AISM (2007) 59:161–170
DOI 10.1007/s10463-006-0105-1

A note on curvature of α-connections of a statistical manifold

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Received: 31 May 2006 / Revised: 15 November 2006 /
Published online: 8 February 2007

Abstract The family of α-connections $\nabla^{(\alpha)}$ on a statistical manifold $\mathcal{M}$ equipped with a pair of conjugate connections $\nabla \equiv \nabla^{(1)}$ and $\nabla^* \equiv \nabla^{(-1)}$ is given as $\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*$. Here, we develop an expression of curvature $R^{(\alpha)}$ for $\nabla^{(\alpha)}$ in relation to those for ∇, ∇^*. Immediately evident from it is that $\nabla^{(\alpha)}$ is equiaffine for any $\alpha \in \mathbb{R}$ when ∇, ∇^* are dually flat, as previously observed in Takeuchi and Amari (*IEEE Transactions on Information Theory* 51:1011–1023, 2005). Other related formulae are also developed.

Keywords Equiaffine connections · Parallel volume form · Ricci tensor · Cubic/skewness form

1 Introduction

Let $\{\mathcal{M}, g\}$ be a Riemannian manifold with a metric g. A pair of torsion-free connections ∇, ∇^* are said to be conjugate with respect to g if

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y).$$

The work was conducted when the author was on sabbatical leave as a visiting research scientist at the Mathematical Neuroscience Unit, RIKEN Brain Science Institute, Wako-shi, Saitama 351-0198, Japan.

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On $\mathcal{M}$, one can further define a parametric family of torsion-free connections $\nabla^{(\alpha)}$ indexed by α ($\alpha \in \mathbb{R}$)

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*, \tag{1}$$

with

$$\nabla^{(1)} = \nabla, \quad \nabla^{(-1)} = \nabla^*,$$

and $\alpha = 0$ denoting the Levi–Civita connection associated with g

$$\nabla^{(0)} = \frac{1}{2}(\nabla + \nabla^*) \equiv \hat{\nabla}.$$

An issue of both theoretical and practical importance for information geometry is to relate the geometry associated with the α-connections to that associated with the (± 1)-connection or 0-connection. For instance, in generalizing the information geometric characterization of mean-field approximation based on (± 1)-projections (Tanaka, 2000), Amari et al. (2001) and then Toyoizumi and Aihara (2006) considered the full-fledged α-projection. In generalizing the Jeffrey's prior in Bayesian inference based on 0-volume form, Takeuchi and Amari (2005) and then Matsuzoe et al. (2006) advanced the notion of α-parallel prior.

Take Takeuchi and Amari (2005) as an example. The authors, realizing that the Riemannian volume form ($\alpha = 0$) has been traditionally taken as the non-informative prior in Bayesian inference on the manifold of probability distributions (so-called Jeffrey's prior), investigated volume forms that are parallel with respect to the α-connections $\nabla^{(\alpha)}$ for general α values. Recall that the necessary and sufficient condition for a torsion-free affine connection to admit (uniquely up to a constant scaling factor) a parallel volume form is that it has symmetric Ricci tensor; such a connection is called an "equiaffine connection" (see, e.g., Nomizu and Sasaki 1994). Takeuchi and Amari (2005) showed that the sufficient condition for a statistical manifold to be equiaffine is that it is "conjugate symmetric" (to be explicated later). They also developed an expression for α-parallel volume form for the exponential family. Matsuzoe et al. (2006) further investigated sufficient conditions of a statistical submanifold to be equiaffine.

It is known that the curvature R of ∇ vanishes if and only if the curvature R^* of ∇^* vanishes; in such case the manifold is called "dually flat". How is the curvature $R^{(\alpha)}$ of $\nabla^{(\alpha)}$ related to R, R^* in general? When is $\nabla^{(\alpha)}$ equiaffine for arbitrary α? In this paper, we provide a simple formula of $R^{(\alpha)}$, from which dual-flatness of ∇, ∇^* is easily seen as a sufficient condition for $\nabla^{(\alpha)}$ to be equiaffine.

Proposition 1 *The curvature tensor $R^{(\alpha)}$ for the α-connection $\nabla^{(\alpha)}$ satisfies*

$$R^{(\alpha)}(X,Y)Z = \frac{1+\alpha}{2}R(X,Y)Z + \frac{1-\alpha}{2}R^*(X,Y)Z + \frac{1-\alpha^2}{4}(K(Y,K(X,Z)) - K(X,K(Y,Z))), \tag{2}$$

where $K(X,Y)$ is the "difference tensor" defined by

$$K(X,Y) = \nabla^*_X Y - \nabla_X Y = 2(\hat{\nabla}_X Y - \nabla_X Y) = 2(\nabla^*_X Y - \hat{\nabla}_X Y). \tag{3}$$

Note that our definition of "difference tensor" differs from that in Simon (2000) by a negative sign, so we use the notation $K(\cdot,\cdot)$.

From (2)

$$R^{(\alpha)}(X,Y)Z - R^{(-\alpha)}(X,Y)Z = \alpha\,(R(X,Y)Z - R^*(X,Y)Z).$$

Taking the contraction $\mathrm{Tr}\,\{X \mapsto R(X,Y)Z\} \equiv \mathrm{Tr}\,R(\cdot,Y)Z$ to yield an analogous expression in terms of Ricci curvature tensor

$$\mathrm{Ric}^{(\alpha)}(Y,Z) = \frac{1+\alpha}{2}\mathrm{Ric}(Y,Z) + \frac{1-\alpha}{2}\mathrm{Ric}^*(Y,Z) + \frac{1-\alpha^2}{4}(\mathrm{Tr}\,K(K(\cdot,Z),Y) - \mathrm{Tr}\,K(\cdot,K(Y,Z))). \tag{4}$$

Since the last term in (4) can be shown to be symmetric in terms of Y, Z, we have

$$\begin{aligned} &\mathrm{Ric}^{(\alpha)}(Y,Z) = \mathrm{Ric}^{(\alpha)}(Z,Y) \\ \Longleftrightarrow\ &\mathrm{Ric}(Y,Z) = \mathrm{Ric}(Z,Y) \\ \Longleftrightarrow\ &\mathrm{Ric}^*(Y,Z) = \mathrm{Ric}^*(Z,Y). \end{aligned}$$

Since Ricci-symmetry is the necessary and sufficient condition for a connection to be equiaffine, we reproduce the following facts that were first shown by Takeuchi and Amari (2005) and then noted in Matsuzoe et al. (2006)

1. $\nabla^{(\alpha)}$ is equiaffine for all $\alpha \in \mathbb{R}$ if $\nabla^{(\alpha_0)}$ is equiaffine for any $\alpha_0 \neq 0$;
2. $\nabla^{(\alpha)}$ is equiaffine for all $\alpha \in \mathbb{R}$ if ∇, ∇^* are dually flat (and hence with vanishing Ricci tensor).

Uohashi (2002) introduced the notion of "α-transitive flatness" of a connection: a connection $\nabla^{(\alpha)}$ is called α-transitively flat if $\nabla \equiv \nabla^{(1)}$ (and hence $\nabla^* \equiv \nabla^{(-1)}$) is curvature-free. It was shown that a statistical manifold that is α-conformally equivalent to a statistical manifold with an α-transitively flat connection can be embedded as a codimension one sub-manifold of the latter manifold. In this context, the second statement above can be rephrased as "all

α-transitively flat connections are equiaffine." We may also refine Uohashi's construction via equiaffine immersion, i.e., requiring the volume form from the codimension one sub-manifold to be induced from the α-volume form.

2 Alpha-curvature and equiaffine statistical manifold

Let $\{\mathcal{M}, g\}$ be a Riemannian manifold and ∇ be a torsion-free connection. Denote $K(X,Y,Z) = (\nabla g)_X(Y,Z)$ where

$$(\nabla g)_X(Y,Z) \equiv Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

The 0–3 tensor $K(\cdot,\cdot,\cdot)$ is called the skewness or cubic form and satisfies $K(\cdot,Y,Z) = K(\cdot,Z,Y)$ by its definition. If $K(\cdot,\cdot,\cdot)$ is further assumed to be totally (i.e., across all three variables) symmetric, which amounts to requiring

$$(\nabla g)_Y(X,\cdot) = (\nabla g)_X(Y,\cdot), \tag{5}$$

then $\{\mathcal{M}, g, K\}$ is said to form a "statistical structure" (Lauritzen, 1987). In the language of modern affine differential geometry (Simon, 2000, p. 913), ∇ and g form a "Codazzi pair" of order 2.

For any torsion-free linear connection ∇, introduce the notion of "conjugate connection" ∇^* defined by

$$g(Y, \nabla^*_X Z) = Xg(Y,Z) - g(\nabla_X Y, Z).$$

It can be verified that ∇^* is indeed a linear connection and that, when ∇g is totally symmetric, ∇^* is torsion-free and vice versa. In this case, $\nabla^* g$ is also totally symmetric, and

$$(\nabla^* g)_X(Y,Z) = -K(X,Y,Z).$$

We note that the notion of "conjugate connection" has been attributed to A.P. Norden in affine differential geometry literature (Simon, 2000) and was independently introduced by (Nagaoka and Amari, 1982) in information geometry literature, where it was called "dual connection" (Lauritzen, 1987).

Now, for $\alpha \in \mathbb{R}$, introduce $\nabla^{(\alpha)}$ by

$$(\nabla^{(\alpha)} g)_X(Y,Z) = \alpha K(X,Y,Z).$$

This defines a family of linear connections $\nabla^{(\alpha)}$ on $\{\mathcal{M}, g\}$, each indexed by an α value, with $\nabla^{(1)} = \nabla$ and $\nabla^{(-1)} = \nabla^*$. The above definition of $\nabla^{(\alpha)}$ is equivalent to that given by (1), with $\nabla^{*(\alpha)} = \nabla^{(-\alpha)}$.

Recall the curvature R of a connection ∇ is the tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

When ∇ is torsion-free,

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$

we have

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z. \tag{6}$$

It can be shown that for a pair of conjugate connections, their curvature tensors satisfy

$$g(R(X, Y)Z, W) + g(Z, R^*(X, Y)W) = 0, \tag{7}$$

and more generally

$$g(R^{(\alpha)}(X, Y)Z, W) + g(Z, R^{(-\alpha)}(X, Y)W) = 0$$

where $R^{(-\alpha)} = R^{*(\alpha)}$.

Recall the difference tensor $K(X, Y)$ introduced in (3), which can be verified to be related to $K(X, Y, Z)$ via

$$g(K(X, Y), Z) = K(X, Y, Z).$$

Torsion-freeness of $\nabla, \nabla^*, \hat{\nabla}$ yields

$$\nabla_X Y - \nabla_Y X = \hat{\nabla}_X Y - \hat{\nabla}_Y X = \nabla^*_X Y - \nabla^*_Y X \; (= [X, Y]),$$

which shows the symmetry of K

$$K(X, Y) = K(Y, X).$$

Through its construction, $K(X, Y)$ satisfies

$$K(X + Y, Z) = K(X, Z) + K(Y, Z) = K(Z, X) + K(Z, Y) = K(Z, X + Y)$$

for any vector fields X, Y, Z on $\mathcal{M}$ and

$$K(fX, Y) = K(Y, fX) = fK(X, Y) = fK(Y, X) \tag{8}$$

for any smooth function f on $\mathcal{M}$. Moreover,

$$\nabla^{(\alpha)}_X Y = \hat{\nabla}_X Y - \frac{\alpha}{2} K(X, Y)$$

with

$$\nabla_X Y = \hat{\nabla}_X Y - \frac{1}{2} K(X, Y), \quad \nabla^*_X Y = \hat{\nabla}_X Y + \frac{1}{2} K(X, Y). \tag{9}$$

We are now in a position to prove the formula (2) of Proposition 1.

Proof of Proposition 1 By definition of the α-connection and of curvature tensor

$$R^{(\alpha)}(X,Y)Z = \left(\frac{1+\alpha}{2}\nabla_X + \frac{1-\alpha}{2}\nabla^*_X\right)\left(\frac{1+\alpha}{2}\nabla_Y + \frac{1-\alpha}{2}\nabla^*_Y\right)Z$$
$$-\left(\frac{1+\alpha}{2}\nabla_Y + \frac{1-\alpha}{2}\nabla^*_Y\right)\left(\frac{1+\alpha}{2}\nabla_X + \frac{1-\alpha}{2}\nabla^*_X\right)Z$$
$$-\left(\frac{1+\alpha}{2}\nabla_{[X,Y]} + \frac{1-\alpha}{2}\nabla^*_{[X,Y]}\right)Z.$$

From the linearity of $\nabla.Z$ operator on Z,

$$R^{(\alpha)}(X,Y)Z = \left(\frac{1+\alpha}{2}\right)^2 R(X,Y)Z + \left(\frac{1-\alpha}{2}\right)^2 R^*(X,Y)Z$$
$$+\frac{1-\alpha^2}{4}\left(\nabla_X\nabla^*_Y + \nabla^*_X\nabla_Y - \nabla_Y\nabla^*_X - \nabla^*_Y\nabla_X - \nabla_{[X,Y]} - \nabla^*_{[X,Y]}\right)Z.$$

Substituting (9), the last parentheses becomes

$$2\hat{R}(X,Y)Z + \frac{1}{2}(K(Y,K(X,Z)) - K(X,K(Y,Z)))$$

where $\hat{R} \equiv R^{(0)}$ is the Riemann curvature tensor, i.e. the curvature of the Levi–Civita connection $\hat{\nabla} \equiv \nabla^{(0)}$. Therefore

$$R^{(\alpha)}(X,Y)Z = \left(\frac{1+\alpha}{2}\right)^2 R(X,Y)Z + \left(\frac{1-\alpha}{2}\right)^2 R^*(X,Y)Z$$
$$+\frac{1-\alpha^2}{2}\left(\hat{R}(X,Y)Z + \frac{1}{4}(K(Y,K(X,Z)) - K(X,K(Y,Z)))\right).$$

Taking $\alpha = 0$ in the above expression to solve for $\hat{R}(X,Y)Z$

$$\hat{R}(X,Y)Z = \frac{1}{2}R(X,Y)Z + \frac{1}{2}R^*(X,Y)Z + \frac{1}{4}(K(Y,K(X,Z)) - K(X,K(Y,Z))),$$

and substitute it back gives rise to (2). □

Under local coordinates,

$$\nabla_{\partial_i}\partial_j = \sum_l \Gamma^l_{ij}\partial_l, \tag{10}$$
$$R(\partial_i,\partial_j)\partial_l \equiv \sum_k R^k_{lij}\partial_k$$

with

$$R^k_{lij} = \partial_i \Gamma^k_{lj} - \partial_j \Gamma^k_{li} + \sum_m (\Gamma^k_{mi}\Gamma^m_{lj} - \Gamma^m_{li}\Gamma^k_{mj}), \tag{11}$$

and

$$K(\partial_i, \partial_j) = \sum_k K^k_{ij}\partial_k = \sum_k (\Gamma^{*k}_{ij} - \Gamma^k_{ij})\partial_k$$

with

$$K(\partial_i, K(\partial_j, \partial_l)) = K\left(\partial_i, \sum_m K^m_{jl}\partial_m\right) = \sum_m K^m_{jl} K(\partial_i, \partial_m) = \sum_{m,k} K^m_{jl} K^k_{im}\partial_k,$$

where we have used the property (8). We then obtain an expression of (2) in local coordinates

$$R^{(\alpha)k}_{lij} = \frac{1+\alpha}{2} R^k_{lij} + \frac{1-\alpha}{2} R^{*k}_{lij} + \frac{1-\alpha^2}{4}\left(\sum_m K^m_{il}K^k_{jm} - \sum_m K^k_{im}K^m_{jl}\right). \tag{12}$$

A statistical manifold $\mathcal{M}$ is said to be "conjugate symmetric" (Lauritzen, 1987) if $R^k_{lij} = R^{*k}_{lij}$, that is, when the curvatures of the pair of conjugate connections are equal. Expression (12) implies that $R^{(\alpha)k}_{lij}$ is an even function of α, $R^{(\alpha)k}_{lij} = R^{(-\alpha)k}_{lij} = R^{(*\alpha)k}_{lij}$, and hence is conjugate symmetric, if and only if ∇ is conjugate symmetric, as pointed out in Lauritzen (1987).

Take the contraction to get the Ricci tensor $R_{ij} = \sum_k R^k_{ikj}$,

$$R^{(\alpha)}_{ij} = \frac{1+\alpha}{2} R_{ij} + \frac{1-\alpha}{2} R^*_{ij} + \frac{1-\alpha^2}{4}\left(\sum_{m,k} K^m_{ik}K^k_{jm} - \sum_{m,k} K^m_{ij}K^k_{km}\right). \tag{13}$$

The last term in the parentheses is obviously symmetric with respect to i, j. The first two terms vanish, $R_{ij} = R^*_{ij} = 0$, when ∇ (and consequently ∇^* as well) is flat. In this case, $R^{(\alpha)}_{ij} = R^{(\alpha)}_{ji}$ so $\nabla^{(\alpha)}$ is equiaffine for any α. This is to say that the family of α-connections with its $\alpha = \pm 1$ members as a pair of dually flat connections are equiaffine. More generally, $R^{(\alpha)}_{ij} = R^{(\alpha)}_{ji}$ for any α so long as ∇ is equiaffine, $R_{ij} = R_{ji}$ (and consequently ∇^* is equiaffine, $R^*_{ij} = R^*_{ji}$). Given that the Levi–Civita connection $\hat{\nabla} \equiv \nabla^{(0)}$ is always equiaffine, we conclude that the entire family of α-connections $\nabla^{(\alpha)}$ are equiaffine if any of its non-Levi–Civita member is equiaffine, as pointed out in Takeuchi and Amari (2005).

Recall that a statistical manifold with an equiaffine connection ∇ admits a unique parallel volume form ω, that is $\nabla\omega = 0$. Here, a volume form is a skew-symmetric multi-linear map from $n = \dim \mathcal{M}$ linearly independent vectors to a non-zero scalar, and "parallel" is in the sense that $(\partial_i\omega)(\partial_1,\ldots,\partial_n) = 0$ where

$$(\partial_i\omega)(\partial_1,\ldots,\partial_n) \equiv \partial_i(\omega(\partial_1,\ldots,\partial_n)) - \sum_{l=1}^{n}\omega(\ldots,\nabla_{\partial_i}\partial_l,\ldots).$$

In fact, the following lemma can be proven.

Lemma 1 *Let two torsion-free connections $\nabla, \widetilde{\nabla}$ be equiaffine, with corresponding parallel volume form $\omega, \widetilde{\omega}$. Then the connection $a\nabla + b\widetilde{\nabla}$ is equiaffine for all $a, b \in \mathbb{R}$, with parallel volume form given by (apart from a scaling constant) $\omega^a\widetilde{\omega}^b$.*

Proof Contracting (11) to get the Ricci curvature, we can see that

$$R_{ij} = R_{ji} \iff \partial_i\left(\sum_l \Gamma_{jl}^l\right) = \partial_j\left(\sum_l \Gamma_{il}^l\right).$$

One therefore sees that $a\Gamma_{ij}^l + b\widetilde{\Gamma}_{ij}^l$ is Ricci symmetric when Γ_{ij}^l and $\widetilde{\Gamma}_{ij}^l$ are, proving that it is equiaffine. Next, applying (10), the equiaffine condition becomes

$$\begin{aligned}\partial_i(\omega(\partial_1,\ldots,\partial_n)) &= \sum_{l=1}^{n}\omega\left(\ldots,\sum_k \Gamma_{il}^k\partial_k,\ldots\right)\\ &= \sum_l\sum_k \Gamma_{il}^k\,\delta_{kl}\,\omega(\partial_1,\ldots,\partial_n) = \omega(\partial_1,\ldots,\partial_n)\sum_l \Gamma_{il}^l\end{aligned}$$

(where δ_{kl} is the Kronecker delta), or

$$\sum_l \Gamma_{il}^l = \partial_i(\log\omega). \tag{14}$$

Clearly,

$$\sum_l(a\Gamma_{il}^l + b\widetilde{\Gamma}_{il}^l) = a\,\partial_i(\log\omega) + b\,\partial_i(\log\widetilde{\omega}) = \partial_i(\log(\omega^a\,\widetilde{\omega}^b)).$$

Therefore, the equiaffine volume form is $\omega^a\widetilde{\omega}^b$. □

As a consequence of Lemma 1, we have (by taking $a = \frac{1+\alpha}{2}, b = \frac{1-\alpha}{2}$)

Proposition 2 *The α-volume form $\omega^{(\alpha)}$ of an α-connection $\nabla^{(\alpha)}$ satisfies*

$$\omega^{(\alpha)} = (\omega)^{\frac{1+\alpha}{2}}\,(\omega^*)^{\frac{1-\alpha}{2}} \tag{15}$$

with $\omega^{(1)} = \omega$, $\omega^{(-1)} = \omega^*$ the volume forms associated with ∇, ∇^*, respectively.

In general, when a pair of conjugate connections $\nabla^{(\alpha_0)}, \nabla^{(-\alpha_0)}$ $(\alpha_0 \neq 0)$ is given, the α-connection is expressed as

$$\nabla^{(\alpha)} = \frac{\alpha_0 + \alpha}{2\alpha_0} \nabla^{(\alpha_0)} + \frac{\alpha_0 - \alpha}{2\alpha_0} \nabla^{(-\alpha_0)}.$$

The expression of the α-parallel volume form $\omega^{(\alpha)}$ is related to the $(\pm\alpha_0)$-parallel volume forms $\omega^{(\pm\alpha_0)}$ via

$$\omega^{(\alpha)} = \left(\omega^{(\alpha_0)}\right)^{\frac{\alpha_0+\alpha}{2\alpha_0}} \left(\omega^{(-\alpha_0)}\right)^{\frac{\alpha_0-\alpha}{2\alpha_0}}.$$

An alternative expression of (15) was given by (Matsuzoe et al., 2006)

$$\omega^{(\alpha)} = \mathrm{e}^{-\frac{\alpha}{2}\phi}\hat{\omega},$$

where $\hat{\omega} = \omega^{(0)}$ is the scalar curvature associated with $\hat{\nabla}$ and

$$\phi = \log \frac{\omega^*}{\omega}.$$

One easily verifies

$$\omega^{(\alpha)}\,\omega^{(-\alpha)} = \omega\,\omega^* = \hat{\omega}^2,$$

as given in (Simon, 2000, p. 913).

From the Ricci tensor, one can obtain the scalar curvature $\sigma = \sum_{j,l} g^{jl} R_{lj}$. In particular, the α-scalar curvature has the following expression:

Proposition 3 *The scalar curvature $\sigma^{(\alpha)}$ for $\nabla^{(\alpha)}$ is related to σ of ∇ via*

$$\sigma^{(\alpha)} = \sigma + \frac{1-\alpha^2}{4} K, \tag{16}$$

where

$$K = \sum_{m,k,i,j} g^{ij} \left(K^m_{ik} K^k_{jm} - K^m_{ij} K^k_{km} \right).$$

Proof First, writing (7) in component form gives

$$\sum_k g_{km} R^k_{lij} + \sum_k g_{lk} R^{*k}_{mij} = 0.$$

Multiplying $g^{tm} g^{sl}$ and sum over l, m indices

$$\sum_l g^{ls} R^t_{lij} + \sum_m g^{mt} R^{*s}_{mij} = 0.$$

From (6), $R(X,Y)Z = -R(Y,X)Z$ or, in local coordinates, $R^t_{lij} = -R^t_{lji}$. Contracting i with s and j with t in the above expression, we have

$$\sigma = \sigma^*.$$

Next, multiplying g^{ij} and then summing over i,j in (13) yields

$$\sigma^{(\alpha)} = \frac{1+\alpha}{2}\sigma + \frac{1-\alpha}{2}\sigma^* + \frac{1-\alpha^2}{4}K.$$

Setting $\sigma = \sigma^*$ in the above yields (16). □

Acknowledgments The work reported in this paper was presented at the 5th Workshop on Differential Geometry and Information Geometry held at Nagoya University. The author thanks Shunichi Amari and Hiroshi Matsuzoe for discussions of many topics on affine and information geometry. The writing of the paper is supported by NSF grant MSBS#0631541.

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