BAYESIAN SOLUTIONS TO AN OPTIMUM ALLOCATION PROBLEM IN STRATIFIED TWO-PHASE SAMPLING WHEN AN OVERALL WITHIN VARIANCE IS TO BE ESTIMATED

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1. Introduction

The framework of the problem that we address ourselves to in this paper is the following. Suppose a population has been divided into H strata, and that the distribution of a certain characteristic of interest of items belonging to the hth stratum, $h=1, \cdots, H$, is $N(\mu_h, \sigma_h^2)$, where both μ_h and σ_h^2 are unknown. Let the proportion of the population that lies in the hth stratum be denoted by π_h, $0<\pi_h<1$, $\sum_{h=1}^{H} \pi_h=1$. Suppose interest is in estimating the overall within variance

$$\sigma^2=\sum_{h=1}^{H} \pi_h \sigma_h^2 , \tag{1.1}$$

and that there is available a total budget of C (C is in monetary units) to carry out the following two-phase sampling scheme. (We denote the cost per observation of sampling in stratum h by c_h.)

For the first phase, it is agreed to expend $100\alpha\%$ of the available budget $(0<\alpha<1)$, and n_h independent observations $(x_{h1}, \cdots, x_{hn_h})=\boldsymbol{x}_h'$ are taken from stratum h, $h=1, \cdots, H$, with $\boldsymbol{x}_h$ independent of $\boldsymbol{x}_k$, all $h\neq k$, and

$$\sum_{h=1}^{H} c_h n_h=\alpha C . \tag{1.2}$$

In the second phase, we "repeat" the above procedure, that is, we select N_h independent observations $(y_{h1}, \cdots, y_{hN_h})=\boldsymbol{y}_h'$ from stratum h, with $\boldsymbol{y}_h$ independent of $\boldsymbol{y}_k$ and all $\boldsymbol{y}_h$ independent of all $\boldsymbol{x}_h$, and such that

$$\sum_{h=1}^{H} c_h N_h=(1-\alpha)C . \tag{1.3}$$

The problem that we wish to solve may now be stated as follows.

Suppose we are given the results of the first phase sampling. How then can we "best" choose the N_h, where the N_h are subject to the condition (1.3)? By "best", we mean that an estimate of a parameter of interest (e.g., (1.1)), should have as small a variance as possible. The answer to this question depends on the method of attack and in this paper we use two methods or approaches, which are Bayesian in character.

This problem of optimally (or best) allocating the N_h, is an old, honored and much discussed problem (see for example, Neyman [7], [8], Cochran [1], Ericson [4], and Draper and Guttman [2], [3], amongst others). Much of the discussion in the literature, however, is given over to the optimal allocation when the overall population mean $\mu=\sum_{h=1}^{H}\pi_h\mu_h$ is of primary interest.

However, it is frequently the case that information about σ^2, defined by (1.1), is desired—in fact, very often, separately budgeted pilot studies are carried out to gain information about σ^2, followed by a separate investigation about μ. With the assumption then, that σ^2 is of primary interest, we proceed to attack the problem of optimally allocating the N_h. Indeed, the methods of attacking this problem are Bayesian, and follow the methods of Draper and Guttman [2]. The latter were interested in the allocation problem when μ is of primary interest, so that we will be in a position to contrast the "optimum allocation for μ "with the" optimal allocation for σ^2."

2. Optimum allocation for the overall within variance

We turn our attention, then, to the case where $\sigma^2=\sum_{h=1}^{H}\pi_h\sigma_h^2$ is of primary interest. We proceed as in Draper and Guttman [2], and suppose first that the N_h are known, and the samples $\{\boldsymbol{x}_h\}$ and $\{\boldsymbol{y}_h\}$ have been observed. When this is so, then we have that the two-phase posterior of σ^2 is such that

$$\text{(2.1)} \qquad \mathrm{E}[\sigma^2|\{\boldsymbol{x}_h\},\{\boldsymbol{y}_h\}]=\sum_{h=1}^{H}\pi_h\,\mathrm{E}[\sigma_h^2|\boldsymbol{x}_h,\boldsymbol{y}_h]\,.$$

But from (A.13b) of the Appendix in Guttman [5], we have that

$$\text{(2.2)} \qquad \mathrm{E}[\sigma^2|\{\boldsymbol{x}_h\},\{\boldsymbol{y}_h\}]=\sum_{h=1}^{H}\frac{\pi_h SS_h}{N_h+n_h-3}\,,$$

where

$$\text{(2.2a)} \qquad SS_h=(n_h-1)s_h^2+(N_h-1)w_h^2+n_hN_h(\bar{y}_h-\bar{x}_h)^2/(n_h+N_h)\,,$$

with

(2.2b)
$$(n_h-1)s_h^2=\sum_{j=1}^{n_h}(x_{hj}-\bar{x}_h)^2\,,\qquad (N_h-1)w_h^2=\sum_{j=1}^{N_h}(y_{hj}-\bar{y}_h)^2$$
$$\bar{x}_h=n_h^{-1}\sum_{j=1}^{n_h}x_{hj}\,,\qquad \bar{y}_h=N_h^{-1}\sum_{j=1}^{N_h}y_{hj}\,.$$

Hence, (2.1) may be written as

(2.3)
$$\mathrm{E}\,[\sigma^2|\{\boldsymbol{x}_h\},\,\{\boldsymbol{y}_h\}]=\sum_{h=1}^{H}\frac{\pi_h SS_h}{N_h+n_h-3}\,.$$

But (2.3) is a function of the yet unobserved y's, through the SS_h's, and has preposterior variance given by

(2.4)
$$\mathrm{V}_{\{\boldsymbol{y}_h\}|\{\boldsymbol{x}_h\}}\{\mathrm{E}\,[\sigma^2|\{\boldsymbol{x}_h\},\,\{\boldsymbol{y}_h\}]\}=\sum_{h=1}^{H}\frac{\pi_h^2}{(N_h+n_h-3)^2}\mathrm{V}(SS_h|\boldsymbol{x}_h)\,.$$

Now it is shown in (A.33) of the Appendix of Guttman [5] that

(2.5)
$$\mathrm{V}(SS_h|\boldsymbol{x}_h)=2N_h(N_h+n_h-3)(u_h^2)^2/(n_h-5)$$

where

(2.5a)
$$u_h^2=(n_h-1)s_h^2/(n_h-3)\,,$$

so that (2.4) may be written as

(2.6)
$$2\sum_{h=1}^{H}\frac{N_h\pi_h^2(u_h^2)^2}{(N_h+n_h-3)(n_h-5)}=2\sum_{h=1}^{H}\frac{\pi_h^2(u_h^2)^2}{n_h-5}-2\sum_{h=1}^{H}\frac{(n_h-3)\pi_h^2(u_h^2)^2}{(n_h-5)(N_h+n_h-3)}\,.$$

Hence, maximizing (2.4) is equivalent to minimizing

(2.7)
$$2\sum_{h=1}^{H}\frac{(n_h-3)\pi_h^2(u_h^2)^2}{(n_h-5)(N_h+n_h-3)}\,.$$

Using a Lagrange multiplier to incorporate the restriction $\sum_{h=1}^{H}c_hN_h=(1-\alpha)C$, we arrive at the optimum selection of the N_h, given by

(2.8)
$$N_{h,v}=\frac{C-3\sum_{h=1}^{H}c_h}{c_h}q_{h,v}-(n_h-3)$$

where, interestingly

(2.9)
$$q_{h,v}=\pi_hu_h'^2c_h^{1/2}\Big/\sum_{h=1}^{H}\pi_hu_h^{2\prime}c_h^{1/2}\,,$$

with $u_h'^2=u_h^2\sqrt{(n_h-3)/(n_h-5)}$. We are here tacitly assuming that the total budget, C, allowed for the two-phase investigation is such that

$C>3\sum_{h=1}^{H} c_h$. For later purposes, however, we assume here that $(1-\alpha)C >3\sum_{h=1}^{H} c_h$.

Notice the form of $q_{h,v}$ of (2.9)—it says that the "best" allocation of the N_h for a variance involves $q_{h,v}$, which in turn involves previous information about a variance, specificially, $u_h'^2=u_h^2\sqrt{(n_h-3)/(n_h-5)}$.

The reader may have noticed that the allocation formula (2.8)-(2.9) may produce some negative results. We handle this by using a procedure due to Draper and Guttman [2], as follows.

Let the set of indices for which $N_{h,v}$ are positive be denoted by $(+)$, and the set of indices for which $N_{h,v}$ are negative or zero be denoted by $(+)^c$. Let $N'_{h,v}$ denote the second allocation. Then

$$N'_{h,v}=0 \quad \text{if} \quad h\in(+)^c, \tag{2.10}$$

and (2.10) implies that we will take no more observations from groups for which $N_{h,v}\leqq 0$. This in term implies that we should allocate the remaining observations to the remaining groups, that is, those belonging to $(+)$. If we now minimize (2.7), subject to $\sum_{(+)} c_h N_h=(1-\alpha)C$, where $N_h>0$, we find

$$N'_{h,v}=\frac{[(1-\alpha)C+\sum_{(+)} n_h c_h-3\sum_{(+)} c_h]}{c_h}q'_{h,v}-(n_h-3) \quad \text{if} \quad h\in(+) \tag{2.11}$$

where

$$q'_{h,v}=\frac{\pi_h u_h^{2\prime} c_h^{1/2}}{\sum_{h\in(+)} \pi_h u_h^{2\prime} c_h^{1/2}}, \quad \text{with} \quad u_h^{2\prime}=u_h^2\sqrt{(n_h-3)/(n_h-5)}. \tag{2.11a}$$

Again, if any of the $N'_{h,v}$ are negative or zero, for $h\in(+)$, we would set $N''_{h,v}=0$ and then reallocate for the groups that still remain, i.e., which have positive $N_{h,v}$ and $N'_{h,v}$. Eventually, this procedure terminates with some zeroes and with some positive values for the N's, and this procedure minimizes (2.7), subject to the constraints

$$N_h\geqq 0, \quad \sum_h c_h N_h=(1-\alpha)C, \tag{2.12}$$

as may be proved via a theorem of Kuhn and Tucker [6].

3. Preposterior estimators

From the previous section, once the optimal $N_{h,v}$ are known, we would take the second phase samples $\boldsymbol{y}_h$ of size $N_{h,v}$, $h\in(+)$, and compute (2.3) which would take the form

$$\text{(3.1)} \qquad E[\sigma^2 | \{x_h\}, \{y_h\}] = \sum_{h \in (+)} \frac{\pi_h SS_h}{N_{h,v} + n_h - 3} + \sum_{h \in (+)} \pi_h u_h^2 .$$

This is the posterior expectation of the overall within variance σ^2, given all the observations, and is the Bayes estimator of σ^2 which we would quote. Note that this estimator is a function of the $\{x_h\}$.

A different sort of estimator may also be considered. This type of estimator is discussed by Raiffa and Schlaifer ([9], p. 104). We have, using (A.7b) of the Appendix in Guttman [5], that posterior to the first phase sampling.

$$\text{(3.2)} \qquad E[\sigma^2 | \{x_h\}] = \sum_{h=1}^{H} \pi_h u_h^2 .$$

Suppose we have not as yet performed the second phase sampling. Using the result (A.24a) of Guttman [5], it is easy to see that

$$\text{(3.3)} \qquad E\left[\sum_{h=1}^{H} \pi_h w_h^2 | \{x_h\} \right] = \sum_{h=1}^{H} \pi_h u_h^2 .$$

That is both $\sum_{h=1}^{H} \pi_h \sigma_h^2$, a *posteriori* after the first phase, and $\sum_{h=1}^{H} \pi_h w_h^2$, *preposteriori* before the second phase, have expectation $\sum_{h=1}^{H} \pi_h u_h^2$, so that $\sum_{h=1}^{H} \pi_h w_h^2$ is a natural preposterior estimator of $\sigma^2 = \sum \pi_h \sigma_h^2$. This estimator of σ^2 has the virtue that it only uses the second phase sampling—however, as in the previous section, we may utilize the first phase sampling to give us the *optimal* sample sizes necessary for the second phase sampling, as follows.

We have that the preposterior variance of our natural preposterior estimator is

$$\text{(3.4)} \qquad V\left[\sum_{h=1}^{H} \pi_h w_h^2 | \{x_h\} \right] = \sum_{h=1}^{H} \pi_h^2 V[w_h^2 | \{x_h\}] ,$$

which, from (A.24b) of Guttman [5], may be written as

$$\text{(3.5)} \qquad 2 \sum_{h=1}^{H} \frac{\pi_h^2 (u_h^2)^2}{n_h - 5} \left[1 + \frac{n_h - 3}{N_h - 1} \right] .$$

If we now minimize (3.5), subject to the constraint $\sum_{h=1}^{H} c_h N_h = (1-\alpha)C$, we find the allocation given by

$$\text{(3.6)} \qquad N_{h,v}^* = \frac{\left[(1-\alpha)C - \sum_{h=1}^{H} c_h \right]}{c_h} q_{h,v} + 1$$

where $q_{h,v}$ has been defined in (2.9). It is interesting to note that the allocation (3.6) always gives positive numbers (recall, that we have made the assumption that C is such that $(1-\alpha)C>3\sum_{h=1}^{H} c_h)$, whereas the allocation (2.8) could lead to negative numbers and subsequent re-allocations as outlined in Section 2.

4. Some numerical results

We now illustrate the results obtained in this paper for the optimum allocation when σ^2 is of primary interest. While we are at it, we will contrast these results for the optimum allocation when μ is of primary interest. We will let $H=6$, $c_h=1$, and $\sum c_h(n_h+N_h)=240$. Now once the values α, n_h, π_h, u_h are available, the results follow immediately. We shall denote groups of n_h by letters A, B, C as shown in Table 1, groups of π_h by α', β, γ, ε, ϕ, η as shown in Table 2, and groups of u_h used by a, b, c, d, e, f, g as shown in Table 3.

Table 1. Groups of n_h used.

α	n_1	n_2	n_3	n_4	n_5	n_6	Notation
0.50	20	20	20	20	20	20	A
0.50	10	10	10	30	30	30	B
0.25	6	6	6	6	18	18	C

Table 2. Groups of π_h used.

π_1	π_2	π_3	π_4	π_5	π_6	Notation
1/6	1/6	1/6	1/6	1/6	1/6	α'
1/21	2/21	3/21	4/21	5/21	6/21	β
6/21	5/21	4/21	3/21	2/21	1/21	γ
1/18	1/18	1/18	5/18	5/18	5/18	ε
5/18	5/18	5/18	1/18	1/18	1/18	ϕ
3/12	2/12	1/12	1/12	2/12	3/12	η

Table 3. Groups of u_h used.

u_1	u_2	u_3	u_4	u_5	u_6	Notation
6	5	4	3	2	1	a
1	2	3	4	5	6	b
4	4	4	3	3	3	c
3	3	3	4	4	4	d
1	1	4	4	16	16	e
16	16	4	4	1	1	f
16	4	1	1	4	16	g

In what follows, $N_{h,v}$ denotes the values arising from (2.8); if some of these are zero or negative, $N'_{h,v}$, $N''_{h,v}$ etc. denote values arising from the procedure implicit in (2.11). Similarly, $N_{h,m}$ denotes the optimal values of N_h when the overall mean $\mu=\sum_{h=1}^{H}\pi_h\pi_h$ is of interest, and, as found by Draper and Guttman [2], is given by

$$N_{h,m}=\frac{C}{c_h}q_{h,m}-n_h \tag{4.1}$$

with

$$q_{h,m}=\pi_h u_h c_h^{1/2}\Big/\sum_{h=1}^{H}\pi_h u_h c_h^{1/2}\,. \tag{4.1a}$$

The results (4.1)–(4.1a) were found by minimizing

$$\sum_{h=1}^{H}\frac{\pi_h^2 u_h^2}{n_h+N_h}\quad\text{subject to}\quad\sum_{h=1}^{H}c_h N_c=(1-\alpha)C\,. \tag{4.2}$$

If some of the $N_{h,m}$ given by (4.1) are zero or negative, $N'_{h,m}$, $N''_{h,m}$ etc., denote the values arising from the appropriate re-allocation procedure, and are given by, for example

$$\begin{aligned} N'_{h,m}&=0 \qquad \text{if}\quad h\in(+)^c \\ N'_{h,m}&=\frac{[(1-\alpha)C+\sum_{(+)}n_h c_h]}{c_h}q_{h,m}-n_h \qquad \text{if}\quad h\in(+) \end{aligned} \tag{4.3}$$

where $q_{h,m}$ is defined in (4.1a). The details of the derivation of these results are given in Draper and Guttman [2]. Also $N^*_{h,v}$ denotes the values arising from (3.6), and, finally, $N^*_{h,m}$ denotes the values arising from optimally allocating for the mean using a preposterior analysis, and, as found in Draper and Guttman [2], is given by

$$N^*_{h,m}=\frac{(1-\alpha)C}{c_h}q_{h,m} \tag{4.4}$$

where $q_{h,m}$ is defined in (4.1a). Of course, the values $N^*_{h,v}$ and $N^*_{h,m}$ are always positive. The result (4.4) is obtained by minimizing

$$\sum_{h=1}^{H}\frac{\pi_h^2 u_h^2}{n_h}\left(1+\frac{n_h}{N_h}\right),\qquad\text{subject to}\quad\sum_{h=1}^{H}c_h N_h=(1-\alpha)C\,. \tag{4.5}$$

To keep Table 4 of manageable size, we tabulate the results for only 12 of the 126 possible conditions obtainable from Tables 1–3; the complete set is available from the author.

Table 4. Calculated sample sizes for optimum allocation when (i) the variance σ^2 is of primary interest and (ii) when the mean μ is of primary interest, using a Bayesian approach (N, N' etc.) and a preposterior analysis (N^*), for some conditions of Tables 1-3.

$A\ \phi\ e$	$N_{h,v}$	−15.20	−15.20	11.74	−11.25	74.96	74.96
	$N'_{h,v}$	0.00	0.00	6.11	0.00	56.95	56.95
	$N_{h,m}$	− 1.82	− 1.82	52.73	− 5.45	38.18	38.18
	$N'_{h,m}$	0.00	0.00	49.23	0.00	35.38	35.38
	$N^*_{h,v}$	1.92	1.92	15.76	3.95	48.22	48.22
	$N^*_{h,m}$	9.09	9.09	36.36	7.27	29.09	29.09
$A\ \phi\ g$	$N_{h,v}$	156.48	− 6.16	−16.32	−16.86	−14.83	17.70
	$N'_{h,v}$	111.33	0.00	0.00	0.00	0.00	8.67
	$N_{h,m}$	132.38	18.10	−10.47	−18.10	−12.38	10.48
	$N'_{h,m}$	104.14	11.03	0.00	0.00	0.00	4.83
	$N^*_{h,v}$	90.08	6.57	1.35	1.07	2.11	18.82
	$N^*_{h,m}$	76.19	19.05	4.76	.95	3.81	15.24
$A\ \eta\ e$	$N_{h,v}$	−16.49	−16.66	−14.30	−14.30	69.31	112.46
	$N'_{h,v}$	0.00	0.00	0.00	0.00	44.60	75.40
	$N_{h,m}$	−12.26	−14.84	− 9.68	− 9.68	62.58	103.87
	$N'_{h,m}$	0.00	0.00	0.00	0.00	44.00	76.00
	$N^*_{h,v}$	1.26	1.17	2.38	2.38	45.32	67.48
	$N^*_{h,m}$	3.87	2.58	5.16	5.16	41.29	61.94
$A\ \eta\ g$	$N_{h,v}$	89.43	−12.57	−16.86	−16.86	−12.57	89.43
	$N'_{h,v}$	60.00	0.00	0.00	0.00	0.00	60.00
	$N_{h,m}$	81.05	− 3.16	−17.89	−17.89	− 3.16	81.05
	$N'_{h,m}$	60.00	0.00	0.00	0.00	0.00	60.00
	$N^*_{h,v}$	55.65	3.28	1.07	1.07	3.28	55.65
	$N^*_{h,m}$	50.53	8.42	1.05	1.05	8.42	50.53
$B\ \phi\ e$	$N_{h,v}$	− 5.00	− 5.00	25.07	−21.37	63.14	63.14
	$N'_{h,v}$	0.00	0.00	20.34	0.00	49.83	49.83
	$N_{h,m}$	8.18	8.18	62.73	−15.45	28.18	28.18
	$N'_{h,m}$	6.94	6.94	57.74	0.00	24.19	24.19
	$N^*_{h,v}$	2.03	2.03	17.47	3.89	47.29	47.29
	$N^*_{h,m}$	9.09	9.09	36.36	7.27	29.09	29.09
$B\ \phi\ g$	$N_{h,v}$	170.07	4.07	− 6.31	−26.88	−25.06	4.10
	$N'_{h,v}$	123.03	1.13	0.00	0.00	0.00	− 4.16
	$N''_{h,v}$	119.12	.88	0.00	0.00	0.00	0.00
	$N_{h,m}$	142.38	28.10	− .48	−28.10	−22.38	.48
	$N'_{h,m}$	107.24	19.31	0.00	0.00	0.00	6.55
	$N''_{h,m}$	102.00	18.00	0.00	0.00	0.00	0.00
	$N^*_{h,v}$	91.93	6.68	1.36	1.06	2.00	16.97
	$N^*_{h,m}$	76.19	19.05	4.76	.95	3.81	15.24

$B\ \eta\ e$	$N_{h,v}$	− 6.43	− 6.62	− 3.94	−24.31	59.11	102.17
	$N'_{h,v}$	0.00	0.00	0.00	0.00	42.60	77.40
	$N_{h,m}$	− 2.26	− 4.84	.32	−19.68	52.58	93.87
	$N'_{h,m}$	0.00	0.00	− .95	0.00	42.38	78.57
	$N''_{h,m}$	0.00	0.00	0.00	0.00	42.00	78.00
	$N^*_{h,v}$	1.29	1.20	2.57	2.38	45.22	67.33
	$N^*_{h,m}$	3.87	2.58	5.16	5.16	41.29	61.94
$B\ \eta\ g$	$N_{h,v}$	106.32	− 2.28	− 6.85	−26.87	−22.85	72.53
	$N'_{h,v}$	74.99	0.00	0.00	0.00	0.00	45.01
	$N_{h,m}$	91.05	6.84	− 7.89	−27.89	−13.16	71.05
	$N'_{h,m}$	68.46	3.08	0.00	0.00	0.00	48.46
	$N^*_{h,v}$	59.19	3.42	1.08	1.07	3.13	52.11
	$N^*_{h,m}$	50.53	8.42	1.05	1.05	8.42	50.53
$C\ \phi\ e$	$N_{h,v}$	− .38	− .38	38.93	5.39	68.22	68.22
	$N'_{h,v}$	0.00	0.00	38.79	5.36	67.93	67.93
	$N_{h,m}$	12.18	12.18	66.73	8.55	40.18	40.18
	$N^*_{h,v}$	3.05	3.05	33.87	7.57	66.23	66.23
	$N^*_{h,m}$	13.64	13.64	54.55	10.91	43.64	43.64
$C\ \phi\ g$	$N_{h,v}$	182.16	8.57	− 2.28	− 2.86	−13.56	7.97
	$N'_{h,v}$	166.40	7.59	0.00	0.00	0.00	6.01
	$N_{h,m}$	146.38	32.10	3.52	− 4.10	−10.38	12.48
	$N'_{h,m}$	136.81	29.70	2.93	0.00	0.00	10.56
	$N^*_{h,v}$	146.12	10.07	1.57	1.11	2.13	19.00
	$N^*_{h,m}$	114.29	28.57	7.14	1.43	5.71	22.86
$C\ \eta\ e$	$N_{h,v}$	− 2.20	− 2,47	1.28	1.28	69.85	112.27
	$N'_{h,v}$	0.00	0.00	1.18	1.18	68.05	109.58
	$N_{h,m}$	1.74	− .84	4.32	4.32	64.58	105.87
	$N'_{h,m}$	1.71	0.00	4.29	4.29	64.29	105.43
	$N^*_{h,v}$	1.63	1.42	4.35	4.35	67.50	100.75
	$N^*_{h,m}$	5.81	3.87	7.74	7.74	61.94	92.90
$C\ \eta\ g$	$N_{h,v}$	128.34	2.47	− 2.83	− 2.83	−11.61	66.45
	$N'_{h,v}$	117.95	2.04	0.00	0.00	0.00	60.01
	$N_{h,m}$	95.05	10.84	− 3.89	− 3.89	− 1.16	83.05
	$N'_{h,m}$	90.92	10.15	0.00	0.00	0.00	78.92
	$N^*_{h,v}$	103.94	5.29	1.13	1.13	3.66	64.84
	$N^*_{h,m}$	75.79	12.63	1.58	1.58	12.63	75.79

A comparison of the final allocations for $N_{h,m}$ and $N_{h,v}$ may be made as follows. Very often, one finds in the literature the statement that the optimal allocation for the mean is very flat. To test this, we have computed some relative efficiencies, specifically (see (4.2))

$$\text{R.E.}\,[\{N_{h,v}\} : \{N_{h,m}\}] = \sum_{h=1}^{H} \frac{\pi_h^2 u_h^2}{N_{h,m}+n_h} \Big/ \sum_{h=1}^{H} \frac{\pi_h^2 u_h^2}{N_{h,v}+n_h} \tag{4.6}$$

where the $\{N_{h,m}\}$ and $\{N_{h,v}\}$ are the "final" allocations determined by (4.1)–(4.1a) or (4.3), and (2.8)–(2.9) or (2.11), respectively. Of course, the numerator of (4.6) is the minimum of $\sum_{h=1}^{H} \frac{\pi_h^2 u_h^2}{N_h+n_h}$, over all choices of N_h, subject to $\sum_{h=1}^{H} c_h N_h=(1-\alpha)C$. Also, we have computed (see (4.5))

$$\text{R.E.}\,[\{N_{h,v}^*\} : \{N_{h,m}^*\}] = \sum_{h=1}^{H} \frac{\pi_h^2 u_h^2}{n_h}\left(1+\frac{n_h}{N_{h,m}^*}\right) \Big/ \sum_{h=1}^{H} \frac{\pi_h^2 u_h^2}{n_h}\left(1+\frac{n_h}{N_{h,v}^*}\right) . \tag{4.7}$$

We have computed these relative efficiencies for all possible 126 conditions, but in Table 5 below, only list the values corresponding to the conditions used in Table 4.

Table 5. Efficiencies in percent.

	(4.6)	(4.7)		(4.6)	(4.7)
$A\ \phi\ e$	72.49	74.11	$B\ \eta\ e$	99.99	95.88
$A\ \phi\ g$	96.36	92.36	$B\ \eta\ g$	99.10	96.40
$A\ \eta\ e$	99.99	96.54	$C\ \phi\ e$	73.69	90.41
$A\ \eta\ g$	100.00	95.72	$C\ \phi\ g$	83.50	97.40
$B\ \phi\ e$	74.96	79.07	$C\ \eta\ e$	98.56	97.69
$B\ \phi\ g$	84.79	95.41	$C\ \eta\ g$	92.37	97.26

It turns out that for the efficiencies (4.6) and (4.7), and that of the 126 conditions that may obtain using Tables 1–3, that the range of values is 70–100%, but that most lie in the 90–100% range. An inspection of Table 5, with a glance at Tables 1–3, seems to indicate that the 'worst' conditions are those for which discrepant π_h's and discrepant u_h's (the latter, discrepant in the opposite direction) are encountered—e.g., ϕ is the condition 5/18, 5/18, 5/18, 1/18, 1/18, 1/18, while e is the condition 1, 1, 4, 4, 16, 16. Notice the efficiencies when ϕ and e are together—they are the 'lowest' obtained. Note too that the lowest are "all that not bad", and in general, for a wide variety of conditions, an optimal allocation dictated by interest in the overall within variance seems not too inappropriate for use when μ is also of interest, so that a recommendation to use $\{N_{h,v}\}$ seems well advised as a multi-purpose design.

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