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Decompositions of binomial ideals

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Abstract We present *Binomials*, a package for the computer algebra system Macaulay 2, which specializes well-known algorithms to binomial ideals. These come up frequently in algebraic statistics and commutative algebra, and it is shown that significant speedup of computations like primary decomposition is possible. While central parts of the implemented algorithms go back to a paper of Eisenbud and Sturmfels, we also discuss a new algorithm for computing the minimal primes of a binomial ideal. All decompositions make significant use of combinatorial structure found in binomial ideals, and to demonstrate the power of this approach we show how *Binomials* was used to compute primary decompositions of commuting birth and death ideals of Evans et al., yielding a counterexample for their conjectures.

Keywords Algebraic statistics · Binomial ideals · Commuting birth and death ideals · Computational commutative algebra · Primary decomposition

1 Introduction

A monomial ideal is an ideal generated by monomials, a binomial ideal is one whose generators can be chosen as binomials. A *pure difference* ideal is an ideal whose generators are all differences of monic monomials. For monomial ideals, central concepts like Gröbner bases, irreducible and primary decompositions, etc. can be defined directly on the exponent vectors of the monomials generating the ideal. In this sense the whole theory is very combinatorial. For binomial ideals the situation is more complicated,

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but essentially it can be made combinatorial too. Starting with Eisenbud and Sturmfels (1996), the combinatorial theory of binomial ideals has developed into a branch of combinatorial commutative algebra which has many connections to different areas of mathematics (Miller and Sturmfels 2005).

The interest in binomial ideals is motivated by the frequency with which one encounters them. For instance, commutative semigroup rings are exactly the quotients of polynomial rings by pure difference binomial ideals (Gilmer 1984). Toric ideals, which are binomial prime ideals, are the defining ideals of toric varieties as defined by Fulton (1993). This fact is central in the field of algebraic statistics, where closures of discrete exponential families, such as graphical or hierarchical models, have been recognized to be nonnegative real parts of toric varieties (Geiger et al. 2006). Also, binomial ideals which are not prime occur there. Conditional independence models are defined through a set of polynomial equations in the elementary probabilities, and studying primary decompositions of the corresponding ideals is of natural interest (Drton et al. 2009; Fink 2009; Herzog et al. 2009). For instance, as Eisenbud and Sturmfels (1996) have shown, the minimal primes of binomial ideals are essentially toric ideals, and therefore a conditional independence model is a union of exponential families. Knowing the primary decomposition, a piecewise parameterization of the model is instantly available.

This paper deals with the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$, over a field $\mathbb{K}$ of characteristic zero. Choices for $\mathbb{K}$ are the rationals $\mathbb{Q}$, their cyclotomic extensions $\mathbb{Q}(\xi_l)$, or the complex numbers $\mathbb{C}$. Primary decompositions of binomial ideals are not necessarily binomial as is easily seen on the ideal $\langle x^3 - 1\rangle$, which over $\mathbb{Q}$ decomposes as $\langle x-1\rangle \cap \langle x^2+x+1\rangle$. If $\mathbb{K}$ is algebraically closed, however, binomial primary decompositions exist. When speaking of primary decompositions in this paper, we always mean primary decomposition into binomial ideals, and we have to extend the coefficient field where needed. For the software package we have restricted even further; we consider only pure difference binomial ideals. In that case, the primary decompositions into binomials will be shown to exist with coefficients in cyclotomic extensions of $\mathbb{Q}$. In many applications it suffices to study this case. Examples include the semigraphoid ideal (Hemmecke et al. 2008), conditional independence ideals, commuting birth and death ideals of Sect. 2, and almost any other binomial ideal considered in algebraic statistics.

This paper is structured as follows: in Sect. 1.1, we study a systematic way of approximating binomial ideals by cellular binomial ideals. Then, in Sect. 1.2 we give an algorithm for finding the solutions of zero-dimensional pure difference binomial ideals and apply it to saturation of partial characters. In Sect. 1.3, we give a new algorithm for computing the minimal primes of a binomial ideal. Section 2 contains results on large primary decompositions that have been carried out with our software *Binomials*. We show a counterexample to Conjectures 5.3 and 5.9 in Evans et al. (2010). Finally, Sect. 3 concludes the paper with future research directions.

Throughout the paper we use notation that tries to coincide with that of Eisenbud and Sturmfels (1996). We assume familiarity with basic notions of commutative algebra. A very pedagogical introduction is the book of Cox et al. (1996), while Eisenbud (1995) covers everything from the very basics to current research topics. In keeping with the introductory nature of this work, each of the following sections contains examples of how to do the discussed computations with the help of *Binomials*. These examples are thought of as a motivation and do not cover all of the functionality that is implemented. They are produced with version 0.5.4 of *Binomials*. The reader is encouraged to download the package, use it, and report experiences to the author. An online help is integrated.

Example 1 (Installation) *Binomials* and an auxiliary package for cyclotomic fields, called *Cyclotomic*, are available under the URL:

$$\texttt{http://personal-homepages.mis.mpg.de/kahle/bpd/} \tag{1}$$

It is recommended to install the latest version of Macaulay 2 (Eisenbud et al. 2001) before using *Binomials*. To get started, run Macaulay 2, then load the package with

```
i1 : load "Binomials.m2"
```

The additional packages *FourTiTwo* and *Cyclotomic* are needed. The first is included in Macaulay 2 as of version 1.2, while the latter can be obtained together with *Binomials*. To make the documentation available the package should be installed:

```
i2:installPackage("Binomials",RemakeAllDocumentation=>true)
```

After running this, help can be accessed with

```
i3 : help "Binomials"
```

1.1 Cell decompositions of binomial varieties

Our analysis of a binomial variety starts with the decomposition of $\mathbb{K}^n$ into the 2^n algebraic tori interior to the coordinate planes. Each of the coordinate planes is defined by a subset $\mathcal{E} \subseteq \{1, \ldots, n\}$ of the indeterminate's indices. We denote the algebraic torus corresponding to $\mathcal{E}$ by

$$(\mathbb{K}^*)^{\mathcal{E}} := \left\{(x_1, \ldots, x_n) \in \mathbb{K}^n : x_i \neq 0, i \in \mathcal{E} \text{ and } x_j = 0, \forall j \notin \mathcal{E}\right\}. \tag{2}$$

Geometrically, for a binomial ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$, we study *cellular decompositions*. Their components are the intersections of primary components which have generic points in a given cell $(\mathbb{K}^*)^{\mathcal{E}}$. The central definition is

Definition 1 A proper binomial ideal $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ is called *cellular* if each variable x_i is either a nonzerodivisor or nilpotent modulo I.

In this paper a *variable* is always a variable in a polynomial ring, random variables are not mentioned explicitly. Primary ideals I are cellular as *every element* of $\mathbb{K}[x_1, \ldots, x_n]/I$ is either nilpotent or a nonzerodivisor. The following explicit representation of cellular ideals is only a reformulation of the definition but useful in many ways.

Lemma 1 *A binomial ideal $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ is cellular if and only if there exists a set $\mathcal{E} \subseteq \{1, \ldots, n\}$ of variable indices of $\mathbb{K}[x_1, \ldots, x_n]$ such that*

1. $I = \left(I : (\prod_{i \in \mathcal{E}} x_i)^\infty\right)$,
2. *For every $i \notin \mathcal{E}$, there exists a nonnegative integer d_i such that the ideal $\left\langle x_i^{d_i} : i \notin \mathcal{E}\right\rangle$ is contained in I.*

We call the set $\mathcal{E}$ the *cell indices* and the variables $\{x_i : i \in \mathcal{E}\}$, which are exactly the nonzerodivisors modulo I, the *cell variables*. We denote by $M(\mathcal{E})$ the ideal generated by the noncell variables, i.e. the variables $\{x_i : i \notin \mathcal{E}\}$. For any vector $d = (d_i)_{i \notin \mathcal{E}}$ of natural numbers we denote $M(\mathcal{E})^d := \langle x_i^{d_i} : i \notin \mathcal{E}\rangle$. With this notation, another useful representation of cellular ideals is given by the following Lemma. In Eisenbud and Sturmfels (1996) the ideal on the right hand side of (3) is denoted $I_{\mathcal{E}}^{(d)}$.

Lemma 2 *A binomial ideal I is cellular if and only if there exist a set $\mathcal{E} \subseteq \{1, \ldots, n\}$ and an exponent vector d, such that*

$$I = \left(\left(I + M(\mathcal{E})^d\right) : \left(\prod_{i \in \mathcal{E}} x_i\right)^\infty\right). \tag{3}$$

Radicals of cellular binomial ideals have a nice combinatorial structure, defined by the set $\mathcal{E}$, and a partial character, which we introduce next. For this let $\emptyset \neq \mathcal{E} \subseteq \{1, \ldots, n\}$ be any nonempty subset of the indices of variables and define the shorthand $\mathbb{K}[\mathcal{E}] := \mathbb{K}[x_i : i \in \mathcal{E}]$.

Definition 2 A *partial character* is a pair $(\mathcal{L}, \sigma)$, consisting of an integer lattice $\mathcal{L} \subseteq \mathbb{Z}^{\mathcal{E}}$ and a map $\sigma : \mathcal{L} \to \mathbb{K}^*$, that is a homomorphism from the additive group $\mathcal{L}$ to the multiplicative group $\mathbb{K}^*$. For each integer lattice $\mathcal{L} \subseteq \mathbb{Z}^{\mathcal{E}}$, we define its *saturation*

$$\mathrm{Sat}(\mathcal{L}) := \{m \in \mathbb{Z}^{\mathcal{E}} : dm \in \mathcal{L} \text{ for some } d \in \mathbb{Z}\}. \tag{4}$$

A lattice $\mathcal{L} \subseteq \mathbb{Z}^{\mathcal{E}}$ is called *saturated* if it satisfies $\mathcal{L} = \mathrm{Sat}(\mathcal{L})$. A partial character $(\mathcal{L}, \sigma)$ is called *saturated* if $\mathcal{L} = \mathrm{Sat}(\mathcal{L})$, and it is called a *saturation* of a partial character $(\mathcal{L}', \sigma')$, provided that $\mathcal{L} = \mathrm{Sat}(\mathcal{L}')$ and $\sigma'(l) = \sigma(l)$, $\forall l \in \mathcal{L}'$.

Often it is convenient to denote by L an integer matrix having the lattice $\mathcal{L}$ as its right image $\mathcal{L} := \{Lm : m \in \mathbb{Z}^{\mathcal{E}}\}$. Thus, the columns of L span the lattice, and we abuse notation speaking of the partial character (L, σ) in this case. To each partial character $(\mathcal{L}, \sigma)$ we associate a *lattice ideal*:

$$I_+(\sigma) := \langle x^{m^+} - \sigma(m)x^{m^-} : m \in \mathcal{L}\rangle \subseteq \mathbb{K}[\mathcal{E}]. \tag{5}$$

Here, we have decomposed $m \in \mathbb{Z}^{\mathcal{E}}$ into its positive part $m_i^+ := \max\{m_i, 0\}$, and negative part m^-, so that $m = m^+ - m^-$. We also have used monomial notation $x^m := \prod_{i \in \mathcal{E}} x_i^{m_i}$. In the notation of (5), the lattice is always implicitly understood from σ.

It follows from Theorem 2.1 of Eisenbud and Sturmfels (1996) that a lattice ideal is prime if and only if its partial character is saturated. More generally, all associated primes of a lattice ideal arise from saturations of its partial character. A nice characterization is that a proper binomial ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a lattice ideal if and only if $I = \left(I : (\prod_{i=1}^n x_i)^\infty\right)$. This fact can be used to compute a minimal generating set of a lattice ideal when only the partial character is given, a problem considered for instance in Hoşten and Sturmfels (1995), Bigatti et al. (1999), Hemmecke and Malkin (2009).

A cellular binomial ideal is a lattice ideal on a subset of the variables. For instance, it follows from Lemma 2 that radical cellular binomial ideals $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ are of the form $I = M(\mathcal{E}) + I_+(\sigma)$ for some partial character (L, σ) on $\mathbb{Z}^{\mathcal{E}}$. Now, assuming that $\mathbb{K}$ is algebraically closed, the associated primes of $M(\mathcal{E}) + I_+(\sigma)$ are given by

$$P_\tau = M(\mathcal{E}) + I_+(\tau), \tag{6}$$

where τ runs through all saturations of σ. In particular, a radical cellular binomial ideal is equidimensional. If $\mathbb{K}$ is not algebraically closed, it may contain only some, or even no saturations of (L, σ). In Sect. 1.3 we give an algorithm that computes the minimal primes of a binomial ideal by directly finding a cellular decomposition of the radical of I into radical cellular ideals.

If the monomials in a cellular binomial ideal I are of higher order, then we only have that $I \cap \mathbb{K}[\mathcal{E}]$ is a lattice ideal. However, the associated primes might have partial characters supported on different lattices. The key theorem for computing associated primes of cellular binomial ideals is

Theorem 1 (Eisenbud and Sturmfels 1996, Theorem 8.1) *Let $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be a cellular binomial ideal in the cell variables $\mathcal{E}$. Let $P = M(\mathcal{E}) + I_+(\sigma)$ be an associated prime of I, then there exists a monomial x^m in the variables not in $\mathcal{E}$ and a partial character τ on $\mathbb{Z}^{\mathcal{E}}$ whose saturation is σ, such that*

$$(I : x^m) \cap \mathbb{K}[\mathcal{E}] = I_+(\tau). \tag{7}$$

Note that the associated primes of a cellular binomial ideal are cellular binomial ideals for the same cell variables. To compute them, one considers all quotients of I modulo the standard monomials in the variables outside $\mathcal{E}$. There are only finitely many, as I is cellular and contains $M(\mathcal{E})^d$ for some nonnegative integer vector d. This theorem reduces the computation of associated primes to cellular decomposition and saturation of partial characters.

We now review an algorithm for computing cellular decompositions due to Ojeda and Piedra (2000). It is based on the following approximation scheme for arbitrary ideals in any Noetherian ring:

Lemma 3 (Eisenbud and Sturmfels 1996, Proposition 7.2) *Let I be an ideal in a Noetherian ring S and $g \in S$ such that $(I : g) = (I : g^\infty)$. Then*

1. $I = (I : g) \cap (I + \langle g \rangle)$.
2. $\mathrm{Ass}(S/(I : g)) \cap \mathrm{Ass}(S/(I + \langle g \rangle)) = \emptyset$.

3. *A minimal primary decomposition of I consists of the primary components of $(I : g)$ and those primary components of $I + \langle g \rangle$ that correspond to associated primes of I.*

Given any noncellular binomial ideal I, we can find a variable x_i that is a zerodivisor but not nilpotent modulo I. A power $s > 0$ of that variable satisfies the conditions on g in Lemma 3 and we can write

$$I = (I : x_i^s) \cap (I + \langle x_i^s \rangle), \tag{8}$$

where the ideals on the right-hand side are both binomial and properly containing I. This can be turned into a simple algorithm for cellular decomposition, formulated by Ojeda and Piedra (2000). The authors also provided an implementation in Macaulay 2, parts of which are used in the *Binomials* package.

Algorithm 1 (*Cellular decomposition*) Input: I, a binomial ideal.
Output: A cellular decomposition of I.

1. If I is cellular, return I.
2. Choose a variable that is a zerodivisor but not nilpotent modulo I.
3. Determine the power s such that $(I : x_i^s) = (I : x_i^\infty)$.
4. Iterate with $(I : x_i^s)$ and $I + \langle x_i^s \rangle$.

Step 1 is carried out as follows. First determine the nilpotent variables by checking for which x_i one has $(I : x_i^\infty) = \mathbb{K}[x_1, \ldots, x_n]$. Denoting the remaining variables' indices as $\mathcal{E}$, I is cellular iff $\left(I : (\prod_{i \in \mathcal{E}} x_i)^\infty\right) = I$. Termination of Algorithm 1 is ensured since $\mathbb{K}[x_1, \ldots, x_n]$ is Noetherian and the two ideals $(I : x_i^s)$ and $I + \langle x_i^s \rangle$ properly contain I. Correctness follows from Lemma 3. Also note that cellular components of pure difference binomial ideals are pure difference binomial ideals.

Example 2 (Cellular decomposition) We study an ideal from Eisenbud and Sturmfels (1996). Let $S = \mathbb{Q}[x_1, \ldots, x_5]$ and $I = \langle x_1 x_4^2 - x_2 x_5^2, x_1^3 x_3^3 - x_4^2 x_2^4, x_2 x_4^8 - x_3^3 x_5^6 \rangle$.

```
i1 : S = QQ[x1,x2,x3,x4,x5];
i2 : I = ideal(x1*x4^2-x2*x5^2, x1^3*x3^3-x4^2*x2^4,
          x2*x4^8-x3^3*x5^6);
i3 : toString BCD I
o3 = {ideal(x1*x4^2-x2*x5^2, x1^3*x3^3-x2^4*x4^2,
              x2^3*x4^4-x1^2*x3^3*x5^2,
              x2^2*x4^6-x1*x3^3*x5^4,
              x2*x4^8-x3^3*x5^6),
        ideal(x1^2,x1*x4^2-x2*x5^2,
        x2^5,x5^6,x2^4*x4^2,x4^8)}
i4 : ap = binomialAssociatedPrimes I; toString ap
o4 = {ideal(x1*x4^2-x2*x5^2, x1^3*x3^3-x2^4*x4^2,
              x2^3*x4^4-x1^2*x3^3*x5^2,
              x2^2*x4^6-x1*x3^3*x5^4,
              x2*x4^8-x3^3*x5^6),
        ideal(x2,x5,x4,x1)}
i5 : intersect (ap#0,ap#1) == I
o5 = false
i6 : binomialRadical I == intersect (ap#0,ap#1)
```

```
o6 = true
i7 : isCellular (ap#0, returnCellVars=>true)
o7 = {x1, x2, x3, x4, x5}
i8 : isCellular (ap#1, returnCellVars=>true)
o8 = {x3}
```

In this listing we have suppressed some output. First we compute a cellular decomposition with `BCD`. It has two components. The first ideal is the toric ideal $\left(I : (\prod_{i=1}^{n} x_i)^{\infty}\right)$, which is prime. It is a general feature of the implementation of *Binomials* that, when the input has no monomial generators, the first ideal of the output of cellular and primary decompositions, as well as minimal and associated primes, is always the toric ideal. We also compute the associated primes. The second one is embedded, and we confirm that I is not radical. Also note that the binomial generator $x_1x_4^2 - x_2x_5^2$ in the second cellular component reduces to zero as soon as one takes the radicals of the monomials. Finally, we confirm that the associated primes are cellular, and show the set of variables with respect to which they are cellular, using `isCellular` with the option `returnCellVars`. The cell variables could have been computed directly together with the cellular decomposition by running the long version `binomialCellularDecomposition`, again with the option `returnCellVars` set to `true`.

Theorem 1 shows that saturation of partial characters is a crucial ingredient for computing associated primes of a binomial ideal. We therefore study the properties of saturations of partial characters. In the current implementation of *Binomials* any operation that needs extension of the coefficient field of the polynomial ring is only implemented for pure difference binomial ideals. It will be shown that in this case cyclotomic field extensions suffice.

1.2 Solving pure difference binomial ideals

In this section we give a fast algorithm for solving pure difference binomial ideals of dimension zero. It is not surprising that such a procedure utilizes only the exponents of the generators. We denote by ξ_l the primitive l-th root of unity $\exp\left\{\frac{2\pi \mathrm{i}}{l}\right\} \in \mathbb{C}$. The field extension of $\mathbb{Q}$ that is obtained by adjoining such a root of unity is called a *cyclotomic field* and denoted by $\mathbb{Q}(\xi_l)$. It can be constructed by taking the quotient of a univariate polynomial ring modulo the principal ideal generated by the minimal polynomial of ξ_l, the *cyclotomic polynomial* (Hungerford 1974, Chapter V).

Proposition 1 *Given a zero-dimensional pure difference binomial ideal I, there exists a primitive root of unity ξ_l such that all complex solutions of I are contained in the cyclotomic field $\mathbb{Q}(\xi_l)$.*

The proof is given after the following Lemma, also of interest for the implementation.

Lemma 4 *The complex solutions of the univariate equation*

$$x^n = \xi_m^k, \tag{9}$$

are given by the following roots of unity

$$x_0 = \xi_{mn}^{k}, \quad x_1 = \xi_{mn}^{m+k}, \ldots, x_{n-1} = \xi_{mn}^{(n-1)m+k}. \tag{10}$$

Proof The $x_0, \ldots, x_{n-1}$ are n distinct roots of (9), which is of degree n. □

Proof (Proposition 1) The standard method of reducing a multivariate problem to a univariate problem applies. The general framework is described, for instance, in Chapter 3 of Cox et al. (1996). Choose an elimination term order, such as lexicographic order, and compute a Gröbner basis of I. This Gröbner basis consists of pure difference binomials since all S-polynomials are pure difference binomials. Furthermore, at least one of the binomials of this Gröbner basis is univariate as I is zero-dimensional and we have chosen an elimination order. The solutions of this univariate equation exist in a cyclotomic field by Lemma 4. We continue to extend the partial solution that we have found, substituting the variable for its value in the remaining elements of the Gröbner basis. We obtain a univariate equation in another variable. The final solution exists in the cyclotomic field containing all the roots of unity that are encountered in the course of the algorithm. □

Of course, the procedure that was just described is also valid for other fields $\mathbb{K}$. In the general case, field extensions have to be carried out by computing the minimal polynomial of the element to be adjoined and one has to do computations over the algebraic numbers. While possible in principle, this quickly becomes infeasible in practice, since both the computations become lengthy and it becomes more and more tedious to produce output in a human-readable form.

We are now ready to formulate the algorithm for computing the variety of a zero-dimensional pure difference binomial ideal. The first thing that needs to be accounted for is the possibility of 0 as a solution, potentially with multiplicities. We take care of this by means of cellular decomposition. Each cellular binomial ideal I can be written as $I = \left((I + M(\mathcal{E})^d) : \left(\prod_{i\in\mathcal{E}} x_i\right)^\infty\right)$, and $I \cap \mathbb{K}[\mathcal{E}]$ is a lattice ideal. The solutions of I take the value zero at the variables outside $\mathcal{E}$ and each solution has a multiplicity of $\prod_{i\notin\mathcal{E}} d_i$, where the d_i need to be chosen minimal.

Algorithm 2 (*Solving pure difference binomial ideals*)
Input: A zero-dimensional pure difference binomial ideal I.
Outputs: The root of unity that needs to be adjoined to $\mathbb{Q}$ and the list of the solutions of I.

1. Compute a cellular decomposition of I.
2. For each cellular component:
   (a) Set the noncell variables to zero, and determine the product $D = \prod_{i\notin\mathcal{E}} d_i$ of the minimal powers of the noncell variables.
   (b) Compute a lexicographic Gröbner basis and solve the lattice ideal of the cellular component, adjoining roots of unity where necessary.
   (c) Save each solution D times.
3. Compute the least common multiple m of the powers of the adjoined roots of unity and construct the cyclotomic field $\mathbb{Q}(\xi_m)$.
4. Output the list of collected solutions as elements of $\mathbb{Q}(\xi_m)$.

This algorithm is the main ingredient for saturating partial characters, which we treat after an example.

Example 3 (Solving pure difference binomial ideals) We solve a simple pure difference binomial ideal to introduce the syntax.

```
i1 : S = QQ[x,y,z];
i2 : I = ideal (x^2-y,y^3-z,x*y-z);

i3 : binomialSolve I
BinomialSolve created a cyclotomic field of order 3.

o3 = {{1, 1, 1}, {- ww_3 - 1, ww_3, 1},
      {ww_3, - ww_3 - 1, 1},
      {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

i4 : degree I
o4 = 6
```

In the implementation, generic names consisting of `ww` and the order are assigned to roots of unity. Note that the square of the third root of unity `ww_3` is represented as `-ww_3-1` by means of its minimal polynomial over $\mathbb{Q}$. A cellular decomposition reveals that this ideal has two components, one of which is of degree 3 with associated prime $\langle x, y, z\rangle$. The function `binomialSolve` outputs the solutions with the correct multiplicities. If this is not desired, one can pass to the radical before solving, or directly compute the binomial minimal primes.

Saturations of partial characters exist only over algebraically closed fields. This is evident, for instance, from the partial character $((2), 2 \mapsto -1)$, consisting of the rank 1 lattice spanned by the integer 2, and the character that maps 2 to $-1 \in \mathbb{C}$. The saturations are pairs $(\mathbb{Z}, \tau)$ that satisfy $\tau(2) = \tau(1)^2 = -1$. This example is merely a combinatorial version of factorizing the polynomial $x^2 + 1$, which is the same as performing the primary decomposition of its principal ideal. The following algorithm to saturate a partial character is the general version of the example's principle.

Algorithm 3 (*Saturation of a partial character*)
Input: A partial character (L, σ), where L is a matrix whose columns are minimal generators of a lattice in $\mathbb{Z}^d$.
Output: All distinct saturations $(\mathrm{Sat}(L), \tau_i),\ i = 1, \dots, n$.

1. Compute the saturation $L' := \mathrm{Sat}(L)$.
2. Express the generators of L in terms of the generators of L', by solving the matrix system

$$L = L'K, \tag{11}$$

   for the square matrix $K = (k_{ij})_{i,j=1,\dots,r}$, where $r := \mathrm{rk}(L) = \mathrm{rk}(L')$ denotes the rank of the lattices.
3. Write l_j, l'_j, and k_j for the columns of L, L', and K, respectively. Introduce new variables $\tau_i := \tau(l'_i), i = 1, \dots, r$, for the values that τ takes on the columns

of L'. Using again monomial notation $\tau^m := \prod_{i=1}^r \tau_i^{m_i}$, compute the following zero-dimensional lattice ideal in $\mathbb{Q}[\tau_1, \ldots, \tau_r]$

$$J := \left(\left\langle \tau^{k_j^+} - \sigma(l_j)\tau^{k_j^-} : j = 1, \ldots, r \right\rangle : \left(\prod_{i=1}^r \tau_i \right)^\infty \right), \tag{12}$$

for the given values $\sigma(l_j)$.

4. Solve J (over a suitable extension of $\mathbb{Q}$) and output L' together with the list of solutions of J.

Proof (Correctness) Computing the saturation of a lattice should be viewed as an integer valued analogue of taking the orthogonal complement twice. It can be carried out in Macaulay 2, for instance, by computing the minimal syzygies of the syzygies among the generators of L. The coefficient matrix K that solves the system (11) exists and is unique over $\mathbb{Z}$, as L is a sublattice of L' and we assumed that the columns of L' are a minimal set of generators of the corresponding lattice. The ranks of L and L' coincide by definition. The ideal J is constructed as follows: for each generator l of L we get a relation $l = L' \cdot k$, to which we apply the homomorphism τ, remembering that τ and σ are required to coincide on the generators of L. The entries of K are integers, thus we get the Laurent binomial ideal

$$\left\langle \sigma(l_j) - \prod_{i=1}^r \tau_i^{k_{ij}} : j = 1, \ldots, r \right\rangle, \tag{13}$$

whose intersection with $\mathbb{Q}[\tau_1, \ldots, \tau_r]$ is exactly J. That J is zero-dimensional follows since the quotient L'/L is a finite group. For details see Corollary 2.2 in Eisenbud and Sturmfels (1996). □

The number of distinct saturations equals the order of the finite group $\mathrm{Sat}(L)/L$, which can be computed by diagonalizing the matrix L, representing the inclusion $\mathbb{Z}^r \to \mathbb{Z}^d$. The Macaulay 2 command for this diagonalization is `smithNormalForm`. Finally, for computing primary decompositions of pure difference binomial ideals, we only need to solve such ideals during the saturation.

Proposition 2 *The saturation of a partial character that occurs during primary decomposition of a pure difference binomial ideal involves only solving pure difference binomial ideals.*

Proof Any cellular component of a pure difference binomial ideal is pure difference again. So we can assume that I is cellular. Now, each partial character consists of a lattice and the constant map $l \mapsto 1$. Therefore, the ideal J in Algorithm 3 is a pure difference binomial ideal. □

1.3 Minimal primes of binomial ideals

In this section, we describe a new algorithm for computing the minimal primes of a binomial ideal. It is based on a variant of cellular decomposition, given in Algorithm 1.

As we have seen previously, the associated primes and thereby the minimal primes of a binomial ideal come in groups, associated to the cellular components of I. Our approach is to directly compute a cellular decomposition of the radical of I.

Algorithm 4 (*Minimal primes of a binomial ideal*)
Input: A binomial ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$.
Output: The binomial minimal primes of I.

1. Determine whether I is cellular.
   (a) If yes, compute the radical $(I \cap \mathbb{K}[\mathcal{E}]) + M(\mathcal{E}) = M(\mathcal{E}) + I_+(\sigma)$ and its partial character (L, σ). Compute the saturations $(\tau_i)_{i=1}^{l}$ of σ and save the ideals

$$P(\tau_i) = M(\mathcal{E}) + I_+(\tau_i). \tag{14}$$

   (b) If not, determine a variable x_i that is a zerodivisor, but not nilpotent modulo I, and iterate with the ideals $I + \langle x_i \rangle$ and $(I : x_i^{\infty})$.
2. From all primes collected, remove redundant ones to find a minimal prime decomposition of $\mathrm{Rad}(I)$.

Proof (Termination and correctness) Termination of this algorithm follows as the ambient ring is Noetherian and $I + \langle x_i \rangle$ and $(I : x_i^{\infty})$ strictly contain I. The radical of I is the intersection of the ideals $I_{\mathcal{E}}$ in (4.2) of Eisenbud and Sturmfels (1996). We encounter a decomposition of $\mathrm{Rad}(I)$ into such ideals in the course of the algorithm, as the iteration is ultimately producing cellular components of the radical of I. Thus, like in their Algorithm 9.2, correctness has been proved in Section 4 of Eisenbud and Sturmfels (1996). For cellular ideals the minimal primes have the form (14), and the collection of all minimal primes of all cellular ideals contains the minimal primes of the original ideal by Lemma 3 □

This algorithm differs from the cellular decomposition algorithm only in the recursion step, where we continue with $I + \langle x_i \rangle$ instead of $I + \langle x_i^s \rangle$. In this way we do not achieve a decomposition of I, but only of the radical of I. Fortunately, this algorithm can be significantly faster than cellular decomposition since adding variables, instead of higher powers of variables, allows the Gröbner basis engine to do more simplifications during the computation.

Example 4 (Binomial minimal primes) We continue where we left off in Example 2.

```
i16 : toString binomialMinimalPrimes I
o16 = {ideal(x1*x4^2-x2*x5^2, x1^3*x3^3-x2^4*x4^2,
           x2^3*x4^4-x1^2*x3^3*x5^2, 2^2*x4^6-x1*x3^3*
           x5^4,x2*x4^8-x3^3*x5^6)}
```

The result consists only of the toric ideal, confirming that the monomial prime is embedded. Although not visible from the output, the second associated prime was not computed on the way to this result. In particular, the minimal primes are not extracted from a list of associated primes.

1.4 Primary decomposition

The original primary decomposition algorithm of Eisenbud and Sturmfels (1996) was refined by Ojeda and Piedra (2000). The computation starts with a cellular decomposition, a first approximation of primary decomposition. It is interesting to identify cases in which the cellular decomposition is already a primary decomposition. Results in this direction are contained in Eisenbud and Sturmfels' paper, and also in Altmann (2000). Note that in these cases a pure difference binomial ideal has a primary decomposition into pure difference binomial ideals, which is, in particular, independent of the coefficient field.

In the general case, for each cellular component the associated primes need to be determined. Then finding the primary component can be achieved as follows. From an associated prime P of a cellular binomial ideal I, extract the "binomial part" $P^{(b)} = P \cap \mathbb{K}[\mathcal{E}]$. Then $I + P^{(b)}$ has P as its unique minimal prime. Computing the primary component over P is carried out by means of a localization operation called Hull, removing the embedded primary components of $I + P^{(b)}$. The refinement of Ojeda and Piedra (2000) is to show that $I + P^{(b)}$ suffices in this procedure, while Eisenbud and Sturmfels originally suggested to add a sufficiently high monomial power. A combinatorial description of the resulting primary components is given in Dickenstein et al. (2008); however, it seems difficult to use these results for computation.

A few remarks on primary decompositions in Eisenbud and Sturmfels (1996) and Ojeda and Piedra (2000) are necessary. Corollary 6.5 of Eisenbud and Sturmfels (1996) shows that $\mathrm{Hull}(I)$ is a binomial ideal if I is a cellular binomial ideal. This corollary is used in the proof of Theorem 7.1′ to deduce that $\mathrm{Hull}(R_i)$ is binomial, where R_i is the sum of a monomial ideal and $I + P^{(b)}$ from above. However, it is not checked whether R_i is in fact cellular, as required by the corollary. Example 5 shows a noncellular R_i that arises in the decomposition of the ideal of adjacent (2×2)-minors of a generic (5×5)-matrix. The computations necessary to check the example can be carried out easily with *Binomials*.

Example 5 In the ring $\mathbb{Q}[a, b, \ldots, o]$ consider the ideal

$$\begin{aligned} I = (\, & ln - ko, lm - jo, km - jn, l^2, kl, jl, k^2, jk, ik - hl, \\ & fk - cl, j^2, ij - gl, hj - gk, fj - al, cj - ak, fh - ci, \\ & fg - ai, cg - ah, f^2, cf, af, ce - bf, ae - df, c^2, ac, ab - cd, a^2). \end{aligned}$$

This ideal is cellular with respect to $\mathcal{E} = \{b, d, e, g, h, i, m, n, o\}$, and has four associated primes, which are pure difference. The binomial part of the unique minimal associated prime is

$$P^{(b)} = (in - ho, im - go, hm - gn).$$

Then $I + P^{(b)}$ has two cellular components whose sets of cell variables are $\mathcal{E}$ and $\{b, d, e, m, n, o\}$, respectively.

Using Theorem 7.1′, in Algorithm 9.7 of Eisenbud and Sturmfels (1996) it is asked to compute $\mathrm{Hull}(R_i)$, using Algorithm 9.6. This, however, requires a cellular ideal as its input. The algorithm can be corrected easily since the operation Hull is called only for ideals whose radical is prime. The associated primes of such an ideal have the radical as their unique minimal element, and as Hull removes embedded primary components, instead of $\mathrm{Hull}(R_i)$ we can compute $\mathrm{Hull}(Q_i)$ of any other ideal $Q_i \supseteq R_i$ that has the same minimal prime. In particular we can choose $Q_i = \left(R_i : (\prod_{i\in\mathcal{E}} x_i)^\infty\right)$, the "cellularization" of R_i. Summarizing, in Algorithm 9.7 of Eisenbud and Sturmfels (1996) Step 3.3 should be replaced by 3.3′ Compute Hull $\left(R_i : \left(\prod_{i\in\mathcal{E}} x_i\right)^\infty\right)$ using Algorithm 9.6.

Unfortunately, also in Theorem 3.2 of Ojeda and Piedra (2000), Corollary 6.5 of Eisenbud and Sturmfels (1996) is used to deduce that $\mathrm{Hull}(I + (P \cap \mathbb{K}[\mathcal{E}]))$ is binomial and primary. Again, this is wrong as $I + (P \cap \mathbb{K}[\mathcal{E}])$ is not necessarily cellular. The result can be saved by first cellularizing as explained above. The implementation in *Binomials* incorporates these modifications and is demonstrated next.

Example 6 (Binomial primary decomposition) We compute the primary decomposition of $I = \langle x^2 - y, y^2 - z, z^2 - x\rangle \in \mathbb{Q}[x, y, z]$.

```
i1 : S = QQ[x,y,z]
i2 : I = ideal(x^2-y,y^2-z^2,z^2-x)
i3 : dim I
o3 = 0
i4 : degree I
o4 = 8

i5 : bpd = BPD I
[ . . . ]

o6 = {ideal(z+ww_6-1,y-ww_6+1,x+ww_6),
      ideal(z+ww_6,y+ww_6,x-ww_6+1), ideal(z+1,y-1,x-1),
      ideal(z-1,y-1,x-1), ideal(z-ww_6,y+ww_6,x-ww_6+1),
      ideal(z-ww_6+1,y-ww_6+1,x+ww_6), ideal(y,x,z^2)}

i7 : intersect bpd == sub (I, ring bpd#0)
o7 = true
```

The function `BPD` is a shorthand for `binomialPrimaryDecomposition`, which can also be used in the long form and offers some options. The primary decomposition of I into binomial ideals exists in $\mathbb{Q}(\xi_6)[x, y, z]$, so `BPD` created this cyclotomic field, calling the primitive sixth root of unity `ww_6`. Observe that the ideal has a double zero at the origin. In `i7` we intersect the result to confirm that the decomposition is correct. The result of the intersection is defined over the extended polynomial ring $\mathbb{Q}(\xi_6)[x, y, z]$, and can be compared to I only after mapping it to that ring.

This concludes our overview of the functionality of *Binomials* and we move on to the discussion of some large primary decompositions.

2 A nonradical commuting birth and death ideal

In this section we study the commutative algebra of discrete time commuting birth and death ideals. One-dimensional birth and death processes are among the simplest

Markov chains that are considered in modeling random processes (Latouche and Ramaswami, 1999). In the discrete time case, many of their properties can be derived from the explicit spectral theory of transition matrices. Evans et al. (2010) give motivation to consider generalized processes that correspond to Markov chains on multidimensional lattices, and as most of the one-dimensional theory does not apply there, the authors strive to identify subclasses with nice properties. The work suggests commuting birth and death processes which are defined by transition matrices having the property that transitions in the different dimensions commute. After reformulation, these conditions can be seen to result in binomial conditions on the entries of the transition matrices, i.e., a binomial ideal. The toric component of this binomial ideal nicely relates to an underlying matroid as discussed in the paper. Determining primary decompositions of commuting birth and death ideals poses interesting challenges in combinatorial commutative algebra.

Computational results given in this section tend to be very large. We have therefore stored them on a web page, which also contains additional scripts to reproduce the results:

$$\texttt{http://personal-homepages.mis.mpg.de/kahle/cbd/} \tag{15}$$

We now define the binomial ideals under consideration. The ambient polynomial ring has variables corresponding to the edges of a regular grid. For fixed integers $n_1, \ldots, n_m$, let

$$E := \prod_{i=1}^{m} \{0, \ldots, n_i - 1\}, \tag{16}$$

be the usual m-dimensional bounded regular grid with edges between vertices that differ by ± 1 in exactly one coordinate. Here, it is sufficient to consider only the cases $m = 2, 3$. For each edge in the grid we define two variables, one for each direction. In the two-dimensional case the authors used the notation $\mathbb{K}[R, L, D, U]$ to denote a polynomial ring in the variables

$$\begin{aligned} &\left\{R_{ij} : 0 \le i < n_1, 0 \le j \le n_2\right\} \cup \left\{L_{ij} : 0 < i \le n_1, 0 \le j \le n_2\right\} \cup \\ &\left\{D_{ij} : 0 \le i \le n_1, 0 < j \le n_2\right\} \cup \left\{U_{ij} : 0 \le i \le n_1, 0 \le j < n_2\right\}, \end{aligned} \tag{17}$$

where R_{ij} is supposed to represent a *right* move starting at position ij and so on. In the case $m = 3$ one can, in a natural way, extend the set of variables by introducing letters F and B and three indices for each indeterminate. The set of commuting birth and death processes is defined by the binomial equations (3.1) of Evans et al. (2010). These equations arise in quadruples, coming from squares in the graph E, by which we mean induced subgraphs G of E that are isomorphic to the usual square. Denoting its vertices by $\left\{(u, v), (u + e_i, v), (u, v + e_j), (u + e_i, v + e_j)\right\}$, the corresponding ideal encodes that the two paths joining opposite vertices are equivalent:

$$\begin{aligned} I^G := \langle & U_{(u,v)} R_{(u,v+e_j)} - R_{(u,v)} U_{(u+e_i,v)}, \\ & D_{(u,v+e_j)} R_{(u,v)} - R_{(u,v+e_j)} D_{(u+e_i,v+e_j)}, \\ & L_{(u+e_i,v+e_j)} D_{(u,v+e_j)} - D_{(u+e_i,v+e_j)} L_{(u+e_i,v)}, \\ & L_{(u+e_i,v)} U_{(u,v)} - U_{(u+e_i,v)} L_{(u+e_i,v+e_j)} \rangle. \end{aligned} \tag{18}$$

The commuting birth and death ideal is the sum of all I^G, where G runs through the induced squares of E.

$$I^E := \sum_{G \text{ square in } E} I^G. \tag{19}$$

In the case $m = 2, 3$ these ideals have been denoted $I^{(n_1,n_2)}$ and $I^{(n_1,n_2,n_3)}$ by Evans et al. (2010).

Example 7 The graph E for $m = 2$ and $n_1 = n_2 = 1$ is just a square and $I^{(1,1)}$ is generated by four binomials

$$\begin{aligned} I^{(1,1)} = \langle & U_{00}R_{01} - R_{00}U_{10}, \quad R_{01}D_{11} - D_{01}R_{00}, \\ & D_{11}L_{10} - L_{11}D_{01}, \quad L_{10}U_{00} - U_{10}L_{11} \rangle\,. \end{aligned} \tag{20}$$

If $m = 3$ and $n_1 = n_2 = n_3 = 1$, E is the 3-cube and the squares arise from facets. Thus, $I^{(1,1,1)}$ is generated by 24 pure difference binomials, 4 for each facet.

On the web page (15) one can download Python scripts that generate Macaulay 2 code for the rings and ideals in the cases $m = 2, 3$. The following shows an example how to use the script `Imn.py` on the command line to generate $I^{(2,2)}$:

```
> ./Imn.py 2 2
-- Macaulay 2 Code for the Commuting Birth and Death Ideal:
-- m = 2, n = 2
S = QQ[R00,U00,R01,D01,U01,R02,D02,R10,L10,U10,R11,L11,D11,
       U11,R12,L12,D12,L20,U20,L21,D21,U21,L22,D22];
I = ideal
(U00*R01-R00*U10,R01*D11-D01*R00,D11*L10-L11*D01,L10*U00
 -U10*L11,U01*R02-R01*U11,R02*D12-D02*R01,D12*L11-L12*D02,
 L11*U01-U11*L12,U10*R11-R10*U20,R11*D21-D11*R10,D21*
 L20-L21*D11,L20*U10-U20*L21,U11*R12-R11*U21,R12*D22-D12*
 R11,D22*L21-L22*D12,L21*U11-U21*L22);
```

In Evans et al. (2010), the authors discuss the primary decompositions of $I^{(2,2)}$, $I^{(1,1,1)}$, and smaller examples. They state that these computations could not be carried out with the standard implementations, but were derived in an interactive session. The current implementation of *Binomials* computed the 199 prime components of $I^{(2,2)}$ in 100 seconds and took 123 seconds to decompose $I^{(1,1,1)}$ on the author's 1.6 GHz laptop. As mentioned before, computing the minimal primes directly is even faster and can be completed in half of the time.

Based on their results, Evans, Sturmfels, and Uhler conjectured

Conjecture 1 For any grid E, the ideal I^E is radical, its prime decomposition consists of pure toric ideals and is independent of the coefficient field.

Here a *pure toric ideal* is an ideal generated by variables and pure difference binomials. Evans et al. (2010) prove that every associated prime of $I^{(1,n)}$ is a pure toric ideal. However, using *Binomials* we have derived the following counterexample to radicality.

Theorem 2 *The ideal $I^{(2,3)}$ is the intersection of 2,638 primary binomial ideals whose properties are given in Table 1. Among these are ten components that are not prime, and thus $I^{(2,3)}$ is not radical. The ten associated primes of these components are all embedded and of codimension 20. The radical* $\mathrm{Rad}(I^{(2,3)})$ *is the intersection of 2,628 minimal primes and given by the following ideal:*

$$\begin{aligned} I^{(2,3)} + \langle & D_{01}R_{03}R_{10}L_{12}U_{21}L_{22}D_{23} - U_{01}R_{03}L_{10}R_{13}D_{21}L_{23}D_{23}, \\ & U_{00}R_{02}R_{12}L_{13}L_{20}D_{22}U_{22} - R_{00}D_{02}R_{13}L_{13}U_{20}U_{22}L_{23}, \\ & R_{00}U_{01}R_{03}L_{10}R_{13}U_{20}L_{23}D_{23} - U_{01}R_{03}^2R_{13}L_{13}U_{20}L_{23}D_{23}, \\ & R_{00}D_{02}L_{10}R_{13}L_{13}D_{21}U_{22}L_{23} - D_{02}R_{03}R_{13}L_{13}^2D_{21}U_{22}L_{23}, \\ & U_{00}R_{02}R_{03}R_{12}L_{13}L_{20}D_{22}D_{23} - R_{00}D_{02}R_{03}R_{13}L_{13}U_{20}L_{23}D_{23}, \\ & U_{00}R_{03}R_{10}L_{12}L_{20}U_{21}L_{22}D_{23} - U_{00}R_{03}L_{12}R_{13}U_{21}L_{22}L_{23}D_{23}, \\ & R_{00}D_{03}L_{11}R_{13}U_{20}L_{21}D_{22}L_{23} - U_{00}R_{03}L_{12}R_{13}L_{20}L_{22}D_{22}D_{23}, \\ & R_{01}U_{02}L_{10}R_{11}R_{13}D_{21}U_{21}L_{23} - D_{01}R_{02}R_{10}R_{12}L_{13}U_{21}U_{22}L_{23}, \\ & D_{01}R_{02}R_{10}R_{12}L_{13}L_{20}D_{22}U_{22} - D_{01}R_{02}R_{12}R_{13}L_{13}D_{22}U_{22}L_{23}, \\ & D_{01}R_{03}R_{10}L_{12}L_{13}U_{21}L_{22}U_{22} - U_{01}R_{03}L_{10}R_{13}L_{13}D_{21}U_{22}L_{23}\rangle. \end{aligned} \tag{21}$$

One should note the two squares of variables in the third and fourth generator of $\mathrm{Rad}(I^{(2,3)})$. To produce these results one can use the functions `BPD` and `binomialMinimalPrimes`. The author's computer determined the minimal primes in approximately 4 h. Taking the intersection of these primes took another hour on a 2.8 GHz AMD Opteron. Care has to be taken when computing intersections of many primes. In Macaulay 2 versions 1.2 and below, using the command `intersect` directly on a large list of primes will not terminate. If one does the intersection manually with a loop, intersecting only two ideals at a time, everything is fine. Computing the cellular and primary decomposition was more delicate. It took several days and used about 5 GB of RAM. In fact, the original computation of the cellular decomposition was done with a slightly different algorithm which only works if the toric component is isolated. We first computed the toric component T independently with the tool 4ti2 (4ti2 team 2007) and then removed it by computing the saturation $(I^{(2,3)} : T^{\infty})$. The cellular decomposition of this ideal was easier to compute. Surprisingly this is not always the case. For some ideals I, with toric component T, the saturation $(I : T^{\infty})$ is just too complicated to be computed with Macaulay 2. In some cases, simply doing the cellular decomposition with Algorithm 1 is faster.

Table 1 Statistics on the primary components of $I^{(2,3)}$ sorted by codimension

Codimension	16	17	17	18	18	19	19	20	20	21	21	22	22
No. of components	1	14	2	107	91	356	612	527	550	212	120	38	8
Gen. max degree	1	1	4	1	6	1	5	1	4	1	2	1	3
Degree	1	1	64	1	4012	1	144	1	36	1	12	1	3
Monomial	y	y	n	y	n	y	n	y	n	y	n	y	n

Monomial components have been separated from binomial ones as indicated in the row "monomial". The row "gen. max degree" gives the maximal degree of a generator in this codimension while "degree" refers to the maximal degree among components. The toric component is generated in degree 6, of codimension 18 and degree 4,012

Table 2 Prime decompositions of $I^{(1,n)}$

n	1	2	3	4	5	6
No. of components	3	11	40	139	466	1528

To complete this computational study, we have also investigated the ideals $I^{(1,n)}$ for $n \leq 6$. It was not possible to find a counterexample there.

Theorem 3 *The ideals* $I^{(1,n)}$, $n = 1, \dots, 6$ *are radical. The respective numbers of prime components are given in Table 2.*

Concluding this section we find that the conjecture turned out to be false in full generality. It might however hold for the ideals $I^{(1,n)}$, and the associated primes could still be pure toric ideals for all I^E.

3 Conclusion and further directions

We have presented algorithms for binomial ideals together with an implementation in Macaulay 2. It covers the case of pure difference binomial ideals, and it remains a future task to extend it to other cases, in particular to finite fields.

A natural area for application of this software is the field of algebraic statistics, where analyzing the solutions of polynomial equations is of central importance. As mentioned in the introduction, describing conditional independence models is naturally connected to primary decomposition and also a very actively pursued research direction in algebraic statistics. The author hopes to facilitate experimentation with the availability of *Binomials*.

Many operations that can be carried out with binomial ideals have been translated to operations on exponent vectors, or on the associated partial characters. By "making them combinatorial", significant speedups can be achieved. The computation of the associated primes is an example. Computing binomial primary components is more delicate; the Hull operation is a bottleneck. Dickenstein et al. (2008) give an explicit lattice point characterization of binomial primary components, but it seems not easy to use these results for computation. In the examples we have considered here, the Hull operation only marginally contributes to the total computation time. This is due to the fact that most of the components in our decompositions are prime ideals. In this case most of the computation time is spent on cellular decomposition, which, in turn, consists of many ideal saturations. Thus, from the author's point of view, software for binomial ideals would greatly benefit from a solution to the following problem:

Problem 1 Develop a specialized algorithm to compute, for any (cellular) binomial ideal I, the "partially saturated" ideal

$$I : \left(\prod_{i \in \mathcal{E}} x_i\right)^{\infty}. \tag{22}$$

The software 4ti2 implements the *project-and-lift* algorithm, a fast algorithm for computing the saturation

$$I : \left(\prod_{i=1}^{n} x_i\right)^{\infty}. \tag{23}$$

It seems natural to extend the program to solve the above problem, and *Binomials* is prepared to incorporate it upon availability.

Finally, a natural approach to continue this work is to investigate decompositions that are finer than cellular decompositions, but not as fine as primary decompositions. In this direction one could aim at a separation of the combinatorial operations like cellular decomposition, and the field dependent operations like saturation of partial characters. The combinatorial operations should be connected to the combinatorics of the underlying semigroup ring. One can ask for the finest decomposition of a pure difference binomial ideal into pure difference binomial ideals, even if it is not primary. This might be interesting for applications where factorization of univariate polynomials is not of great importance. For example, if a component is generated by $x^{19} - 1$, we would like the algorithm to stop, since we know the result of this decomposition, and don't want the 19 cases to clutter up the output. It will be the subject of future work to investigate these possibilities.

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