GENERATING DOUBLY EXPONENTIAL RANDOM NUMBERS

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Summary

A useful technique for computer transformation of random numbers from the uniform distribution into those from a given probability distribution was devised Marsaglia [5]. In this note we apply his technique to the transformation into doubly exponential random numbers. Remarks on the other two extreme value distributions are included.

1. The technique and the extreme value distributions

Let N be a positive integer random variable (r.v.) with probability generating function $g(s)$, and let Z be a r.v. with distribution function $F(z)$. Then the extreme values of a sample of random size N,

$$Y_1=\min(Z_1,\cdots,Z_N) \quad \text{and} \quad Y_2=\max(Z_1,\cdots,Z_N),$$

have the following distribution functions respectively:

$$\text{(1.1)} \qquad \begin{aligned} &\mathrm{P}(Y_1\leqq y)=1-g(1-F(y)),\\ &\mathrm{P}(Y_2\leqq y)=g(F(y)). \end{aligned}$$

As a special case, if N is the zero-truncated Poisson r.v. with parameter ν; that is,

$$\text{(1.2)} \qquad \mathrm{P}(N=n)=(e^{\nu}-1)^{-1}\nu^n/n!, \qquad n=1,2,\cdots,$$

then

$$\text{(1.3)} \qquad \begin{aligned} &\mathrm{P}(Y_1\leqq y)=(1-e^{-\nu F(y)})/(1-e^{-\nu}),\\ &\mathrm{P}(Y_2\leqq y)=(e^{\nu F(y)}-1)/(e^{\nu}-1). \end{aligned}$$

Further, if Z's are uniform r.v.'s, we have

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LEMMA 1. *$(U_1, \cdots, U_N)$ is a random sample from the* (0, 1) *uniform distribution of size N, which is the r.v. with probabilities* (1.2). *Then*

$$Y = \nu \min (U_1, \cdots, U_N)$$

has the exponential distribution truncated on $(0, \nu)$:

$$\mathrm{P}(Y \leqq y) = (1 - e^{-y})/(1 - e^{-\nu}), \qquad 0 < y < \nu . \tag{1.4}$$

Marsaglia [5] considered the generation of exponential random numbers. Sibuya [6] examined how to choose the value of the parameter and extended the technique for generating gamma random numbers.

We apply, here, the technique to the doubly exponetial distribution

$$\Phi_1(x) = \exp(-e^{-x}), \qquad -\infty < x < \infty . \tag{1.5}$$

A table of random numbers from the doubly exponential distribution as well as those from the other extreme value distributions, i.e., Frechet's

$$\Phi_2(x) = \exp(-x^{-k}), \qquad 0 < x < \infty, \quad k > 0 , \tag{1.6}$$

and Weibull's

$$\Phi_3(x) = 1 - \exp(-x^{k}), \qquad 0 < x < \infty, \quad k > 0 , \tag{1.7}$$

was computed by Goldstein [3]. Concerning the details of these distributions, refer to Gumbel [4].

2. The transformation method

As a direct result of (1.3) we have a genesis of $\Phi_1(x)$:

THEOREM 1. *$(Y_1, \cdots, Y_L)$ is a random sample from the truncated or untruncated exponential distribution*

$$\mathrm{P}(Y \leqq y) = (e^{-A} - e^{-y})/(e^{-A} - e^{-B}), \qquad -\infty < A < y < B \leqq \infty . \tag{2.1}$$

Its size L is the zero-truncated Poisson r.v. with parameter λ and independent of the Y's. Then

$$X = \max (Y_1, \cdots, Y_L) - \log \Lambda$$

where

$$\Lambda = \lambda / (e^{-A} - e^{-B})$$

is the truncated doubly exponential r.v.:

$$\mathrm{P}(X \leqq x) = \frac{\exp(-e^{-x}) - \exp(-\Lambda e^{-A})}{\exp(-\Lambda e^{-B}) - \exp(-\Lambda e^{-A})} , \tag{2.2}$$

$$-\infty < A - \log \Lambda < x < B - \log \Lambda \leqq \infty .$$

It should be noted that the denominator is the probability of $\Phi_1(x)$ on the carrier of X.

We combine theorem 1 and lemma 1 to get theorem 2 which gives an algorithm for transforming uniform random numbers into doubly exponential ones.

THEOREM 2. *Let* $\{U_{ij}\}$ *be the sequence of* $(0, 1)$ *uniform r.v.'s,* $\{N_j\}$ *be the sequence of zero-truncated Poisson r.v.'s with parameter* ν, *and* L *be the zero-truncated Poisson r.v. with parameter* λ. *We assume all r.v.'s are mutually independent. Then*

$$X=\nu \max_{i=1,\dots,L} (\min_{j=1,\dots,N_i} U_{ij})-\log \Lambda ,$$

where

$$\Lambda=\lambda/(1-e^{-\nu}) ,$$

is the truncated doubly exponential r.v.:

$$\mathrm{P}(X\leqq x)=\frac{\exp(-e^{-x})-\exp(-\Lambda)}{\exp(-\Lambda e^{-\nu})-\exp(-\Lambda)} ,$$
$$-\log \Lambda<x<-\log \Lambda+\nu .$$

3. Choice of the parameters' values

A criterion to determine the values of ν and λ is the expected number of uniform random numbers which are consumed to generate one X:

$$m=\mathrm{E}(\textstyle\sum_{i=1}^{L} N_i)=\lambda\nu/(1-e^{-\lambda})(1-e^{-\nu}) . \tag{3.1}$$

Figure 1 shows the value of m as a function of the truncation points expressed by the probability integrals,

$$p_0=\exp(-\Lambda) \quad \text{and} \quad p_1=\exp(-\Lambda e^{-\nu}) , \qquad 0<p_0<p_1<1 .$$

Note that for a fixed value of p_1-p_0, the value of m is close to its minimum value for a wide range of (p_0, p_1).

If we make p_1-p_0 approach to 1 to improve the accuracy of the random number, then m increases fast. One way to keep the expected number of uniform random numbers moderate is to use several generating systems for the random numbers on consecutive intervals. We partition the interval $0<p_0<p_k<1$ into subintervals $(p_0, p_1), \cdots, (p_{k-1}, p_k)$, choosing them with frequencies $p_1-p_0, \cdots, p_k-p_{k-1}$ respectively, and apply the above generation system to each subinterval, then the expected number of uniform random numbers m decreases at the cost of complexity of the computer program.

For simplicity we use subintervals of the same length on the X domain:

$$-\infty<-\log \Lambda<-\log \Lambda+\nu<\cdots<-\log \Lambda+k\nu<\infty .$$

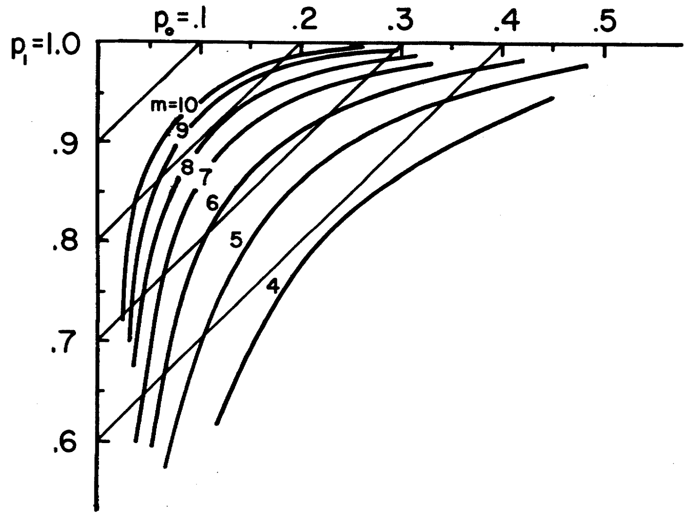

Fig. 1. Contour of m

In other words we use N's from the same distribution for each subinterval. The value of λ is not constant now and for the ith subinterval

$$\lambda_i = \Lambda_i(1-e^{-\nu}), \tag{3.2}$$

where

$$-\log \Lambda_i = -\log \Lambda + (i-1)\nu .$$

The expected number of consumed uniform random numbers is

$$m = \nu(1-e^{-\nu})^{-1} \sum_{i=1}^{k} (p_i - p_{i-1})\lambda_i(1-e^{-\lambda_i})^{-1} . \tag{3.3}$$

For a given value of $p = p_k - p_0$, p_0 should be chosen to minimize m. Numerical results show that, in this case also, the value of m is close to its minimum value, unless p_0 is too small or too large (close to $1-p$). Figure 2 shows how m^*=(the minimum value of m with respect to p_0), a function of p, decreases when k increases. Table 1 shows the values of parameters for $p=.9999$.

To cover the whole range of distribution we choose a large enough value of p. Or we cover the lower part by the other generation method:

$$X = -\log(-\log U + e^{-c}), \qquad -\infty < X < c , \tag{3.4}$$

or

$$X = -\log(Y + e^{-c}), \qquad -\infty < X < c , \tag{3.5}$$

where U is the uniform r.v. and Y is the exponential r.v.

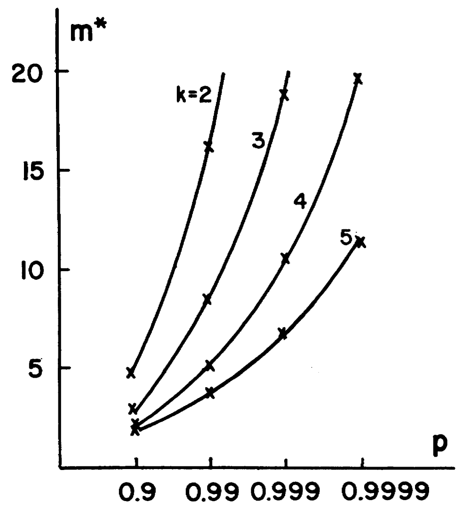

Fig. 2. m^* for k subintervals

Table 1.

The parameters for the generating system covering $p=p_k-p_0=0.9999$ by k subintervals optimized with respect to p_0, $a_i=-\log \Lambda_i$

$k=3$

i	p_i	p_i-p_{i-1}	a_i	λ_i	$\nu\lambda_i/(1-e^{-\nu})(1-e^{-\lambda_i})$
0	.000042	—	−2.311	—	—
1	.834256	.834214	1.708	9.90754	40.55
2	.996750	.162494	5.728	0.177960	4.47
3	.999942	.003192	9.747	0.003197	4.10
		$\nu=4.01949$		$m^*=34.57$	

$k=5$

i	p_i	p_i-p_{i-1}	a_i	λ_i	$\nu\lambda_i/(1-e^{-\nu})(1-e^{-\lambda_i})$
0	.000015	—	−2.407	—	—
1	.348992	.348977	−0.051	10.053100	26.17
2	.905032	.556040	2.305	.952921	4.04
3	.990586	.085554	4.661	.090327	2.72
4	.999104	.008518	7.017	.008562	2.61
5	.999915	.000811	9.373	.000812	2.60
		$\nu=2.35610$		$m^*=11.63$	

We can cover the upper part by the generation method:

$$X=-\log(-\log(1+DU)+e^{-d}), \qquad d<X<\infty, \tag{3.6}$$

where

$$D=\exp(e^{-d})-1\,,$$

or by applying theorem 1 with $A=0$ and $B=\infty$ (the shift of A has no effect when $B=\infty$); that is,

(3.7) $$X=\max(Y_1,\cdots,Y_L)-\log\lambda\,,\qquad -\log\lambda<X<\infty\,.$$

Combining (3.5) or (3.7) with Marsaglia's method for generating exponential random numbers is equivalent to increasing the number of subintervals in the above discussion.

4. The other extreme value distributions

The distribution functions (1.6) and (1.7) satisfy the relation

$$1-\Phi_2(x^{-1})=\Phi_3(x)\,.$$

Then, if we get random numbers from one of these distributions, we get those from the other by inversion. It will be easier to generate the Weibull random numbers. There are geneses of $\Phi_2(x)$ and $\Phi_3(x)$ similar to theorem 1. In the following theorems L is the Poisson r.v. with λ, and all r.v.'s are mutually independent.

THEOREM 3. *If Y's are truncated or untruncated Pareto r.v.'s:*

(4.1) $$\mathrm{P}(Y\leqq y)=(A^{-k}-y^{-k})/(A^{-k}-B^{-k})\,,\qquad 0<A<y<B\leqq\infty\,,$$

then

$$X=\max(Y_1,\cdots,Y_L)\Lambda^{-1/k}\,,$$

where

$$\Lambda=\lambda/(A^{-k}-B^{-k})\,,$$

has the truncated Frechet distribution:

(4.2) $$\mathrm{P}(X\leqq x)=\frac{\exp(-X^{-k})-\exp(-\Lambda A^{-k})}{\exp(-\Lambda B^{-k})-\exp(-\Lambda A^{-k})}\,,$$

$$0<A\Lambda^{-1/k}<x<B\Lambda^{-1/k}\,.$$

THEOREM 4*. *If Y's are truncated or untruncated power distribution r.v.'s:*

(4.3) $$\mathrm{P}(Y\leqq y)=(y^k-A^k)/(B^k-A^k)\,,\qquad 0\leqq A<y<B\,,$$

then

* This theorem was also obtained independently by K. Wakimoto and communicated to the author.

$$X = \min(Y_1, \cdots, Y_L)\Lambda^{1/k} ,$$

where

$$\Lambda = \lambda/(B^k - A^k) ,$$

has the truncated Weibull distribution:

$$\text{(4.4)} \qquad \mathrm{P}(X \leqq x) = \frac{\exp(-\Lambda A^k) - \exp(-x^k)}{\exp(-\Lambda A^k) - \exp(-\Lambda B^k)} ,$$

$$0 \leqq A\Lambda^{1/k} < x < B\Lambda^{1/k} .$$

If $A=0$ and k is a positive integer, the r.v. Y in (4.3) can be generated from uniform random numbers U's by

$$Y = \max(U_1, \cdots, U_k) , \qquad 0 < Y < 1 .$$

(The change of B has no meaning when $A=0$, so we put $B=1$.) If $A=0$ and $k=1/2, 1/3, \cdots$, Y can be generated by

$$Y = U^{1/k} .$$

For general k the techniques devised by Bankövi [1] and Békéssy [2] for generating the power series r.v. and the beta r.v. are recommended.

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