} b^r \\
&\times \mathrm{detr}_k(c\mathbf{I}_j + \mathbf{X}_1\mathbf{C}_1\mathbf{X}_1' + \mathbf{X}_1\mathbf{C}_2\mathbf{X}_2' + \mathbf{X}_2\mathbf{C}_3\mathbf{X}_1' + \mathbf{X}_2\mathbf{C}_4\mathbf{X}_2').
\end{aligned}$$

Taking expectations over $\mathbf{X}_1, \mathbf{X}_2$ in the equality above we have

$$\begin{aligned}
&\mathbb{E}(\mathrm{detr}_i(a\mathbf{P} + b\mathbf{Q} + c\mathbf{I}_j + \mathbf{X}_1\mathbf{C}_1\mathbf{X}_1' + \mathbf{X}_1\mathbf{C}_2\mathbf{X}_2' + \mathbf{X}_2\mathbf{C}_3\mathbf{X}_1' + \mathbf{X}_2\mathbf{C}_4\mathbf{X}_2')) \\
&= \sum_{k=0}^{i} \frac{j!}{(j-i+k)!} \sum_{r=0}^{i-k} \binom{\nu}{i-k-r} \binom{\eta}{r} a^{i-k-r} b^r \sum_{l=0}^{k} \binom{j-l}{k-l} c^{k-l} \\
&\times \mathbb{E}(\mathrm{detr}_l(\mathbf{X}_1\mathbf{C}_1\mathbf{X}_1' + \mathbf{X}_1\mathbf{C}_2\mathbf{X}_2' + \mathbf{X}_2\mathbf{C}_3\mathbf{X}_1' + \mathbf{X}_2\mathbf{C}_4\mathbf{X}_2')),
\end{aligned}$$

which ends the proof. □

Lemma 7 *Let*

$$\begin{aligned}
&\mathbf{P} \sim \mathrm{Wishart}_j(\nu, \mathbf{I}_j), \quad \mathbf{Q} \sim \mathrm{Wishart}_j(\eta, \mathbf{I}_j), \\
&\mathbf{X}_1 \sim \mathrm{Normal}_{j\times m}(\mathbf{0}, \boldsymbol{\Sigma} \otimes \mathbf{I}_j), \quad \mathbf{X}_2 \sim \mathrm{Normal}_{j\times m}(\mathbf{0}, \mathbf{I}_{jm}), \\
&\mathbf{H} \sim \mathrm{Normal}_{j\times j}(\mathbf{0}, M(\mathbf{I}_j))
\end{aligned}$$

independently, a, b, c, h be fixed scalars, and $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$ be fixed $m \times m$ matrices. Then

$$\mathbb{E}(\det(a\mathbf{P}+b\mathbf{Q}+c\mathbf{I}_j+\mathbf{X}_1\ \mathbf{C}_1\mathbf{X}_1'+\mathbf{X}_1\mathbf{C}_2\mathbf{X}_2'+\mathbf{X}_2\mathbf{C}_3\mathbf{X}_1'+\mathbf{X}_2\mathbf{C}_4\mathbf{X}_2'+h\mathbf{H}))$$
$$=\sum_{i=0}^{[\frac{j}{2}]}\frac{(-1)^i(2i)!}{2^i i!}h^{2i}\sum_{k=0}^{j-2i}\frac{j!}{(2i+k)!}\sum_{r=0}^{j-2i-k}\binom{\nu}{j-2i-k-r}$$
$$\times\binom{\eta}{r}a^{j-2i-k-r}b^r\sum_{l=0}^{k}\binom{j-l}{k-l}c^{k-l}$$
$$\times\ \mathbb{E}(\mathrm{detr}_l(\mathbf{X}_1\mathbf{C}_1\mathbf{X}_1{}'+\mathbf{X}_1\mathbf{C}_2\mathbf{X}_2{}'+\mathbf{X}_2\mathbf{C}_3\mathbf{X}_1{}'+\mathbf{X}_2\mathbf{C}_4\mathbf{X}_2{}')).$$

Proof Holding $\mathbf{P}, \mathbf{Q}, \mathbf{X}_1, \mathbf{X}_2$ fixed and using Lemma 5 we obtain

$$\mathbb{E}(\det(a\mathbf{P}+b\mathbf{Q}+c\mathbf{I}_j+\mathbf{X}_1\ \mathbf{C}_1\mathbf{X}_1'+\mathbf{X}_1\mathbf{C}_2\mathbf{X}_2'+\mathbf{X}_2\mathbf{C}_3\mathbf{X}_1'+\mathbf{X}_2\mathbf{C}_4\mathbf{X}_2'+h\mathbf{H}))$$
$$=\sum_{i=0}^{[\frac{j}{2}]}\frac{(-1)^i(2i)!}{2^i i!}h^{2i}\mathrm{detr}_{j-2i}(a\mathbf{P}+b\mathbf{Q}+c\mathbf{I}_j+\mathbf{X}_1\mathbf{C}_1\mathbf{X}_1'$$
$$+\mathbf{X}_1\mathbf{C}_2\mathbf{X}_2'+\mathbf{X}_2\mathbf{C}_3\mathbf{X}_1'+\mathbf{X}_2\mathbf{C}_4\mathbf{X}_2').$$

Now, taking expectations and using Lemma 6 we have the desired result. □

Lemma 8 *Let*

$$\mathbf{X}_1\sim\mathrm{Normal}_{j\times m}(\mathbf{0},\mathbf{\Sigma}\otimes\mathbf{I}_j),\quad \mathbf{X}_2\sim\mathrm{Normal}_{j\times m}(\mathbf{0},\mathbf{I}_{jm}),$$

and $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$ *be fixed symmetric* $m\times m$ *matrices.. Then*

$$\mathbb{E}(\det(\mathbf{X}_1\mathbf{C}_1\mathbf{X}_1{}'+\mathbf{X}_1\mathbf{C}_2\mathbf{X}_2{}'+\mathbf{X}_2\mathbf{C}_3\mathbf{X}_1{}'+\mathbf{X}_2\mathbf{C}_4\mathbf{X}_2{}'))$$
$$=\begin{cases}\mathrm{tr}(\mathbf{C}_1\mathbf{\Sigma}+\mathbf{C}_4), & j=1,\\ \mathrm{tr}(\mathbf{C}_1\mathbf{\Sigma}+\mathbf{C}_4)^2-\mathrm{tr}(\mathbf{C}_1\mathbf{\Sigma}\mathbf{C}_1\mathbf{\Sigma} & \\ +\mathbf{C}_4^2+\mathbf{C}_2\mathbf{C}_3\mathbf{\Sigma}+\mathbf{C}_3\mathbf{C}_2\mathbf{\Sigma}), & j=2.\end{cases}$$

Proof It follows from the fact that the rows of $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent vectors from $\mathrm{Normal}_m(\mathbf{0},\mathbf{\Sigma})$ and $\mathrm{Normal}_m(\mathbf{0},\mathbf{I}_m)$, respectively, and using the well-known identity

$$\mathbb{E}(x_ix_jx_kx_l)=\sigma_{ij}\sigma_{kl}+\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk},$$

$i, j, k, l = 1,\ldots,r$, where $\sigma_{ij}=\mathbb{E}(x_ix_j)$. □

A.1 Zonal polynomials

For any $m\times m$ symmetric matrix $\mathbf{B}$ and any multi-indices $\kappa=(k_1,k_2,\ldots,k_m)$, $k_1\geq k_2\geq\cdots\geq k_m$, denote by $\mathbf{C}_\kappa(\mathbf{B})$ the zonal polynomials corresponding to κ (see details in Muirhead 1982), defined by

$$\mathrm{tr}(\mathbf{B})^k=\sum_{|\kappa|=k}\mathbf{C}_\kappa(\mathbf{B}). \tag{33}$$

For any real a and natural k, define $(a)_k = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$, and for the multi-index $\kappa = (k_1, k_2, \dots, k_m)$,

$$(a)_\kappa = \prod_{i=1}^{m} \left(a - \frac{1}{2}(i-1)\right)_{k_i}.$$

For any real $a > \frac{1}{2}(m-1)$, $\Gamma_m(a)$ denotes the multivariate Gamma function,

$$\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(a - \frac{1}{2}(i-1)\right).$$

Lemma 9 *If* $\kappa = (k_1, \dots, k_m)$, $|\kappa| = k$ *then for* $a > k_1 + (m+1)/2$,

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}_m} \det(\mathbf{X})^{a-(m+1)/2} \det(\mathbf{I}_m - \mathbf{X})^{b-(m+1)/2} \mathbf{C}_\kappa(\mathbf{X}^{-1})\, d\mathbf{X}$$
$$= \frac{\left(-(a+b) + \frac{m+1}{2}\right)_\kappa}{\left(-a + \frac{m+1}{2}\right)_\kappa} \frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)} \mathbf{C}_\kappa(\mathbf{I}_m), \tag{34}$$

where

$$\mathbf{C}_\kappa(\mathbf{I}_m) = 2^{2k} k! \left(\frac{m}{2}\right)_\kappa \frac{\prod_{i<j}^{p}(2k_i - 2k_j - i + j)}{\prod_{i=1}^{p}(2k_i + p - i)!},$$

and p *denotes the nonzero indices in* κ.

Proof It is proved by using following similar ideas to that of Theorem 7.2.10 and Theorem 7.2.13 in Muirhead (1982). □

As a particular case we have that if $\mathbf{B} \sim Beta_m(\frac{1}{2}\nu, \frac{1}{2}\eta)$ then for $\nu > 2k_1 + m + 1$,

$$\mathbb{E}(\mathbf{C}_\kappa(\mathbf{B}^{-1})) = \frac{\left(-\frac{\nu+\eta}{2} + \frac{m+1}{2}\right)_\kappa}{\left(-\frac{\nu}{2} + \frac{m+1}{2}\right)_\kappa} \mathbf{C}_\kappa(\mathbf{I}_m). \tag{35}$$

A.2 Numerical computation of $c_i(y)$

According to results in Yeh (1974) and Zabell (1979) one has

$$\mathbb{E}(\mathbf{C}_\kappa(\mathbf{B}^{-1}) \mid \det(\mathbf{B}) = x) = \frac{\int_{-\infty}^{\infty} \mathbb{E}(C_\kappa(\mathbf{B}^{-1}) e^{i \det(\mathbf{B}) t}) e^{-ixt} dt}{2\pi f_{\det(\mathbf{B})}(x)},$$

where $f_{\det(\mathbf{B})}(x)$ denotes the probability density function of $\det(\mathbf{B})$ evaluated at x and $i = \sqrt{-1}$. Then, approximating $e^{i \det(\mathbf{B}) t}$ by its Taylor series expansion up to order r

it is obtained

$$\mathbb{E}\left(C_\kappa(\mathbf{B}^{-1})\mathrm{e}^{\mathrm{i}\det(\mathbf{B})t}\right) \approx \sum_{j=0}^{r} \frac{(it)^j}{j!} \mathbb{E}\left(C_\kappa(\mathbf{B}^{-1})\det(\mathbf{B})^j\right),$$

which, by (34) and (35) gives

$$\mathbb{E}\left(C_\kappa(\mathbf{B}^{-1})e^{\det(\mathbf{B})t}\right) \approx \frac{\left(-\frac{\nu+\eta}{2}+\frac{m+1}{2}\right)_\kappa}{\left(-\frac{\nu}{2}+\frac{m+1}{2}\right)_\kappa} \mathbf{C}_\kappa(\mathbf{I}_m) \sum_{j=0}^{r} \frac{(it)^j}{j!} p_m(j),$$

where

$$p_m(a) = \begin{cases} \frac{\Gamma_m(\frac{\nu}{2}+a)\Gamma_m(\frac{\nu+\eta}{2})}{\Gamma_m(\frac{\nu}{2})\Gamma_m(\frac{\nu+\eta}{2}+a)}, & m \geq 1 \\ 0, & m \leq 0. \end{cases}.$$

Hence,

$$\mathbb{E}\left(\mathbf{C}_\kappa(\mathbf{B}^{-1}) \mid \det(\mathbf{B}) = x\right) \approx \frac{\left(-\frac{\nu+\eta}{2}+\frac{m+1}{2}\right)_\kappa}{\left(-\frac{\nu}{2}+\frac{m+1}{2}\right)_\kappa} \frac{\mathbf{C}_\kappa(\mathbf{I}_m)}{2\pi f_{\det(\mathbf{B})}(x)} \overline{\int_{-\infty}^{\infty} \sum_{j=0}^{r} \frac{p_m(j)\overline{(it)^j}}{j!} \mathrm{e}^{\mathrm{i}xt} \mathrm{d}t},$$

where $\overline{a}$ denotes the conjugate of the complex number a. Finally, our approximation follows from taking the real part of the integral $\int_{-\infty}^{\infty} \sum_{j=0}^{r} \frac{p_m(j)\overline{(it)^j}}{j!} \mathrm{e}^{\mathrm{i}xt} \mathrm{d}t$, which is numerically approximated by inverse Discrete Fourier Transform (FFT) and taking into account that $x = e^{-y}$.

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