Ann Inst Stat Math (2010) 62:1113–1142
DOI 10.1007/s10463-008-0206-0

# Tilted Edgeworth expansions for asymptotically normal vectors

**Christopher S. Withers · Saralees Nadarajah**

Received: 17 May 2007 / Revised: 3 April 2008 / Published online: 10 September 2008


**Abstract** We obtain the Edgeworth expansion for $P(n^{1/2}(\hat{\theta} - \theta) < x)$ and its derivatives, and the *tilted Edgeworth* (or *saddlepoint* or *small sample*) expansion for $P(\hat{\theta} < x)$ and its derivatives where $\hat{\theta}$ is any vector estimate having the standard cumulant expansions in powers of $n^{-1}$.

**Keywords** Cornish and Fisher · Cumulants · Distribution · Edgeworth · Expansions · Lagrange inversion · Tilted Edgeworth

## 1 Introduction and summary

We obtain the Edgeworth expansion in powers of $n^{-1/2}$ for $P(n^{1/2}(\hat{\theta} - \theta) < x)$ and its derivatives, and the tilted Edgeworth expansion (also known as the saddlepoint or small sample expansion) for $P(\hat{\theta} < x)$ and its derivatives, where $\hat{\theta}$ is any vector estimate having the standard cumulant expansions in powers of $n^{-1}$. The general term in these expansions is given explicitly in terms of Bell polynomials.

The univariate Edgeworth expansion is given in Theorem 1. The univariate expansions for $P(\hat{\theta} < x)$ and its derivatives are given in Sect. 3 in particular (21)–(22), with proofs in Sect. 4. Multivariate extensions are given in Sect. 5 in particular (44).

C. S. Withers
Applied Mathematics Group,
Industrial Research Limited, Lower Hutt, New Zealand

S. Nadarajah (✉)
School of Mathematics,
University of Manchester, Manchester M13 9PL, UK
e-mail: mbbsssn2@manchester.ac.uk

Section 6 gives expansions for $\hat{\theta}$ a function of a sample mean. We believe that all of the results presented in the paper including the examples are new.

Nearly all of the literature on saddlepoint methods in statistics has been for the sample mean or functions of it. Concerning densities, Daniels (1954, 1980) gave the first two terms of the expansions for the density of a sample mean (or sum) from a univariate sample for both continuous and discrete random variables extended to multivariate continuous random variables by Barndorff-Nielsen and Cox (1979). Blackwell and Hodges (1959) and Good (1957) gave results for the density of the mean of discrete univariate random variables, extended to multivariate discrete random variables by Good (1961). Easton and Ronchetti (1986) gave an example of the effect of setting cumulants of order greater than 4–0: the results are surprisingly good considering that the results are not even asymptotically correct. Fraser (1988) considered the density of MLEs and 1 to 1 functions of them (the MLE can be regarded as a function of a sample mean). Jensen (1988) gives a very clear account of both the density and tail probability of a univariate sample mean, as well as an extension to sums of the form $S_N$ where $N$ is an independent Poisson random variable, and also to a normal autoregressive estimate. Shuteev (1990) essentially studies the density of a smooth function of a univariate sample mean. Field and Ronchetti (1990) suggest that for general estimates one can approximate the cumulants, or for estimates that are smooth functions of an empirical distribution, one can approximate it using the first functional derivative (But however well these may perform empirically, their asymptotic behaviour will be incorrect if not done properly. Compare Remark 4 of Easton and Ronchetti (1986), Wang (1990a) who uses the second functional derivative, and Ohman-Strickland and Casella (2002)). They also touch on confidence intervals. Barndoff-Nielsen (1990) approximates the density and tail probability of an MLE.

Concerning tail probabilities, Lugannani and Rice (1980) and Robinson (1982) gave approximations for a univariate sample mean, compared in Daniels (1983, 1987). Daniels (1983) also gives tail and density results for a univariate maximum likelihood estimate. Barndoff-Nielsen (1990) views the tilted estimate as involving the MLE of a 1 parameter exponential family, and extends this to more general exponential families. Diciccio and Field (1990) consider bivariate tail probabilities. Fraser (1990) gives a variation of Lugannani and Rice (1980). Wang (1990b) gives tail probabilities for bivariate means for both continuous and discrete random variables. Robinson et al. (1990) give tail probabilities and mixed density/tail probabilities for a vector mean with some elements continuous and others lattice. Unlike the previous results, they give full expansions in powers of $n^{-1/2}$ or $n^{-1}$, as do we. They give full regularity conditions for the remainder after $s$ terms to have a given order of magnitude. Kolassa (1991) proposes approximating a cumulant generating function by a close one for which the saddlepoint is analytic. Routledge and Tsao (1997) showed that one can differentiate the result of Lugannani and Rice (1980) to obtain that of Daniels (1954). Conditional densities and tail probabilities have been studied by Skovgaard (1987) and Jing and Robinson (1994). Applications are many. They include Daniels and Young (1991) to a studentized mean, Pazman (1991) and Spady (1991) to regression, Butler et al. (1992) to the generalized variance and Wilk's statistic, Jensen (1992) to the signed likelihood statistic, Wang (1992) to autoregression, Maesono and Penev (1998) to improve on the Cornish–Fisher expansion for a quantile, Kuonen (1999) to quadratic forms in normal random variables, Giles (2001) to the Anderson-Darling statistic, Kathman and Terrell (2002) on Poisson approximations, and many others. For other related work, see Riordan (1949), Fisher and Cornish (1960), Bhattacharya and Ghosh (1978), Honda (1978), Pedersen (1979), Lazakovich (1982), Skovgaard (1989), Jensen (1992), Jing et al. (1994), Maesono and Penev (1998), Kolassa (2000), Anderson and Robinson (2001), Jing et al. (2002), Yang and Kolassa (2002), Anderson and Robinson (2003), Jermiin et al. (2003), Kolassa (2003a,b), Helmers et al. (2004), Jing et al. (2004), Robinson and Wang (2005), and Kolassa and Robinson (2007). A review article is Skovgaard (2001).

This paper is the first time that general term of the Edgeworth expansions has been given for either the density or the distribution. It is also the first time general tilted expansions have been obtained. Tilted expansions have been obtained in the past for the sample mean or functions of it. For example, Monti (1993) obtains a titled expansion for the sample mean up to the second order by expanding the saddlepoint approximation. Booth et al. (1994) give tilted Edgeworth expansions of a sample mean from a distribution on $k$ points. Their Theorem 3.1 shows, at least in some cases, that the tilted Edgeworth expansion to say $N$ terms is valid when the ordinary Edgeworth expansion is. Kakizawa and Taniguchi (1994) obtain tilted expansions for $P(\hat{\theta} < x)$ under the assumption that $\hat{\theta}$ has a cumulant expansion in powers of $n^{-1}$, similar to the assumption made in this paper. However, the results of Kakizawa and Taniguchi (1994) are limited to the univariate case and the expansions are given only up to the order $n^{-3/2}$. Furthermore, Kakizawa and Taniguchi (1994) make no considerations of the derivatives of $P(\hat{\theta} < x)$ and the only example discussed is limited to the Gaussian AR process. Attempts by other authors have generally assumed that the cumulant generating function for $\hat{\theta}$ is known, and have used its saddlepoint (rather than the asymptotic saddlepoint) or used an approximation for the cumulant generating function and saddlepoint. For example, Gatto and Ronchetti (1996) claim to approximate $P(m(\bar{X}) < x)$ to $1 + O(n^{-1})$ for $m(\cdot)$ a smooth function. But since they ignore cumulants beyond the fifth their saddlepoint and index of large deviation are not correct asymptotically and their results will not be even $1 + o(1)$. That their example (based on simulations for their logit model with $n = 20$) does so well seems to imply that for this case the missing contributions from the higher cumulants are not critical.

Our general term is expressed using the Bell polynomials (the coefficients of a power series raised to a power). Appendix A gives their uses needed here and a number of other applications including to Cornish–Fisher expansions. The truncated expansion for $P(n^{1/2}(\hat{\theta} - \theta) < x)$ requires the truncated triangular array of cumulant coefficients: a *finite* number of coefficients. By contrast the truncated expansion for $P(\hat{\theta} < x)$ requires the leading diagonals of this array: an infinite number of coefficients. Appendix B give series expansions for the main *index of large deviation* for the univariate case. It is this index that is needed to complete the asymptotic fixed $\alpha$, fixed $\beta$ (or Bahadur) and Bayes (or Chernoff) efficiencies.

Using Withers (1996), the results given here may be extended to repeated integrals of $P(\hat{\theta} < x)$ or – if $\hat{\theta}$ is a vector – to its mixed integrals and differentials, such as $\int^{x_1} dx_1(\partial/\partial x_2)P(\hat{\theta} < x)$. We use the notation $f_n \approx \sum_{i=0}^{\infty} f_{ni}$ to mean that RHS is an asymptotic expansion for LHS which may or may not converge.

## 2 Univariate Edgeworth expansions

Let $\hat{\theta}$ be a real estimate and that its cumulant generating function exists for some non-zero arguments. Suppose that for $r \geq 1$, its $r$th cumulant has the standard expansion

$$\kappa_r(\hat{\theta}) \approx \sum_{j=r-1}^{\infty} a_{rj} n^{-j} \tag{1}$$

for $r \geq 1$ with $a_{10} = \theta$, or the more general expansion

$$\kappa_r(\hat{\theta}) \approx \sum_{k=2r-2}^{\infty} b_{rk} n^{-k/2} \tag{2}$$

for $r \geq 1$ with $b_{10} = \theta$ with the coefficients $\{a_{rj}\}$ or $\{b_{rk}\}$ not depending on $n$, or at least bounded as $n \to \infty$ with $a_{21}$ and $b_{22}$ bounded away from 0 (this form arises for expansions of 1-sided confidence limits: see Withers 1983a,b, 1988). In the next section we shall also assume that the cumulant generating function of $\hat{\theta}$ exists in a neighbourhood of the origin.

*Example 1* If $\hat{\theta} = T_n(F_n)$ is a regular functional of $F_n$, the empirical distribution of an i.i.d. sample of size $n$ on $R^s$, and $T_n$ has a power series in $n^{-1}$ (or $n^{-1/2}$) then (1) (or (2)) holds with leading coefficients given by Withers (1982).

*Example 2* If $\hat{\theta} = t_n(\bar{X}_1, \ldots, \bar{X}_k)$ or $T_n(F_{n_1 1}, \ldots, F_{n_k k})$ is a smooth function or functional of the means or empirical distributions of $k$ independent random samples of sizes $n_1, \ldots, n_k$ and $T_n$ is expandable in powers of $n^{-1}$ (or $n^{-1/2}$) then (1) or (2) holds with $n = min_{i=1}^{k} n_i$. The simplest case is $\hat{\theta} = \sum_{i=1}^{k} c_i \bar{X}_i$; in this case $\kappa_r(\hat{\theta}) = \kappa_r n^{1-r}$ holds with $\kappa_r = \sum_{i=1}^{k} c_i^r (n/n_i)^{1-r} \kappa_r(X_i)$. We may take $n = \min_{i=1}^{k} n_i$ or $n = \sum_{i=1}^{k} n_i$.

The expansion (1) is more familiar, but that of (2) is useful in constructing 1-sided confidence intervals: see Withers (1983b). We assume that $a_{21}$ or $b_{22}$ is bounded away from 0 as $n \to \infty$. The aim of this section is to give an expansion in powers of $n^{-1/2}$ for the distribution of

$$\begin{aligned} Y_n &= (n/b_{22})^{1/2}(\hat{\theta} - \theta - b_{11} n^{-1/2}) \\ &= (n/a_{21})^{1/2}(\hat{\theta} - \theta) \text{ if (1) holds} \\ &= (n/\kappa_2)^{1/2}(\bar{X} - EX) \text{ if } \kappa_r(\hat{\theta}) = \kappa_r n^{1-r} \text{ holds.} \end{aligned}$$

Note that for $r \geq 1$

$$\kappa_r(Y_n) \approx n^{r/2} \sum_{k=2r-2}^{\infty} B_{rk}(Y_n) n^{-k/2} \tag{3}$$

with $B_{10}(Y_n) = B_{11}(Y_n) = 0$ and

$$B_{rk}(Y_n) = b_{rk} b_{22}^{-r/2} \tag{4}$$

otherwise. Let $Y$ be any random variable with cumulants of the form (3) with $B_{10}(Y) = B_{11}(Y) = 0$ and $B_{22}(Y) = 1$. Set

$$B_{rk} = B_{rk}(Y_n) - B_{rk}(Y), \tag{5}$$

$$e_j(t) = \sum_{r=1}^{j+2} B_{r,r+j} t^r / r! \tag{6}$$

and $B = \{B_{rk}\}$ for $j \geq 1$.

Note 1 The usual choice is $Y \sim \mathcal{N}(0, 1)$, the unit normal, so that for $(r, k) \neq (2,2)$, $B_{rk}(Y) = 0$ and $B_{rk} = b_{rk} b_{22}^{-r/2}$. However, one could also choose $Y$ depending on $n$ and asymptotically $N(0, 1)$.

For $x = (x_1, x_2, \ldots)$ a sequence from $R$, let $\tilde{B}_r(x)$ denote the coefficient of $y^r$ in $\exp(\sum_{j=1}^{\infty} x_j y^j)$. See Appendix A for these. Then for $e(t)$ of (6), $\tilde{B}_r(e(t))$ has the form

$$\tilde{B}_r(e(t)) = \sum_{j=0}^{3r} P_{rj} t^j \tag{7}$$

for $r \geq 0$, where $P_{rj} = P_{rj}(B)$ are given as polynomials in $B$ for $r \leq 3$ in Appendix A. That is, $P_{rj}$ is the coefficient of $y^r t^j$ in the expansion for $\exp\{\sum_{j=1}^{\infty} e_j(t) y^j\}$. An explicit expression for $P_{rj}$ for $\bar{X}$ is given by (53).

**Theorem 1** *Suppose that $Y_n$ and $Y$ have densities $p_n(y)$ and $p(y)$ with respect to Lebesgue measure. Set*

$$\begin{aligned} &P_n(y) = P(Y_n < y),\ P(y) = P(Y < y), \\ &H_k(y) = p(y)^{-1}(-\partial/\partial y)^k p(y) \text{ for } k \geq 0,\ H_{-1}(y) = -p(y)^{-1} P(y), \\ &h_{kr}(y) = h_{kr}(y, B) = \sum_{j=0}^{3r} P_{rj} H_{k+j-1}(y) = p(y)^{-1}(-\partial/\partial y)^k \tilde{B}_r(e(-\partial/\partial y)) P(y) \end{aligned} \tag{8}$$

*for $P_{rj}$ of (7). Then*

$$P_n(y) \approx P(y) - p(y) \sum_{r=1}^{\infty} n^{-r/2} h_{0r}(y) = -p(y) \sum_{r=0}^{\infty} n^{-r/2} h_{0r}(y) = P_n(y, B) \text{ say.} \tag{9}$$

*Its kth derivative is*

$$P_n^{(k)}(y) \approx (-1)^{k-1} p(y) \sum_{r=0}^{\infty} n^{-r/2} h_{kr}(y) \tag{10}$$

*for $k \geq 0$.*

*Proof* Note $\kappa_r(Y_n) - \kappa_r(Y) \approx n^{r/2} \sum_{k=2r-2}^{\infty} B_{rk} n^{-k/2}$, so the difference in the cumulant generating functions of $Y_n$ and $Y$ is

$$\triangle_n(t) \approx K_{Y_n}(t) - K_Y(t) = \sum_{j=1}^{\infty} n^{-j/2} e_j(t).$$

So, $\exp(\triangle_n(t)) \approx \sum_{r=0}^{\infty} n^{-r/2} \tilde{B}_r(e(t))$. By the Charlier differential series—see Cornish and Fisher (1937), $P_n(y) = \exp(\triangle_n(-\partial/\partial y))P(y)$, which with (8) proves (9). Differentiating gives (10). □

## 3 Univariate tilted expansions and cumulant functions

Here, we obtain expansions for $P(\hat{\theta} < x)$ and its derivatives. Again we assume that the cumulants of $\hat{\theta}$ can be expanded in the form (2). We give our results in terms of the functions

$$k_j = k_j(t) = \sum_{r=1}^{\infty} b_{r,j+2r-2} t^r / r! \tag{11}$$

for $j \geq 0$, where $b_{rk}$ are the coefficients of (2). We call $k_j$ the *j th cumulant function* or the *cumulant function of order j*. The cumulant generating function of $n\hat{\theta}$ is given in terms of $\{k_j\}$ by

$$n^{-1} K_{n\hat{\theta}}(t) \approx \sum_{j=0}^{\infty} n^{-j/2} k_j(t) \tag{12}$$

since (2) is equivalent to (12) and (11). If (1) holds then $k_j = 0$ for $j$ odd and

$$k_{2j} = \sum_{r=1}^{\infty} a_{r,r+j-1} t^r / r!.$$

The most important cumulant function is that of order 0 given by $k_0(t) = \lim_{n\to\infty} K_{n\hat{\theta}}(t)$. For $\hat{\theta}$ a sample mean, $k_0$ is the cumulant generating function of $X$ and the other cumulant functions are 0.

We assume that $k_0^{(2)}(t) > 0$ (if $\kappa_r(\hat{\theta}) = \kappa_r n^{1-r}$ holds this is equivalent to $\kappa_2 > 0$). Let $t$ in R be an interior point in

$$T = \{t : k_0(t) < \infty\}. \tag{13}$$

In Sect. 4, we show that (11) implies that $k_0^{(2)}(t) > 0$, so there is a unique function $t = t(x) : R \to R$ maximising $xt - k_0(t)$ and satisfying

$$x = k_0^{(1)}(t) \tag{14}$$

at $t = t(x)$. Set

$$I(x) = \sup_{t \in T}(xt - k_0(t)) = xt - k_0(t) \tag{15}$$

at $t = t(x)$ so that $t(x) = I^{(1)}(x)$. We assume that $x < x(k_0) = \sup_{t \in T} k_0^{(1)}(t)$. When (1) holds, Appendix B gives expansions for these of the form $I(x) = J(z_x)$ and $t(x) = a_{21}^{-1/2} J^{(1)}(z_x)$, where $z_x = (x - \theta)/a_{21}^{1/2}$, $J(z) = \sum_{i=2}^{\infty} C_{i-1} z^i / i!$, $J^{(1)}(z) = \sum_{i=1}^{\infty} C_i z^i / i!$, and the coefficients $C_i$ are polynomials in the leading standardised cumulant coefficients.

Since $k_0^{(1)}(0) = \theta$, $x > \theta \Longleftrightarrow t(x) > 0$. Also $y = \theta t$ is the tangent to $y = k_0(t)$ at $t = 0$. Define $H_k(y)$ in terms of $p(y)$, the density of $Y$, as in (8). We now give our basic result. In Sect. 4, we see that it follows from the fact that if $\hat{\theta}_t$ is a random variable with density proportional to $\exp(ntx)$ times the density of $\hat{\theta}$ then its cumulants can be expanded in the basic form (1) or (2) with $a_{rj}$ of (1) or $b_{rk}$ of (2) replaced by

$$a_{rj}(t) = k_{2k-2r+2}^{(r)}(t), \ \ b_{rk}(t) = k_{k-2r+2}^{(r)}(t). \tag{16}$$

Just as (5), (4) and (7) defined $B = \{B_{rk}\}$ and $P_{rj}$ in terms of $\{b_{rk}\}$, we define $B(t) = \{B_{rk}(t)\}$ and $P_{rj}(t)$ in terms of $\{b_{rk}(t)\}$. However, for notational simplicity we drop the dependency on $t$ in this section. That is, in this section $B = \{B_{rk}\}$ and $P_{rj}$ will now refer to $B(t) = \{B_{rk}(t)\}$ and $P_{rj}(t)$.

**Theorem 2** *Fix $x$ in $R$ and set $\alpha_n(x, t) = \exp\{-n(xt - k_0(t)) + n^{1/2} k_1(t) + k_2(t)\}$. Choose $t = t(x)$ of (14) and let $I(x)$ be defined by (15). Set*

$$L = t k_0^{(2)}(t)^{1/2}. \tag{17}$$

*Also, set $y = -k_1^{(1)}(t) k_0^{(2)}(t)^{-1/2}$ and $\bar{\alpha}_n = \alpha_n(x, t) = \exp\{-nI(x) + n^{1/2} k_1(t) + k_2(t)\}$. Then for $x < \theta$,*

$$P(\hat{\theta} < x) \approx \bar{\alpha}_n n^{-1/2} p(y) |L|^{-1} \left\{ 1 + \sum_{i=1}^{\infty} n^{-i/2} \bar{C}_{0i} \right\}, \tag{18}$$

*where $\bar{C}_{0i} = \bar{C}_{0i}(y, t) = \sum_{p=0}^{i} C_{0p} \tilde{B}_{i-p}(d)$, $\tilde{B}_r(d)$ is given by (51) of Appendix A in terms of $\{d_r\}$ for $d_r = k_{r+2}(t)$,*

$$C_{0p} = C_{0p}(y, t) = \sum_{j=0}^{p} (-L)^{-j} D_{p-j,j}$$

*and*

$$D_{rp} = D_{rp}(y, t) = \sum_{j=0}^{3r} P_{rj} H_{j+p}(y) = h_{p+1,r}(y, B(t))$$

*of* (8). *Also, for* $x > \theta$, $P(\hat{\theta} > x)$ = *RHS* (18), *and for* $k \geq 1$, $Q_n(x) = P(\hat{\theta} < x)$ *has kth derivative*

$$Q_n^{(k)}(x) \approx \bar{\alpha}_n n^{k-1/2} p(y)|t|^{k-1} k_0^{(2)}(t)^{-1/2} \sum_{i=0}^{\infty} n^{-i/2} \bar{C}_{ki}, \tag{19}$$

*where*

$$\bar{C}_{ki} = \bar{C}_{ki}(y, t) = C_{ki} \otimes \tilde{B}_i(d) = \sum_{p=0}^{i} C_{kp} \tilde{B}_{i-p}(d),$$

*and*

$$C_{kp} = C_{kp}(y, t) = \sum_{j=0}^{p} \binom{k-1}{j} L^{-j} D_{p-j,j}. \tag{20}$$

Note 1 applies to $B = B(t)$. The first four coefficients in the expansions (18) and (19) are $\bar{C}_{k0} = 1$, $\bar{C}_{k1} = k_3 + C_{k1}$, $\bar{C}_{k2} = k_4 + k_3^2/2 + k_3 C_{k1} + C_{k2}$, and

$$\bar{C}_{k3} = k_5 + k_4 k_3 + k_3^3/6 + (k_4 + k_3^2/2) C_{k1} + k_3 C_{k2} + C_{k3}.$$

Truncating the sum in the expansions (18) or (19) to $O(n^{-r/2})$ requires knowing $k_0, \ldots, k_r$.

**Corollary 1** *Fix* $x$ *in* $R$. *Suppose that* (1) *holds and* $Y$ *of Theorem 1 is symmetric about* 0. *So,* $k_{2j+1} = 0$ *and*

$$k_{2j} = \sum_{r=1}^{\infty} a_{r,r+j-1} t^r / r!.$$

*Define* $t = t(x)$ *by* (14), $I(x)$ *by* (15), $L$ *by* (17), $\bar{\alpha}_n = \exp\{-nI(x) + k_2(t)\}$, *and* $P_{rj} = P_{rj}(t)$ *by* (52) *of Appendix A in terms of* $\{A_{rj} = A_{rj}(t) = B_{r,2j}\}$ *of Theorem 2. Then, for* $x < \theta$,

$$P(\hat{\theta} < x) \approx \bar{\alpha}_n n^{-1/2} p(0) |L|^{-1} \left\{ 1 + \sum_{i=1}^{\infty} n^{-i} \bar{C}_{0,2i} \right\}, \tag{21}$$

*and, for* $x > \theta$, $P(\hat{\theta} > x)$ = *RHS* (21). *Set* $Q_n(x) = P(\hat{\theta} < x)$. *Then, for* $k \geq 1$ *and* $x \neq \theta$,

$$Q_n^{(k)}(x) \approx \bar{\alpha}_n n^{k-1/2} p(0) |L|^{-1} |t|^k \left\{ 1 + \sum_{i=1}^{\infty} n^{-i} \bar{C}_{k,2i} \right\}, \tag{22}$$

*where* $\bar{C}_{k2} = k_4 + C_{k2}$, $\bar{C}_{k4} = k_6 + 3k_4^2/2 + k_4 C_{k2} + C_{k4}$, *and* $C_{k2}$ *and* $C_{k4}$ *are given by* (20) *in terms of*

$$D_{0p} = H_p(0),\ \ D_{1p} = \sum\{P_{1j} H_{j+p}(0) :\ j = 1, 3\},$$
$$D_{2p} = \sum\{P_{2j} H_{j+p}(0) :\ j = 2, 4, 6\}$$

*and*

$$D_{3p} = \{P_{3j} H_{j+p}(0) :\ j = 1, 3, 5, 7, 9\},\ \ D_{40} = \sum\{P_{4j} H_j(0)\ :\ j = 2, 4, \ldots, 12\}.$$

Note 2 Suppose that $Y \sim \mathcal{N}(0, 1)$. Then, for $(r, i) \neq (2, 1)$,

$$A_{ri}(t) = a_{ri}(t)/a_{21}(t)^{r/2},$$

where $a_{ri}(t) = k_{2i-2r+2}^{(r)}(t)$ and $H_{2j}(0) = (-1)^j 1.3.5 \cdots (2j-1) = (-1/2)^j (2j)!/j! = E(-Y^2)^j$. Note $C_{k2}$ needs

$$D_{02} = -1,\ \ D_{11} = -P_{11} + 3P_{13} = -A_{11} + A_{32}/2,$$
$$D_{20} = -P_{22} + 3P_{24} - 3.5P_{26} = -A_{22}/2 - A_{11}^2/2 + A_{43}/8 + A_{11}A_{32}/2 - 5A_{32}^2/24,$$

which are given by (52) in terms of $a_{21} = k_0^{(2)}$, $a_{11} = k_2^{(1)}$, $a_{32} = k_0^{(3)}$, $a_{22} = k_2^{(2)}$ and $a_{43} = k_0^{(4)}$. Similarly, $a_{r,r-1} = k_0^{(r)}(t)$, $a_{rr} = k_2^{(r)}(t)$, $a_{r,r+1} = k_4^{(r)}(t), \cdots$. Note $C_{k4}$ needs

$$D_{04} = 3,\ \ D_{13} = 3P_{11} - 3\cdot 5P_{13} = 3A_{11} - 5A_{32}/2,$$
$$D_{22} = 3P_{22} - 3\cdot 5P_{24} + 3\cdot 5\cdot 7P_{26} = 3(A_{22} + A_{11}^2)/2 - 5(A_{43} + 4A_{11}A_{32})/8 + 35A_{32}^2/24,$$
$$D_{31} = -P_{31} + 3P_{33} - 3\cdot 5P_{35} + 3\cdot 5\cdot 7P_{37} - 3\cdot 5\cdot 7\cdot 9P_{39},$$
$$D_{40} = -P_{42} + 3P_{44} - 3\cdot 5P_{46} + 3\cdot 5\cdot 7P_{48} - 3\cdot 5\cdot 7\cdot 9P_{4,10} + 3\cdot 5\cdot 7\cdot 9\cdot 11P_{4,12},$$

which are given by (52) in terms of the above $A_{ri}$ and $a_{ri}$. The coefficients needed for the density are particularly simple: $C_{1,2i} = D_{2i,0}$.

*Example 3* Suppose that $\hat{\theta} = \theta\chi_n^2/n$, where $\theta > 0$ and $Y \sim \mathcal{N}(0, 1)$. We can take $\theta = 1$ (for if $\theta \neq 1$, replace $x$ by $x/\theta$ in the expansions below). Since $\hat{\theta} > 0$ with probability 1 we can assume $x > 0$. The distribution of $\hat{\theta}$ and its derivatives are given by (21)–(22) in terms of

$$I(x) = (x - 1 - \log x)/2,\ \ k_2(t) = 0,\ \ p(0) = (2\pi)^{-1/2},$$
$$L = (x-1)2^{-1/2},\ \ t = 7(1 - x^{-1})/2,$$

and

$$\bar{C}_{kj} = C_{kj} = \sum_{i=0}^{k-1} (k-1)_i (x-1)^{-i} c_{ji},$$

where $(k-1)_i = (k-1)(k-2)\cdots(k-i) = \Gamma(k)/\Gamma(k-i)$ and $c_{ji} = 2^{i/2} D_{j-i,i}/i!$. Now suppose that $\hat{v}/v \sim \chi_n^2/n$ and we test $H_0 : v = v_0$ versus $H_1 : v > v_0$, rejecting $H_0$ if

$$\hat{v}/v_0 > \chi^2_{n,1-\alpha}/n = x_n$$

say. So, this test has level $1-\alpha$. So, $x_n = 1 + (2/n)^{1/2} z_\alpha + O(n^{-1})$, where $z_\alpha = \Phi^{-1}(1-\alpha)$. Then this test has type 2 error

$$P(\hat{\theta} < x) = (n\pi)^{-1/2} (xe^{1-x})^{n/2} |x-1|^{-1} \{1 + n^{-1} C_{02} + n^{-2} C_{04} + \cdots\}, \quad (23)$$

where $x = x_n v_0/v$, $C_{02} = -1/6 - 2(x-1)^{-1} - 2(x-1)^{-2}$ and

$$C_{04} = 1/72 + (x-1)^{-1}/3 + (25/3)(x-1)^{-2} + 20(x-1)^{-3} + 12(x-1)^{-4}.$$

Similarly, a test of level $1-2\alpha$ of $H_0 : v = v_0$ versus $H_2 : v \neq v_0$ is to reject $H_0$ if

$$\hat{v}/v_0 > \chi^2_{n,1-\alpha}/n = x_n \text{ or } \hat{v}/v_0 < \chi^2_{n,\alpha}/n = y_n$$

say. Set $y = y_n v/v_0$. This test has type 2 error $P(y < \hat{\theta} < x)$. To within a factor $1 + O(e^{-nJ})$, where $J > 0$, this is equal to $P(\hat{\theta} < x)$ given above if $\theta > x$, and to $P(y < \hat{\theta})$ if $\theta < y$. The second probability is given by RHS (23) with $x$ replaced by $y$.

Let $\Gamma(\gamma)$ be a gamma random variable with mean $\gamma$. So, $\Gamma(\gamma)/\gamma = \chi_n^2/n$ with $n = 2\gamma$. So, for $x > 1$,

$$P(\Gamma(\gamma)/\gamma > x) = e^{-\gamma I_1(x)} (2\pi\gamma)^{-1/2} |L_1|^{-1} \{1 + O(\gamma)^{-1}\} \quad (24)$$

for $I_1(x) = 2I(x) = x - 1 - \ln x$ and $L_1 = 2^{1/2} L = x - 1$. Figure 1 illustrates the accuracy of the approximation in (24) for $\gamma = 10$. In general, the approximation performs well for large values of $x$ and $\gamma$.

*Example 4* Suppose that $\hat{\theta} = \left( \sum_{i=1}^M \alpha_i \chi^2_{\lambda_i n + \nu_i} + \sum_{j=1}^N \beta_j \chi^2_{\gamma_j} \right) \Big/ n$. Then

$$\theta = \sum_{i=1}^M \alpha_i \lambda_i \text{ and } \kappa_r(\hat{\theta}) = \sum_{i=r-1}^{r} a_{ri} n^{-i},$$

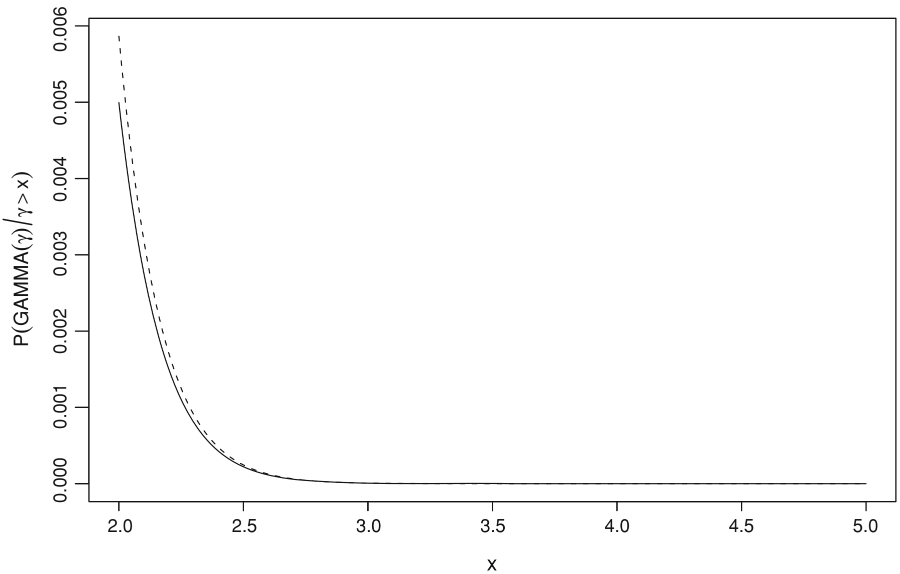


**Fig. 1** Comparison of $P(\Gamma(\gamma)/\gamma > x)$ (*the solid curve*) versus the RHS of (24) (*the broken curve*) for $\gamma = 10$

where

$$a_{r,r-1} = 2^{r-1}(r-1)! \sum_{i=1}^{M} \lambda_i \alpha_i^r, \; a_{rr} = 2^{r-1}(r-1)! \left( \sum_{i=1}^{M} \nu_i \alpha_i^r + \sum_{j=1}^{N} \gamma_j \beta_j^r \right)$$

and

$$k_0 = -2^{-1} \sum_{i=1}^{M} \lambda_i \log(1-2\alpha_i t), \; k_2 = -2^{-1} \sum_{i=1}^{M} \nu_i \log(1-2\alpha_i t) - 2^{-1} \times \sum_{j=1}^{N} \gamma_j \log(1-2\beta_j t).$$

The other $k_j$ are zero. Let $t(x)$ be the unique root of $x = k_0^{(1)}(t) = \sum_{i=1}^M \lambda_i \alpha_i/(1-2\alpha_i t)$ in $(t_-, t_+)$, where $t_- = 2^{-1} \max(\alpha_i^{-1} : \alpha_i < 0) = -\infty$ if all $\alpha_i > 0$ and $t_+ = 2^{-1} \min(\alpha_i^{-1} : \alpha_i > 0) = \infty$ if all $\alpha_i < 0$. For $\{\alpha_i\}$ distinct, there are $M$ roots if $x \neq 0$ and $M-1$ roots if $x = 0$. So, if $M$=1, one needs $x \neq 0$ since $|t| \to \infty$ as $x \to 0$. We assume that $x$ satisfies $1 > \max_{j=1}^M \beta_j t$.

Take $y > 0$. Note $P(\chi_\gamma^2/\chi_n^2 > y) = P(\hat{\theta} < 0)$ for $\hat{\theta} = (\chi_n^2 - y^{-1}\chi_\gamma^2)/n$. By the second to last sentence this case is not dealt with. However, $k_0, t, L$ and $I(x)$ are given by Example 3, and

$$k_2 = -2^{-1}\gamma \log(1+2t/y) = -2^{-1}\gamma \log(1+y^{-1}-x^{-1}y^{-1}),$$
$$k_2^{(1)}(t) = \gamma/(y+2t), \; k_2^{(2)}(t) = -2\gamma/(y+2t)^2.$$

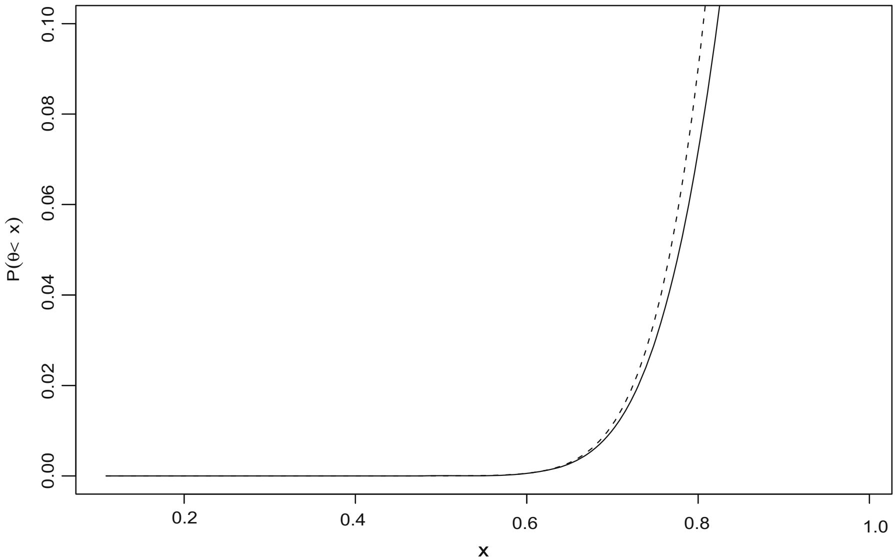


**Fig. 2** Comparison of $P(\hat{\theta} < x)$ (*the solid curve*) versus the RHS of (25) (*the broken curve*) for $y = 9$, $\gamma = 1$, $n = 100$ and $k = 0$, where $\hat{\theta} = (\chi_n^2 - y^{-1}\chi_\gamma^2)/n$

So, for $(1 + y)^{-1} < x < 1$ and $k \geq 0$,

$$P^{(k)}(\hat{\theta} < x) \approx \exp(-nI(x))(1 + y^{-1} - x^{-1}y^{-1})^{-\gamma/2}(\pi n)^{-1/2}(1 - x)^{-1}(n/2)^k \times (x^{-1} - 1)^k \left(1 + \sum_{i=1}^{\infty} C_{k,2i} n^{-i}\right), \tag{25}$$

where $C_{k,2}$ and $C_{k,4}$ are given by Note 2 in terms of $A_{12} = A_{23} = 0$, $A_{11} = 2^{-1/2}\gamma/d$, $A_{22} = -\gamma/d^2$, $A_{33} = 2^{3/2}\gamma/d^3$, $A_{44} = -12\gamma/d^4$ and $A_{r,r-1}$ for $d = xy + x - 1$. Figure 2 illustrates the accuracy of the approximation given by (25) up to the order $1/n$ for $y = 9$, $\gamma = 1$, $n = 100$ and $k = 0$. The approximation performs well for small enough values of $x$. Further investigation suggests that the best performance is achieved when $y$ is large, $\gamma$ is small and $n$ is large.

## 4 Proofs for Sect. 3

Here, we prove and extend Theorem 2 and Corollary 1. Let us write $\int^x$ for $\int_{-\infty}^x$ and $\int_x$ for $\int_x^\infty$. By (2) the cumulant generating function of $n\hat{\theta}$ satisfies

$$n^{-1}K_{n\hat{\theta}}(t) \approx \sum_{j=0}^{\infty} n^{-j/2} k_j = K(t)$$

say. Set $K_{jr} = k_j^{(r)}(t)$ and $K_r = K^{(r)}(t)$. For $t$ satisfying (13) let $\hat{\theta}_t$ be a random variable with distribution

$$P(\hat{\theta}_t < x) = \exp(-nK_0) \int^x \exp(ntx) dP(\hat{\theta} < x).$$

So,

$$P(\hat{\theta} < x) = \exp(nK_0) \int^x \exp(-ntx) dP(\hat{\theta}_t < x). \tag{26}$$

Since

$$K_{n\hat{\theta}_t}(\tau) = K_{n\hat{\theta}}(t+\tau) - K_{n\hat{\theta}}(t), \tag{27}$$

$\kappa_r(n\hat{\theta}_t) = nK_r$, so by (26), $\hat{\theta}_t$ satisfies (2) with $b_{rk}$ replaced by $b_{rk}(t)$ of (16). Since $\hat{\theta}$ is not a constant, $K_2 > 0$ and $K_{02} > 0$. Since $K_{02} > 0$, there is a unique function $t = t(x)$ satisfying (14). Set

$$\begin{aligned} y_{nt}(x) &= (n/b_{22}(t))^{1/2}(x - b_{10}(t) - b_{11}(t)n^{-1/2}) \\ &= (n/K_{02})^{1/2}(x - K_{01} - K_{11}n^{-1/2}), \\ x_{nt}(y) &= K_{01} + K_{11}n^{-1/2} + K_{02}^{1/2}n^{-1/2}y, \text{ its inverse}, \\ Y_{nt} &= y_{nt}(\hat{\theta}_t), \ Q_{nt}(x) = P(\hat{\theta}_t < x). \end{aligned} \tag{28}$$

Theorem 2 follows by choosing $t = t(x)$ in the following more general result (see Theorem 2 for the notation).

**Theorem 3** *For $t < 0$,*

$$Q_n(x) = P(\hat{\theta}_n < x) \approx n^{-1/2}\alpha_n(x,t)p(y)|L|^{-1} \sum_{i=0}^{\infty} n^{-i/2}\bar{C}_i(y,t), \tag{29}$$

*where $y = y_{nt}(x)$ of (28). For $t > 0$, $1 - Q_n(x) = P(\hat{\theta}_n > x) = RHS$ (29). For $t \neq 0$ and $k = 1, 2, 3, \ldots$,*

$$Q_n^{(k)}(x) \approx n^{k-1/2}\alpha_n(x,t)p(y)|t|^{k-1}k_0^{(2)}(t)^{-1/2} \sum_{i=0}^{\infty} n^{-i/2}\bar{C}_{k-1,i}(y,t). \tag{30}$$

*Proof of Theorem 3* Set $y = y_{nt}(x)$, $B_{rk}(t) = b_{rk}(t)b_{22}(t)^{-r/2}$, and $B(t) = \{B_{rk}(t)\}$. Then

$$Q_{nt}(x) = P(Y_{nt} < y) = P_n(y, B(t))$$

of (9). By (26),

$$\begin{aligned} Q_n(x) &= \exp(nK_0)\int^x \exp(-ntx')dQ_{nt}(x') \\ &= \exp(nK_0)\int^y \exp\{-ntx_{nt}(y')\}dP(Y_{nt} < y') \\ &= \int^y \exp\{A_{nt}(y')\}dP_n(y', B(t)), \end{aligned}$$

where $A_{nt}(y') = nK_0 - ntx_{nt}(y') = -nI_0 - n^{1/2}I_1 + K_{20} + \triangle_n - n^{1/2}Ly'$, $I_i = tK_{i1} - K_{i0}$, $\triangle_n \approx \sum_{r=1}^{\infty} n^{-r/2}d_r$ and $d_r = k_{r+2}(t)$. So, for $\tilde{B}_r$ of (51), $\exp(\triangle_n) = \sum_{r=0}^{\infty} n^{-r/2}\tilde{B}_r(d) = \beta_n$ say. So,

$$Q_n(x) = \alpha_n\beta_n J_t(y, n^{1/2}L), \tag{31}$$

where

$$\begin{aligned} \alpha_n &= \exp(-nI_0 - n^{1/2}I_1 + K_{20}), \\ J_t(y,z) &= \int^y \exp(-yz)dP_n(y, B(t)) \approx \sum_{r=0}^{\infty} n^{-r/2}J_{rt}(y,z), \\ J_{rt}(y,z) &= \int^y \exp(-y'z)p(y')h_{1r}(y', B(t))dy' = \sum_{j=0}^{3r} P_{rj}g_j(y,z) \text{ at } P_{rj} = P_{rj}(B(t)), \\ g_j(y,z) &= \int^y \exp(-yz)H_j(y)p(y)dy = -c_{j-1} - zg_{j-1}(y,z) \\ &= -z^{-1}\{c_j + g_{j+1}(y,z)\}, \quad c_k = \exp(-yz)H_k(y)p(y). \end{aligned}$$

So,

$$\begin{aligned} g_j(y,z) &= \sum_{i=1}^{I}(-z)^{-i}c_{j+i-1} + (-z)^I g_{j+I} \text{ for } I \geq 1, \\ &= \sum_{i=1}^{\infty}(-z)^{-i}c_{j+i-1} = \sum_{k=0}^{\infty}(-z)^{-k-1}c_{j+k} \text{ for } z < 0. \end{aligned} \tag{32}$$

So, for $t < 0$, $L < 0$ and $z = n^{1/2}L$,

$$\begin{aligned} Q_n(x) &\approx \alpha_n\beta_n n^{-1/2}|L|^{-1}\sum_{r=0}^{\infty} n^{-r/2}\sum_{k=0}^{\infty}(-n^{1/2}L)^{-k}\sum_{j=0}^{3r} c_{j+k}P_{rj} \\ &= \alpha_n\beta_n \exp(-yn^{1/2}L)p(y)n^{-1/2}|L|^{-1}\sum_{p=0}^{\infty} n^{-p/2}C_{0p}(y,y) = \text{RHS (29)} \end{aligned}$$

since $\alpha_n(x,t) = \alpha_n \exp(-yn^{1/2}L)$ and $I_0 + t(x - K_{01}) = tx - k_0$. Similarly,

$$P(\hat{\theta} > x) = \alpha_n \beta_n J_t^*(y, n^{1/2}L)$$

at $y = y_{nt}(x)$, where

$$\begin{aligned} J_t^*(y,z) &= \int_y \exp(-yz) dP_n(y, B(t)) \approx \sum_{r=0}^{\infty} n^{-r/2} J_{rt}^*(y,z), \\ J_{rt}^*(y,z) &= \sum_{j=0}^{3r} P_{rj} g_j^*(y,z) \text{ at } P_{rj} = P_{rj}(B(t)), \\ g_j^*(y,z) &= \int_y \exp(-yz) H_j(y) p(y) dy = z^{-1} c_j - z^{-1} g_{j+1}^*(y,z) \\ &= \sum_{i=1}^{\infty} (-z)^{-i} c_{j+i-1} = z^{-1} \sum_{k=0}^{\infty} (-z)^{-k} c_{j+k} \end{aligned} \tag{33}$$

for $z > 0$. So, for $t > 0$, $P(\hat{\theta} > x) = \text{RHS}$ (29). Differentiating (31),

$$Q_n^{(1)}(x) \approx \bar{\alpha}_n \beta_n (n/K_{02})^{1/2} \sum_{r=0}^{\infty} n^{-r/2} \partial J_{rt}(y, n^{1/2}L)/\partial y$$

at $y = y_{nt}(x)$. Also, $\partial J_{rt}(y,z)/\partial y = \exp(-yz) p(y) \sum_{j=0}^{3r} P_{rj} H_j(y)$. Similarly,

$$Q_n^{(k)}(x) \approx \bar{\alpha}_n \beta_n (n/K_{02})^{k/2} \sum_{r=0}^{\infty} n^{-r/2} (\partial/\partial y)^k J_{rt}(y, n^{1/2}L)$$

and

$$(\partial/\partial y)^k J_{rt}(y,z) = (-1)^{k-1} \sum_{j=0}^{3r} P_{rj} M_{k-1,j}(y,z),$$

where, by Leibniz rule,

$$\begin{aligned} M_{k-1,j}(y,z) &= (-\partial/\partial y)^{k-1} \{\exp(-yz)(-\partial/\partial y)^j p(y)\} \\ &= \sum_{i=0}^{k-1} \binom{k-1}{i} A_{k-1-i}(y,z) B_{i+j}(y) \end{aligned}$$

for $A_r(y,z) = (-\partial/\partial y)^r \exp(-yz) = z^r \exp(-yz)$ and $B_r(y) = (-\partial/\partial y)^r p(y) = H_r(y)p(y)$. So, for $k \geq 1$ at $y = y_{nt}(x)$,

$$Q_n^{(k)}(x) \approx (-1)^{k-1}\alpha_n(x,t)\beta_n(n/K_{02})^{k/2}(n^{1/2}L)^{k-1}p(y) \times \sum_{p=0}^{\infty} n^{-p/2}C_{kp}(t,y) = \text{RHS (30)}.$$

□

*Proof of Corollary 1* This follows from Theorem 2 because of the following: $y = 0$ and $P_{rj} = 0$ for $r + j$ odd, $H_j(0) = k_j = \tilde{B}_j(d) = \bar{C}_j = C_j = 0$ for $j$ odd and $\bar{C}_{kj} = C_{kj} = 0$ for $k + j$ even. □

Note 3 We can combine the asymptotic expansions (32) and (34) as

$$\int_{y_0}^{y} \exp(-yz)(-\partial/\partial y)^j p(y)dy = \exp(-yz)\sum_{k=0}^{\infty}(-z)^{-1-k}(-\partial/\partial y)^{j+k}p(y)$$

for $z \neq 0$, where $y_0 = \infty\text{sign}(z) = \infty$ if $z > 0$ and $y_0 = \infty\text{sign}(z) = -\infty$ if $z < 0$.

## 5 Multivariate expansions

Here, we give the multivariate versions of the results of Sects. 2–4. Suppose that $\hat{\theta}$ is a random variable in $R^p$ with the standard cumulant expansion

$$\kappa(\hat{\theta}_{i_1}, \ldots, \hat{\theta}_{i_r}) = \sum_{j=r-1}^{\infty} a_j^{i_1\cdots i_r} n^{-j} \qquad (34)$$

for $r \geq 1$ or the extended expansion

$$\kappa(\hat{\theta}_{i_1}, \ldots, \hat{\theta}_{i_r}) \approx \sum_{k=2r-2}^{\infty} b_k^{i_1\cdots i_r} n^{-k/2} \qquad (35)$$

for $r \geq 1$. Equivalently, suppose that the cumulant generating function of $n\hat{\theta}$ can be expanded as (12), that is,

$$n^{-1}K_{n\hat{\theta}}(t) \approx \sum_{j=0}^{\infty} n^{-j/2}k_j(t).$$

In the case of (34), $k_j(t) = 0$ for $j$ odd. The examples of Sect. 2 all extend to $p \geq 1$. Set

$$Y_n = (Y_{ni}) = n^{1/2}(\hat{\theta} - \theta - b_1 n^{-1/2})$$

in $R^p$, where $\theta_i = b_0^i$ and $(b_1)_i = b_1^i$. So, for $r \geq 1$,

$$\kappa(Y_{ni_1}, \ldots, Y_{ni_r}) \approx n^{r/2} \sum_{k=2r-2}^{\infty} b_k^{i_1 \ldots i_r}(Y_n) n^{-k/2}, \tag{36}$$

where $b_0^i(Y_n) = b_1^i(Y_n) = 0$ and otherwise $b_k^{i_1 \ldots i_r}(Y_n) = b_k^{i_1 \ldots i_r}$. Let $Y$ be another random variable in $R^p$ with cumulants expandable as (36) with $Y_n$ replaced by $Y$ and the same asymptotic covariance, say $V = (b_2^{ij}(Y))$. Then

$$\kappa(Y_{ni_1}, \ldots, Y_{ni_r}) - \kappa(Y_{i_1}, \ldots, Y_{i_r}) \approx n^{r/2} \sum_{k=2r-2}^{\infty} B_k^{i_1 \ldots i_r} n^{-k/2},$$

where $B_0^i = B_1^i = B_2^{ij} = 0$ and otherwise $B_k^{i_1 \ldots i_r} = b_k^{i_1 \ldots i_r} - b_k^{i_1 \ldots i_r}(Y_n) = b_k^{i_1 \ldots i_r}$ if $Y \sim \mathcal{N}_p(0, V)$. For $t$ in $C^p$, the difference in their cumulant generating functions is

$$\nabla_n(t) = K_{Y_n}(t) - K_Y(t) \approx \sum_{j=1}^{\infty} n^{-j/2} e_j(t),$$

where $e_j(t) = \sum_{r=1}^{j+2} B_{r+j}^{i_1 \ldots i_r} t_{i_1} \ldots t_{i_r}/r!$ and here and below the repeated pairs of subscripts $i_1, \ldots, i_r$ are implicitly summed over their range $1, \ldots, p$. Set $B = \{B_k^{i_1 \ldots i_r}\}$ and

$$\tilde{P}_{(n)}(t, B) = \exp(\nabla_n(t)) \approx \sum_{r=0}^{\infty} n^{-r/2} \tilde{P}_r(t, B)$$

for $\tilde{P}_r(t, B) = \tilde{B}_r(e(t))$ of (51): $\tilde{P}_1(t, B) = e_1(t)$, $\tilde{P}_2(t, B) = e_2(t) + e_1(t)^2/2, \ldots$. They have the dual forms

$$\tilde{P}_r(t, B) = \sum_{j=0}^{3r} P_r^{i_1 \ldots i_j} t_{i_1} \ldots t_{i_j} = \sum_{|\nu|=0}^{3r} P_{r\nu} t^{\nu},$$

where $\nu = (\nu_1, \ldots, \nu_p)$ lies in $N^p$, $N = \{0, 1, 2, \ldots\}$, $|\nu| = \sum_{i=1}^{p} \nu_i$, and $t^s = t_1^{s_1} \ldots t_p^{s_p}$ for $t$, $s$ in $C^p$. By an extension of Appendix A, the coefficients $P_{r\nu}(B) = P_{r\nu}$ may be written as explicit polynomials in $B$: $P_1^{i_1 \ldots i_r}(B) = B_{r+1}^{i_1 \ldots i_r}, \ldots$. Suppose that $Y_n$ and $Y$ have densities with respect to Lebesgue measure on $R^p$. Then $P_n(y) = P(Y_n < y)$ satisfies

$$P(Y_n < y) = \tilde{P}_{(n)}(-\partial/\partial y, B)\Phi_V(y) = P_n(y, B) \tag{37}$$

say, with density $p_{(n)}(y) = \tilde{P}_{(n)}(-\partial/\partial y, B)\phi_V(y) \approx \sum_{r=0}^{\infty} n^{-r/2} p_r(y, B)$, where $\Phi_V$ and $\phi_V$ are the distribution and density of $Y$ given by

$$p_r(y, B) = \tilde{P}_r(-\partial/\partial y, B)\phi_V(y) = \phi_V(y) \sum_{|\nu|=0}^{3r} P_{r\nu} H_\nu(y, V)$$

and

$$H_\nu(y, V) = H_\nu(y) = \phi_V(y)^{-1}(-\partial/\partial y)^\nu \phi_V(y).$$

Let $k = (k_1, \ldots, k_s) \in N^s$. Then

$$p_{(n)}^{(k)}(y) \approx (-1)^{|k|} \sum_{r=0}^{\infty} n^{-r/2} p_{kr}(y)$$

for

$$p_{kr}(y) = (-\partial/\partial y)^k \tilde{P}_r(-\partial/\partial y, B) = \phi_v(y) \sum_{|\nu|=0}^{3r} P_{r\nu} H_{\nu+k}(y).$$

The cumulant generating function of $n\hat{\theta}$ satisfies $n^{-1} K_{n\hat{\theta}}(t) \approx \sum_{j=0}^{\infty} n^{-j/2} k_j(t)$ $= K(t)$ say, where

$$k_j(t) = \sum_{r=1}^{\infty} B_{j+2r-2}^{i_1 \ldots i_r} t_{i_1} \ldots t_{i_r}/r! = \sum_{r=1}^{\infty} \sum_{|\nu|=r} B_{\nu, j+2r-2} t^\nu/\nu!$$

say. For example, $k_0(t) = B_0^i t_i + B_1^{i_1 i_2} t_{i_1} t_{i_2}/2! + \cdots$. Let $t$ be an interior point in $T$ $= \{t = k_0(t) < \infty\}$. Define the partial derivatives $K_j^{i_1 \ldots i_r} = k_j^{(i_1 \ldots i_r)}(t)$ and $K^{i_1 \ldots i_r} =$ $K^{(i_1 \ldots i_r)}(t)$. Let $\hat{\theta}_t$ be a random variable in $R^s$ with distribution

$$P(\hat{\theta}_t < x) = \exp(-nK(t)) \int^x \exp(nt'x) dP(\hat{\theta} < x).$$

So, $P(\hat{\theta} < x) = \exp(nK(t)) \int^x \exp(-nt'x) dP(\hat{\theta}_t < x)$. The cumulant generating function of $n\hat{\theta}_t$ is again given by (27) so

$$\kappa^{i_1 \ldots i_r}(n\hat{\theta}_t) = nK^{i_1 \ldots i_r}.$$

So, $\hat{\theta}_t$ satisfies (35) with $B = \{b_k^{i_1\ldots i_r}\}$ replaced by $B(t) = \{b_k^{i_1\ldots i_r}(t) = K_{k-2r+2}^{i_1\ldots i_r} = k_{k-2r+2}^{(i_1\ldots i_r)}(t)\}$. Assume $\hat{\theta}$ is not a constant so $(K^{ij})$ and $(K_0^{ij})$ are positive-definite. Set

$$\begin{aligned} y_{nt}(x) &= n^{1/2}(x - (K_0^i) - (K_1^i)n^{-1/2}) = n^{1/2}(x - \dot{k}_0(t) - n^{1/2}\dot{k}_1(t)), \quad (38)\\ x_{nt}(y) &= (K_0^i) + (K_1^i)n^{-1/2} + n^{-1/2}y,\ Y_{nt} = y_{nt}(\hat{\theta}_t),\ Q_{nt}(x)\\ &= P(\hat{\theta}_t < x) = P(Y_{nt} < y_{nt}(x)). \end{aligned}$$

Then $Q_{nt}(x) = P_n(y_{nt}(x), B(t))$ for $P_n(y, B)$ of (37) and

$$P(\hat{\theta} < x) = \exp(nK(t)) \int^{y_{nt}(x)} \exp(A_{nt}(y)) dP_n(y, B(t)),$$

where $I_i = \sum_{j=1}^s t_j K_i^j - k_i$, $\triangle_n = \sum_{r=1}^\infty n^{-r/2} d_r$, $d_r = k_{r+2}$ and

$$A_{nt}(y) = nK(t) - nt'x_{nt}(y) = -nI - n^{1/2}I_1 + k_2(t) + \triangle_n - n^{1/2}t'y.$$

Set $\alpha_n = \exp(-nI - n^{1/2}I_1 + k_2(t))$ and $\beta_n = \exp(\triangle_n)$. So, $P(\hat{\theta} < x) = \alpha_n\beta_n J_t(y, n^{1/2}t)$ at $y = y_{nt}(x)$, where

$$\begin{aligned} J_t(y, z) &= \int^y \exp(-y'z) dP_n(y, B(t)) = \sum_{r=0}^\infty n^{-r/2} J_{rt}(y, z),\\ J_{rt}(y, z) &= \int^y \exp(-y'z) p_r(y, B(t)) \mathrm{d}y = \sum_{|\nu|=0}^{3r} P_{r\nu} g_\nu(y, z, V_t), \end{aligned}$$

and

$$\begin{aligned} P_{r\nu} &= P_{r\nu}(B(t)),\ V_t = \partial^2 k_0(t)/\partial t \partial t',\ g_\nu(y, z, V)\\ &= \int^y \exp(-y'z) H_\nu(y, V)\phi_V(y) \mathrm{d}y. \end{aligned}$$

By (2.18) of Withers (1996), for $z_1 \ldots z_p \neq 0$ and $(y_0)_j = \infty\ \mathrm{sign} z_j$, we have the asymptotic expansion

$$\int_{y_0}^y \exp(-y'z) H_\nu(y, V)\phi_V(y) dy = \exp(-y'z) \sum_{\lambda=\mathbf{0}}^\infty (-z)^{-1_p - \lambda} H_{\nu+\lambda}(y, V)\phi_V(y). \quad (39)$$

(This identity was given in Note 3 for the case $p = 1$).

For $z < 0$, LHS (40) = $g_\nu(y, z, V)$. This proves the following result for the case $t < 0$ in $R^s$.

**Theorem 4** *Define $\tilde{B}_r(d)$ by (51). Set*

$$J_{rt}(y,z) = \exp(-y'z)\phi_{V_t}(y)\sum_{|\nu|=0}^{3r} P_{r\nu}(B(t))\sum_{\lambda=\mathbf{0}}^{\infty}(-z)^{-1_s-\lambda}H_{\nu+\lambda}(y,V_t).$$

*For $t_1\cdots t_s \neq 0$ and $y = y_{nt}(x)$ of (38),*

$$P((\text{sign}\; t_j)(\hat{\theta}_j - x_j) > 0 \;for\; 1 \le j \le s) \approx \alpha_n\beta_n \sum_{r=0}^{\infty} n^{-r/2}J_{rt}(y, n^{1/2}t)$$

$$= \alpha_n(x,t)n^{-s/2}|t_1\cdots t_s|^{-1}\phi_{V_t}(y)\sum_{i=0}^{\infty} n^{-i/2}\bar{C}_{0i}(y,t), \tag{40}$$

*where $\alpha_n(x,t) = \exp\{-n(x't-k_0)+n^{1/2}k_1+k_2\}$, $\bar{C}_{0i}(y,t) = \sum_{j=0}^{i} C_{0j}(y,t)\tilde{B}_{i-j}(d)$ at $d_r = k_{r+2}(t)$, $C_{0j}(y,t) = \sum_{r=0}^{j}(-1)^{j-r}G_{r,j-r}$, $G_{rk} = \sum_{|\lambda|=k} t^{-\lambda}D_{r\lambda}$, and $D_{r\lambda} = \sum_{|\nu|=0}^{3r} P_{r\nu}(B(t))H_{\nu+\lambda}(y,V_t)$.*

*Proof of Theorem 4* In Sect. 3 we gave the proof for $s = 1$ and $t > 0$. Here, we have given the proof for the case $t < 0$ in $R^s$. The proof for $t$ in the other $2^s - 1$ quadrants is the same by virtue of (39). □

**Corollary 2** *Given $x$ in $R^s$, choose $t = t(x)$ to maximise*

$$x't - k_0(t). \tag{41}$$

*So, $x = \partial k_0(t)/\partial t$ at $t = t(x)$. Suppose that $x_j \neq \theta_j$ for $1 \le j \le s$, where $\theta_j = B_0^j$. Set $y(x) = -\partial k_1(t)/\partial t$ at $t = t(x)$. Then at $y = y(x)$*

$$P((\hat{\theta} - x)_j \,.\, \text{sign}\;(\theta - x)_j < 0 \;for\; 1 \le j \le s) = RHS\;(40). \tag{42}$$

*Also at $t = t(x)$, $\alpha_n(x,t) = \exp\{-nI(x) + n^{1/2}k_1 + k_2\}$, where*

$$I(x) = sup_{t\in T}(x't - k_0(t)) = x't(x) - k_0(t(x)) \tag{43}$$

*for $T = \{t : k_0(t) < \infty\}$.*

*Proof* $y = (x-\theta)'t$ is parallel to the tangent plane to $k_0(t) - \theta' t$ at $t = t(x)$. Consider a plot of

$$k_0(t) - \theta' t = B_1^{ij} t_i t_j/2! + \cdots.$$

Its tangent plane at $t = \mathbf{0}$ has gradient $\mathbf{0}$. Its tangent plane at $t = t(x)$ has gradient $x-\theta$, and $I(x)$ is the distance of the point $(k_0(t)-\theta' t, t)$ below the plane $y = (x-\theta)'t$ at $t = t(x)$. So, sign $(x-\theta)_j$ = sign $t_j$ at $t = t(x)$. □

**Corollary 3** *If* (34) *holds and* $Y$ *is symmetric about* 0*, that is* $Y$ *and* $-Y$ *have the same distribution, then provided* $x_i \neq \theta_i$ *for* $i = 1, \ldots, s$,

$$
\begin{aligned}
LHS(42) &= P((\text{sign}\, t_j)(\hat{\theta}_j - x_j) > 0 \text{ for } 1 \leq j \leq s) \\
&\approx \exp\{-nI(x) + k_2(t)\} n^{-s/2} |t_1 \cdots t_s|^{-1} \phi_{V_t}(0) \sum_{i=0}^{\infty} n^{-i} \bar{C}_{0,2i}(0,t)
\end{aligned}
\tag{44}
$$

*for* $I(x)$ *of* (43).

*Proof* Note $H_\lambda(0, V_t) = 0$ for $|\lambda|$ odd, $P_{r\nu} = 0$ for $r + |\nu|$ odd, $k_j = \tilde{B}_j(d) = \bar{C}_j = C_j = 0$ for $j$ odd, and $D_{r\lambda} = 0$ for $r + |\lambda|$ odd. □

Note 4 Putting $P_{r\nu} = P_{r\nu}(B(t))$, the leading terms are given by $\bar{C}_{00}(0,t) = C_{00}(0,t) = D_{00} = P_{00} = 1$ and $\bar{C}_{02}(0,t) = C_{02}(0,t) + k_4$, where $C_{02}(0,t) = G_{02} - G_{11} + G_{20}$

$$
G_{02} = \sum_{|\lambda|=2} t^{-\lambda} D_{0\lambda}, \; G_{11} = \sum_{|\lambda|=1} t^{-\lambda} D_{1\lambda}, \; G_{20} = D_{20} = \sum_{|\nu|=2,4,6} P_{2\nu} H_\nu(0, V_t)
$$

and

$$
D_{0\lambda} = H_\lambda(0, V_t), \; D_{1\lambda} = \sum_{|\nu|=1,3} P_{1\nu} H_{\nu+\lambda}(\theta, V_t).
$$

The next term is $\bar{C}_{04}(0,t) = k_6 + 3k_4^2/2 + k_4 C_{02}(0,t) + C_{04}(0,t)$, where $C_{04}(0,t) = \sum_{r=0}^{4} (-1)^r G_{r,4-r}$,

$$
\begin{aligned}
G_{04} &= \sum_{|\lambda|=4} t^{-\lambda} H_\lambda(0, V_t), \; G_{13} = \sum_{|\lambda|=3} t^{-\lambda} D_{1\lambda}, \; G_{22} = \sum_{|\lambda|=2} t^{-\lambda} D_{2\lambda}, \\
G_{31} &= \sum_{|\lambda|=1} t^{-\lambda} D_{3\lambda}, \; D_{2\lambda} = \sum_{|\nu|=2,4,6} P_{2\nu}(B(t)) H_{\nu+\lambda}(0, V_t), \\
D_{3\lambda} &= \sum_{|\nu|=1,3,\ldots,7,9} P_{3\nu}(B(t)) H_{\nu+\lambda}(0, V_t),
\end{aligned}
$$

and

$$
G_{40} = D_{40} = \sum_{|\nu|=2,4,\ldots,10,12} P_{4\nu}(B(t)) H_\nu(0, V_t).
$$

**Theorem 5** *Take* $y = y_{nt}(x)$ *of* (38) *and* $\alpha_n(x,t)$ *of Theorem* 4*. For* $k \geq \mathbf{0}$ *in* $N^s$ *and* $t_1 \ldots t_s \neq 0$*, the partial derivatives of* $Q_n(x) = LHS$ (40) *can be expanded as*

$$
Q_n^{(k)}(x) = \prod_{j=1}^{s} (\partial/\partial x_j)^{k_j} Q(x) \approx \alpha_n(x,t) n^{|k|-s/2} |t^{k-1_s}| \phi_{V_t}(y) \sum_{i=0}^{\infty} n^{-i/2} \bar{C}_{k,i},
$$

*where*

$$\bar{C}_{ki} = \bar{C}_{ki}(y,t) = \sum_{j=0}^{i} C_{kj}\tilde{B}_{i-j}(d),\ C_{kj} = C_{kj}(y,t)$$
$$= \sum_{r=0}^{j} G_{kr,j-r},\ G_{krs} = \sum_{|\lambda|=s} T_{\lambda k}(-t)^{-\lambda} D_{r\lambda} \tag{45}$$

*and*

$$T_{\lambda k} = \prod_{j=1}^{s}\left\{(-1)^{\lambda_j}\binom{k_j-1}{\lambda_j} I(\lambda_j < k_j) + \delta_{0,k_j} I(\lambda_j \geq k_j)\right\}, \tag{46}$$

*and* $\alpha_n(x,t)$ *and* $D_{r\lambda}$ *as defined in Theorem 4. So, if* $k \geq 1_p$ *then* $G_{krs} = \sum_{|\lambda|=s} \binom{k-1_p}{\lambda} t^{-\lambda} D_{r\lambda}$.

*Proof* Consider the case $t < 0$ in $R^s$. Set $z = n^{1/2}t$ and $\gamma_{nk} = \prod_{i=1}^{p}(\partial y_i/\partial x_i)^{k_i} = n^{|k|/2}$. Differentiating (40) gives

$$Q_n^{(k)}(x) \approx \alpha_n\beta_n\gamma_{nk}\sum_{r=0}^{\infty} n^{-r/2}\sum_{|\nu|=0}^{3r} P_{r\nu}\sum_{\lambda=0}^{\infty}(-z)^{-1_s-\lambda}A_{k,\nu+\lambda},$$

where

$$A_{k,\nu+\lambda} = (\partial/\partial y)^k\{\exp(-y'z)(-\partial/\partial y)^{\nu+\lambda}\phi_V(y)\}$$
$$= \exp(-y'z)\sum_{a=0}^{k}\binom{k}{a}(-z)^{k-a}(-1_s)^a(-\partial/\partial y)^{\nu+\lambda+a}\phi_V(y)$$

by Leibniz' rule. So, $Q_n^{(k)}(x) = \alpha_n\beta_n\gamma_{nk}(-z)^{k-1_s}\exp(-y'z)\phi_{V_t}(y)S_{nk}$, where

$$S_{nk} \approx \sum_{r=0}^{\infty} n^{-r/2}\sum_{\lambda,a=0}^{\infty}(-z)^{-\lambda-a}\binom{k}{a}(-1_s)^a D_{r,\lambda+a} = \sum_{j=0}^{\infty} n^{-j/2} C_{k-1_s,j},$$
$$C_{k-1_s,j} = \sum_{r+|c|=j}(-t)^{-c}D_{rc}T_{ck},$$

and $T_{ck} = \sum_{a=0}^{c}(-1_s)^a\binom{k}{a}$ = RHS (46) at $\lambda = c$ by Note 3. □

**Corollary 4** *Suppose that* (34) *holds and* $Y$ *is symmetric about* 0. *Set* $t = t(x)$ *of* (41)*,* $I(x) = x't - k_0(t)$ *and* $H_\lambda = H_\lambda(0, V_t)$. *Define* $T_{\lambda k}$ *by* (46). *Then for* $k$ *in* $N^p$*,* *and* $x_i \neq \theta_i$ *for* $i = 1, \ldots, p$

$$Q_n^{(k)}(x) = \exp\{-nI(x) + k_2(t)\}n^{|k|-s/2}|t^{k-1_s}|\phi_{V_t}(0)\sum_{i=0}^{\infty} n^{-i}\bar{C}_{k,2i}$$

*for* $I(x)$ *of* (43) *and* $\bar{C}_{k,i} = \bar{C}_{k,i}(0,t)$ *of* (45). *The leading terms are* $\bar{C}_{k,0} = 0$, $\bar{C}_{k,2} = k_4 + C_{k,2}$ *and*

$$\bar{C}_{k,4} = k_6 + 3k_4^2/2 + k_4 C_{k,2} + C_{k,4},$$

*where*

$$C_{k,2} = \sum_{|\lambda|=2} T_{\lambda k} t^{-\lambda} H_\lambda - \sum_{|\lambda|=1} T_{\lambda k} t^{-\lambda} D_{1\lambda} + D_{20}$$

*and*

$$C_{k4} = \sum_{|\lambda|=4} T_{\lambda k} t^{-\lambda} H_\lambda - \sum_{|\lambda|=3} T_{\lambda k} t^{-\lambda} D_{1\lambda} + \sum_{|\lambda|=2} T_{\lambda k} t^{-\lambda} D_{2\lambda} - \sum_{|\lambda|=1} T_{\lambda k} t^{-\lambda} D_{3\lambda} + D_{40}$$

*for* $D_{1\lambda}$, $D_{2\lambda}$, $D_{3\lambda}$ *and* $D_{40}$ *given by Note* 4.

*Proof* This is as for that of Corollary 3. □

## 6 Functions of a sample mean

A weakness of the method so far is the difficulty of obtaining the functions $\{k_i(t)\}$. For $\hat{\theta} = \bar{X}$ these have been given in terms of the cumulant generating function of $X$. But usually only the leading coefficients $\{a_{ri}\}$ are available. Here, we show how to obtain $\{k_i(t)\}$-or rather $P(\hat{\theta} > x)$-directly in terms of the cumulant generating function of $X$ when $\hat{\theta}$ is any smooth function of $\bar{X}$. Let $\bar{X}$ be the mean of an i.i.d. sample of size $n$ from a distribution on $R^k$ with mean $\mu$, covariance $V_X$ and cumulant generating function $K_X(s) = \sum_k \kappa_\lambda s^\lambda/\lambda!$ for $s$ in $C^k$, where $\sum_k$ sums over $\lambda$ in $N^k - \{\mathbf{0}\}$. Let

$$\hat{\theta} = H(\bar{X})$$

for $H : R^k \to R^p$ a function with finite derivatives at $\mu$. The condition $det\ V > 0$ implies $p \leq k$. So, (34) holds with the leading cumulant coefficients $a_j^{i_1\ldots i_r}$ needed for the Edgeworth expansions (9) and (10) given by the appendix to Withers (1982) or alternatively, by the multivariate version of the formula for $a_{rj}$ given in Corollary 3.1 of Withers (1983b).

Take $\theta < x$ in $R^p$ and $Y \sim \mathcal{N}_p(0, V)$. Since

$$P(\hat{\theta} > x) = \int_{H(z)>x} dP(\bar{X} < z), \tag{47}$$

we have two methods of evaluating (47). By (44),

$$P(\hat{\theta} > x) = \exp\{-nI(x) + k_2(t)\}n^{-p/2}|t_1 \dots t_p|^{-1}\phi_{V_t}(0)\sum_{i=0}^{\infty} n^{-i}\bar{C}_{0,2i}(0,t)$$

at $t = t(x)$, where $I(x) = x't - k_0(t)$. Also $\phi_{V_t}(0) = (2\pi)^{-p/2}det(k_0^{(2)}(t))^{-1/2}$, where $k^{(2)}(t) = \partial^2 k(t)/\partial t \partial t'$. Set $\bar{C}_{m,i}(y, t : \hat{\theta}) = \bar{C}_{m,i}(y,t)$ of (45). By Corollary 4 with $k = 1_p$ and $p$ replaced by $k$,

$$\begin{aligned} P(\hat{\theta} > x) &= \int_{H(z)>x} dP(\bar{X} < z) \\ &= (n/2\pi)^{k/2} det(K_X^{(2)}(s))^{-1/2} \int_{H(z)>x} dz \exp\{-nJ(z)\} \\ &\quad \times \sum_{i=0}^{\infty} n^{-i}\bar{C}_{1_p,2i}(z,\ s : \bar{X}), \end{aligned}$$

where $J(z) = z's - K_X(s)$ at $s = s(z)$ such that $z = K_X^{(1)}(s)$.

We now sketch a way of obtaining our expression for LHS (47) from that for RHS (47). Since our functions are analytic we may write

$$s(z) = \sum_k (z-\mu)^{\nu} S_\nu/\nu!,\ \ J_0(z) = J_0(z, s(z)) = \sum_k (z-\mu)^{\nu} J_\nu/\nu!$$

with $S_\nu = (S_{\nu i})$ in $R^k$ and $J_\nu$ in $R$. These coefficients can be obtained from the cumulants of $X$ and it can be shown $J_0(z) = \sum_{|\nu|\geq 2}(z-\mu)^\nu J_\nu/\nu!$ that is, $J_0(z)$ is quadratic at $z = \mu$. The leading term in RHS (47) is

$$\phi_{V_X}(0)n^{k/2}\int \{\exp(-nJ_0(z))dz\ :\ H(z) > x\}. \tag{48}$$

Suppose there is a unique $z = Z = Z_x$ in $R^k$ maximising $J_0(z)$ over $\{z\ :\ H(z) > x\}$ and that it lies in $\{z\ :\ H(z) = x\}$. Then $(Z, w)$ in $R^{k+p}$ is the simultaneous solution of the $K + p$ equations $\dot{J}_0(z) = \dot{H}(Z)w$ and $H(Z) = x$, where $\dot{J}_0(Z) = \partial J_0(Z)/\partial Z$ is $k \times 1$, $\dot{H}(Z) = \partial H(Z)'/\partial Z$ is $k \times p$, and $w$ is a $p \times 1$ Lagrange multiplier.

Let us transform to $u = n(z - Z)$. So, $|\partial u/\partial z'| = n^k$ and

$$nJ_0(z) = n\sum_k (z-Z)^\nu J_0^{(\nu)}(Z)/\nu! = nJ_0(Z) + \sum_{i=0}^{\infty} n^{-i}a_i(u),$$

where $a_0(u) = u'\dot{J}_0(Z)$. Similarly, $H(z) \approx H(Z) + n^{-1}\sum_{i=0}^{\infty} n^{-i}b_i(u)'$, where $b_0(u)' = u'\dot{H}(Z)$ is $1 \times p$. So, RHS (48) is $\phi_{V_X}(0)\exp\{-nJ_0(Z)\}\{L(Z) + O(n^{-1})\}$, where $a = \dot{J}_0(Z)$, $B = \dot{H}(Z)'$ and

$$L(Z) = \int [\exp\{-a_0(u)\} \ : \ b_0(u) \geq 0] = \int_{Bu>0} \exp(-a'u)du.$$

Note $N$ is $p \times k$. Set $v = Bu$. If $p = k$, $det\ B \neq 0$, and $c = (B')^{-1}a$ then

$$L(Z) = \int_{v>0} \exp(-c'v)dv|det\ (\partial v/\partial u')| = det\ B \cdot \prod_{i=1}^{k} c_i|^{-1}$$

assuming $c > 0$ in $R^k$. So, if $p = k$ this gives

$$[x't - k_0(t)]_{t=t(x)} = I(x) = J_0(Z), \tag{49}$$

$$\left[\exp\{k_2(t)\}|t_1 \cdots t_p|^{-1}\phi_V(0)\right]_{t=t(x)} = \phi_{V_X}(0)L(Z_X). \tag{50}$$

Since Appendix B gives $t(x)$ in terms of the coefficients of $k_0(t)$, (49) does *not* provide an alternative method for obtaining $k_0(t)$. However, (50) could in theory be used to obtain $k_2(t)$ by putting $x = t^{-1}(t)$. If $p < k$, (49) gives $L(Z) = \infty$, so this method breaks down.

**Acknowledgments** The authors would like to thank the Editor, the Associate Editor and the two referees for carefully reading the paper and for their comments which greatly improved the paper.

## Appendix A

For $x = (x_1, x_2, \ldots)$ a sequence in $R$ or $C$, and $y$ in $R$ or $C$ define $\tilde{B}_{rj}(x)$ by

$$\left(\sum_{i=1}^{\infty} x_i y^i\right)^j = \sum_{r=j}^{\infty} \tilde{B}_{rj}(x)y^r$$

for $j = 0, 1, 2, \ldots$ and $\tilde{B}_r(x)$ by

$$\exp\left(\sum_{i=1}^{\infty} x_i y^i\right) = \sum_{r=0}^{\infty} \tilde{B}_r(x)y^r.$$

So,

$$\tilde{B}_r(x) = \sum_{j=0}^{r} \tilde{B}_{rj}(x)/j!. \tag{51}$$

Comtet (1974) calls $\tilde{B}_r(x)$ and $\tilde{B}_{rj}(x)$ the complete and partial ordinary Bell polynomials and tables the latter on p. 309 (we use $^\sim$ rather than his $\hat{}$ to avoid confusion with $\hat{\theta}$

as an estimate for $\theta$). Substituting $x_j = e_j(t)$ of (6), that is $e_j(t) = \sum_{r=1}^{j+2} B_{r,r+j} t^r/r!$ for $j \geq 1$, gives

$$
\begin{aligned}
&P_{00} = 1,\ P_{r0} = 0 \text{ for } r \geq 1,\ P_{r1} = B_{1,r+1} \text{ for } r \geq 1,\ P_{13} = B_{34}/6,\\
&P_{22} = (B_{24}+B_{12}^2)/2,\ P_{23}=B_{35}/6+B_{12}B_{23}/2,\ P_{24}=B_{46}/24+B_{12}B_{34}/6+B_{23}^2/8,\\
&P_{25} = B_{23}B_{34}/12,\ P_{26} = B_{34}^2/72,\ P_{32} = B_{25}/2 + B_{12}B_{13},\\
&P_{33} = B_{36}/6 + B_{12}B_{24}/2 + B_{13}B_{23}/2 + B_{12}^3/6,\\
&P_{34} = B_{47}/24 + B_{12}B_{35}/6 + B_{13}B_{34}/6 + B_{23}B_{24}/4 + B_{12}^2 B_{23}/4,\\
&P_{35} = B_{58}/120 + B_{12}B_{46}/24 + B_{23}B_{35}/12 + (B_{24} + B_{12}^2)B_{34}/12 + B_{12}B_{23}^2/8,\\
&P_{36} = (B_{23}B_{46}/4 + B_{34}B_{35}/3 + B_{12}B_{23}B_{34} + B_{23}^3/4)/12,\\
&P_{37} = (B_{34}B_{46}/6 + B_{12}B_{34}^2/3 + B_{23}^2 B_{34}/2)/24,\\
&P_{38} = B_{23}B_{34}^2/144,\ P_{39} = B_{34}^3/1296.
\end{aligned}
$$

Higher order coefficients are easily found using say MAPLE. When (1) holds, (3) simplifies to

$$\kappa_r(Y_n) \approx n^{r/2} \sum_{k=r-1}^{\infty} A_{ri}(Y_n) n^{-i}$$

with $A_{10}(Y_n) = 0$ and $A_{ri}(Y_n) = a_{ri} a_{21}^{-r/2}$ otherwise. If also the cumulants of $Y$ have this form with $A_{10}(Y) = 0$ and $A_{21}(Y) = 1$ then (4) simplifies to

$$\kappa_r(Y_n) - \kappa_r(Y) \approx n^{r/2} \sum_{i=r-1}^{\infty} A_{ri} n^{-i},$$

where $A_{ri} = A_{ri}(Y_n) - A_{ri}(Y)$, so $A_{10} = A_{21} = 0$ and (6) simplifies to $e_{2j}(t) = \sum_{r=1}^{j+1} A_{2r,r+j} t^{2r}/(2r)!$ and $e_{2j-1}(t) = \sum_{r=0}^{j} A_{2r+1,r+j} t^{2r-1}/(2r+1)!$. Also $P_{rj} = 0$ for $r + j$ odd and the non-zero $P_{rj}$ for $r \leq 3$ are

$$
\begin{aligned}
&P_{11} = A_{11},\ P_{13} = A_{32}/6,\\
&P_{22} = (A_{22} + A_{11}^2)/2,\ P_{24} = (A_{43} + 4A_{11}A_{32})/4!,\ P_{26} = A_{32}^2/72,\\
&P_{31} = A_{12},\ P_{33} = (A_{33} + 3A_{11}A_{22} + A_{11}^3)/3!,\\
&P_{35} = (A_{54}/10 + A_{11}A_{43}/2 + A_{22}A_{32} + A_{11}^2 A_{32})/12,\\
&P_{37} = (A_{32}A_{43}/2 + A_{11}A_{32}^2)/72,\ P_{39} = A_{32}^3/1296
\end{aligned}
\tag{52}
$$

(c.f. Sect. 4 of Withers 1984). If $\kappa_r(\hat{\theta}) = \kappa_r n^{1-r}$ holds and $\lambda_r = \kappa_r \kappa_2^{-r/2}$ then the $r$ non-zero $\{P_{rk}\}$ are given by

$$P_{rk} = \tilde{B}_{rj}(\alpha)/j! \tag{53}$$

for $r < k \leq 3r$, $k - r = 2j$ even and $\alpha_j = \lambda_{j+2}/(j+2)!$.

## Appendix B

Here, we assume that (1) holds. So, the saddlepoint $t(x)$ and the index of large deviation $I_0(x)$ of (14) and (15) are determined by the leading cumulant coefficients $\{a_{r,r-1}\}$. In many situations a closed form for the saddlepoint and the index may not exist. In this appendix we give expansions for them amenable to computer calculation directly in terms of the leading coefficients.

Set $(z)_k = z!/(z-k)! = z(z-1)\ldots(z-k+1)$. For $x = (x_1, x_2, \ldots)$, a sequence in $C$, and $y$ in $C$, Comtet (1954) defines the partial exponential Bell polynomial $B_{rj}(x)$ by

$$\left(\sum_{i=1}^{\infty} x_i y^i/i!\right)^j /j! = \sum_{r=j}^{\infty} B_{rj}(x) y^r/r!$$

for $j = 0, 1, 2, \ldots$. Comtet (1954) tables them on pages 307 and 308 and gives simple recurrence formulas for them.

**Theorem B** *Set* $l_r = a_{r,r-1}$, $\nu_r = l_r/l_2^{r/2}$, $\beta_r = \nu_{r+2}/(r+1)$, $C_1 = 1$, $C_i = \sum_{k=1}^{i-1} (-i)_k B_{i-1,k}(\beta)$ *for* $i \geq 2$, $z_x = (x - l_1)/l_2^{1/2} = (x-\theta)/a_{21}^{1/2}$, *and* $J(z) = \sum_{i=2}^{\infty} C_{i-1} z^i/i!$. *So,* $J(z)$ *has first derivative*

$$\dot{J}(z) = \sum_{i=1}^{\infty} C_i z^i/i!.$$

*Then*

$$t(x) = l_2^{-1/2} \dot{J}(z_x) \tag{54}$$

*and*

$$I(x) = J(z_x). \tag{55}$$

*Proof* Set $b_i = l_{i+1}/l_2$ and $y_x = (x - l_1)/l_2$. Write $x = k_0^{(1)}(t)$ as $z_x = \sum_{i=1}^{\infty} b_i t^i/i!$. By page 151 of Comtet (1954), $t(x) = l_2^{-1/2} \sum_{i=1}^{\infty} c_i y_x^i/i!$, where $c_1 = 1$ and $c_i = \sum_{k=1}^{i-1} (-i)_k B_{i-1,k}(b_2/2, b_3/3, \ldots)$ for $i \geq 2$. Now substitute into $I(x) = xt - k_0(t)$ at $t = t(x)$ to obtain $I(x) = \sum_{i=2}^{\infty} d_i y_x^i/i!$, where

$$d_i = i c_{i-1} k_2 - \sum_{r=2}^{i} k_r B_{ir}(c).$$

Putting $C_i = c_i l_2^{(1-i)/2}$ and $D_i = d_i l_2^{-i/2}$, we obtain (54) and $I(x) = \sum_{i=2}^{\infty} D_i z_x^i/i!$. That $d_i = l_2 c_{i-1}$ and $D_i = C_{i-1}$ hold, that is, (55) holds, follows since differentiating (55) gives (54). □

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