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Kamal C. Chanda

Sampling properties of U-statistics for a class of stationary nonlinear processes

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Abstract We consider the sampling properties of U-statistics based on a sample of realization from a class of stationary nonlinear processes which include, in particular, linear, bilinear and finite order volterra processes. It is shown that if the size n of the realization tends to infinity then certain normalized versions of the U-statistics tend to be distributed normally with zero means and finite variances.

Keywords Stationary nonlinear processes · U-statistics · Large sample properties

1 Introduction

Structures of stationary stochastic processes (SSP) have been investigated quite extensively in the past and volumes of literature on SSP's have emerged over a long period of time. Historically, the researchers in this area have adopted, essentially, two different kinds of approach. Starting with the pioneering work in Rosenblatt (1956) which, for the first time, introduced the concept of strong mixing (SM) properties of the SSP, the probabilists have invented several other kinds of mixing conditions to characterize the SSP's. They have used terms such as absolutely regular (ABR), uniformly mixing (UM) and *- mixing among others. Ibragimov (1962) and Ibragimov and Linnik (1971) have investigated weak laws of large numbers relating to sums of observations on stationary SM and UM processes. Other works in this area have appeared in Bradley (1983, 1986, 2001), Davydov (1968), Denker (1986), Doukhan (1994), Oodaira and Yoshihara (1972), Yoshihara (1978), and Withers (1981) among others.

Statistical analyses of time series data, on the other hand, have, traditionally, followed a different path. Statisticians working in this area have characterized

K.C. Chanda
Department of Mathematics and Statistics, Texas Tech University, Lubbock,
TX 79409-1042, USA
E-mail: k.chanda@ttu.edu

the concept of dependence in a SSP through terms like linear (which includes autoregressive moving average (ARMA) schemes), bilinear, Volterra, and threshold autoregressive models (TAR), and have attempted to derive the sampling properties of various statistics computed from samples drawn from these processes. As results of this research several interesting cases of weak and strong laws of large numbers have been explored over the years.

The first attempt to demonstrate that a linear process (LP), under certain mild regularity conditions, satisfies the SM conditions is due to Chanda (1974), Goradetskii (1977) and Withers (1981). ABR properties of a LP have been established by Phan and Tran (1985). A LP can, asymptotically, be treated as a m-dependent sequence where m is unbounded and the central limit theorem (CLT) relating to sums of observations on a LP follows, essentially, from the result in Hoeffding and Robbins (1948) and Berk (1973). It is not necessary to prove this CLT by appealing to the SM conditions of a LP. In fact, it is quite possible that the LP does not satisfy the SM conditions and yet the CLT can be applied to sums of observations on a LP. Two classic examples of such a LP have been mentioned in Rosenblatt (1980) and Goradetskii (1977).

If we follow the path of time series analysts then it is imperative that we know the exact structural model for the time series before we can validly carry out the appropriate data analyses. Such is not the case for probability models which satisfy various mixing conditions. If $\{X_t\}$ is a SSP which has SM properties then any measurable functions $\{g(X_t)\}$, $\{g(X_t, X_{t+s})\}$ for a given s are also SM. But if $\{X_t\}$ is a LP the function $\{g(X_t)\}$ is not ordinarily a LP. Therefore, if we want to analyze the sampling properties of statistics derived from such a process we need to use other statistical tools to carry out the data analysis.

Various weak and strong laws of large numbers pertaining to statistics computed from mixing processes have been established in the past. Ever since the concept of the U-statistics appeared in Hoeffding (1948), several attempts have been made to extend the asymptotic results about these U-statistics to processes which have SM properties. Such extensions have been possible due to the works appearing in Denker and Keller (1983), Sen (1972) and Yoshihara (1976) among others. The most recent result concerning the asymptotic properties of the U-statistics is due to Borovkova, Burton, and Dehling (2001) who assumed that the $\{X_t\}$ is itself a functional of another process $\{\varepsilon_t\}$ which is ABR. But no systematic effort has been made in the past to investigate if the large sample properties of the U-statistics can also be proved for time series which are LP, bilinear (BL), Volterra or TAR models.

Our main objective in the sections that follow is to establish the CLT relating to the U-statistics computed from a general class of stationary nonlinear processes which include the LP, the BL, and the finite order Volterra processes. We should also mention that the regularity conditions that we use in this paper are more general than those in Borovkova et al. (2001) in the sense that the primary process $\{\varepsilon_t\}$ which determines the structure of $\{X_t\}$ can, itself, be a process which is not necessarily even a SM process and unlike in Borovkova et al. (2001) the seed function of the U-statistics needs not be bounded.

2 A general class of nonlinear processes

Statisticians who deal with time series data often encounter situations where the SSP's $\{X_t\}$ do not conform to linear models. Some of the simple nonlinear processes they have used to model these SSP's when linearity is in doubt include BL and Volterra processes (for description of these processes see Priestley 1988). In this paper we introduce a class of stationary nonlinear processes which among others include the BL and the finite order Volterra processes (see Chanda 1991, 2003) and is defined by

$$X_t = \varepsilon_t + \sum_{r=1}^{\infty} W_{r,t}, \tag{1}$$

where we assume that $\{\varepsilon_t\}$ is a sequence of independent and identically distributed (IID) random variables, $W_{r,t}$, $(r \geq 1)$ is a function $f_r(\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots, \varepsilon_{t-r-\omega})$, w is a finite integer ≥ 0, $E|W_{r,t}|^k < Mh_r^k$ for every $r \geq 0 (W_{0,t} = \varepsilon_t)$ and for some $k \geq 1$ and $\{h_r, r \geq 1\}$ such that $\sum_{r=1}^{\infty} r h_r^{1/2} < \infty$ (here and in what follows M is used a generic symbol to denote a universal constant > 0).

For any $\varepsilon > 0$ if we define $A_N = \{|\sum_{r=N}^{\infty} W_{r,t}| > \varepsilon\}$ for $N = 1, 2, \ldots$ we have that $E|\sum_{r=N}^{\infty} W_{rt}| \leq \sum_{r=N}^{\infty} E|W_{r,t}| \leq \sum_{r=N}^{\infty} E^{1/k}|W_{r,t}|^k \leq M \sum_{r=N}^{\infty} h_r$. This implies that $P(\cup_{N=m}^{\infty} A_N) \leq \sum_{N=m}^{\infty} P(A_N) \leq \sum_{N=m}^{\infty} E|\sum_{r=N}^{\infty} W_{r,t}|/\varepsilon \leq M \sum_{N=m}^{\infty} \sum_{r=N}^{\infty} h_r/\varepsilon \leq M \sum_{r=m}^{\infty} r h_r/\varepsilon \leq M(\sum_{r=m}^{\infty} r h_r^{1/2})/\varepsilon \to 0$ as $m \to \infty$. This shows that the infinite sum on the right hand side of (1) converges a.s.

Although we have assumed in (1) that $\{\varepsilon_t\}$ is a sequence of IID random variables it is only for the sake of simplicity of derivation of the results that appear in Sect. 3. We can easily establish the same results when $\{\varepsilon_t\}$, more generally, belongs to a class of stationary processes which can even be non SM and $\{W_{r,t}\}$ is defined in a slightly different manner than in (1). For example, if $\{\varepsilon_t\}$ is, itself, a LP which may not be SM but, otherwise, satisfies some mild regularity conditions and $W_{r,t} = \sum_{j=1}^{r} \lambda_{r,j} \varepsilon_{t-j} \varepsilon_{t-j-1}$ where $\lambda_{r,j} (1 \leq j \leq r, 1 \leq r < \infty)$ are suitably defined constants then we can show (details withheld) that the results of Theorems 4 and 11 below will hold.

3 Sampling properties of U-statistics

Consider the function g which belongs to a general class of functions, with $Eg(X_1) = \theta$, and $E(g(X_1))^2 < \infty$. We are interested in exploring the large sample properties of

$$U_n = n^{-1} \sum_{t=1}^{n} g(X_t), \tag{2}$$

and, in particular, of

$$Z_n = n^{1/2}(U_n - \theta). \tag{3}$$

Our primary objective is to investigate situations (relating to different forms of g) for which $\mathcal{L}(Z_n) \to \mathcal{N}(0, \sigma^2)$ as $n \to \infty$, where σ is a finite positive constant.

Note that the boundedness of the moments of $g(X_1)$ depends very critically on the boundedness of the moments of ε_1 and $g(\varepsilon_1)$, and the relation between $g(X_1)$ and $g(\varepsilon_1)$ through some Taylor series like expansion of $g(X_1)$ in terms of $g(\varepsilon_1)$. To this end let us define

$$J_1(\alpha, \beta) := E(g(\varepsilon_1 + \beta) - g(\varepsilon_1 + \alpha))^2 \tag{4}$$

where $\alpha, \beta(\beta \geq \alpha)$ are arbitrary but fixed real numbers, and assume that

$$J_1(\alpha, \beta) \leq M|\beta - \alpha|^k (1 + |\alpha|^\ell + |\beta|^\ell), \tag{5}$$

where $k \geq 1$ and $\ell \geq 0$ are arbitrary but pure constants. As examples of functions g satisfying relation (5) we may cite the following cases. (i) Let $g(x)$ be a step function satisfying some mild regularity conditions. (ii) Let $g(x)$ be a rational function of the form $g(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomial functions of finite orders and $|Q(x)| \geq C$ for every x and some constant $C > 0$. If in (1) we assume that F_0' (F_0 is the distribution function (DF) of ε_1) then (5) holds with $k = 1$ and $\ell = 0$ whereas (5) holds for (ii) with $k = 2$ and $\ell = 2(p + q + 1)$ if we assume that $E|\varepsilon_1|^s < \infty$ where $s = \max(2p, 2(p + q + 1))$. (iii) If g is a differentiable function with an everywhere bounded derivative g', then (5) holds with $k = 2$ and $\ell = 0$, if $E\varepsilon_1^2 < \infty$.

Let $\theta = Eg(X)$ and set $Y_t = g(X_t) - \theta$. Then we prove the following Lemma which will be useful for establishing Theorem 4 which is stated later.
We also assume from hereon that $w = 0$ (the adjustments that are necessary when $w > 0$ are only marginal).

Lemma 3.1 *Let $\{X_t\}$ satisfy the a.s. representation in (1). Assume that (5) holds, $Eg^2(\varepsilon_1) < \infty$, $E|\varepsilon_1|^{k+\ell} < \infty$, $E|W_{r,1}|^{k+\ell} < Mh_r^{k+\ell} (r \geq 1)$ for some sequence $\{h_r, r \geq 1\}$ of positive numbers such that $\sum_{r=1}^{\infty} rh_r^{1/2} < \infty$. Then $Eg^2(X_1) < \infty$ and*

$$|\sum_{v=1}^{\infty} EY_1Y_{1+v}| < \infty, \tag{6}$$

where $Y_t = g(X_t) - \theta$.

Proof Define $X_{a,t} = \sum_{r=0}^{a-1} W_{r,t}, a \geq 1$. Then by Minkowski's inequality and conditions of Lemma 1, we conclude that $E|X_{a,t}|^p \leq (\sum_{r=0}^{a-1} E^{1/p}|W_{r,t}|^p)^p \leq M(\sum_{r=0}^{a-1} h_r) \leq M$ where p is any positive number $\leq k + \ell$. Similarly we can show that $E|X_1|^p \leq M$ and $E|X_1 - X_{a,1}|^p \leq MH_a^p (H_a = \sum_{r=a}^{\infty} h_r)$. Again by (5) and the fact that $X_1 - \varepsilon_1$ is independent of ε_1 we conclude that $E(g(X_1) - g(\varepsilon_1))^2 = EJ_1(0, X_1 - \varepsilon_1) < M(H_1^k + H_1^{k+\ell}) \leq M$. Therefore $Eg^2(X_1) < \infty$ by (5) and the conditions of Lemma 1. This implies that $EY_1^2 < \infty$ and $|EY_1| < \infty$. Also if $v \geq 1$ then

$$\begin{aligned} |EY_1Y_{1+v}| &= |EY_1(g(X_{1+v}) - \theta)| \\ &= |EY_1(g(X_{1+v}) - g(X_{v,1+v})|. \end{aligned} \tag{7}$$

□

The result above follows from the fact that $X_{v,1+v}$ which involves $\varepsilon_{1+v}, \varepsilon_v, \ldots, \varepsilon_2$ is independent of Y_1 and $EY_1 = 0$. From (5) and the fact that $X_{1+v} - \varepsilon_{1+v}$ and $X_{v,1+v} - \varepsilon_{1+v}$ are both independent of ε_{1+v} we conclude that whenever $v \geq 1$

$$\begin{aligned} |EY_1Y_{1+v}| &\leq E^{1/2}Y_1^2 E^{1/2} J_1(X_{v,1+v} - \varepsilon_{1+v}, X_{1+v} - \varepsilon_{1+v}) \\ &\leq M(H_v^{k/2} H_1^{\ell/2} + H_v^{(k+\ell)/2}) \\ &\leq MH_v^{1/2} \text{ (because } k \geq 1). \end{aligned} \tag{8}$$

(in (8) we have used the Hölder's inequality $E(|X_{1+v}-X_{v,1+v}|^k|X_{v,1+v}-\varepsilon_{1+v}|^\ell) \le E^{k/(k+\ell)}|X_{1+v}-X_{v,1+v}|^{k+\ell}E^{\ell/(k+\ell)}|X_{v,1+v}-\varepsilon_{1+v}|^{k+\ell} \le MH_v^kH_1^\ell$). The relation (8) implies that

$$\sum_{v=1}^{\infty}|EY_1Y_{1+v}| \le M\sum_{v=1}^{\infty}H_v^{1/2} < M\sum_{r=1}^{\infty}rh_r^{1/2} \le M. \tag{9}$$

This completes the proof of Lemma 3.1.

For any arbitrary but fixed integer $m \ge 1$, set $Y_{m,t} = g(X_{m,t}) - \theta_m, \theta_m = Eg(X_{m,t}), Z_{m,t} = Y_t - Y_{m,t}$ and $V_{m,n} = n^{-1/2}\sum_{t=1}^{n} Z_{m,t}$.

We now prove

Lemma 3.2 *Let the conditions of Lemma 3.1 hold. Then*

$$\mathcal{L}\left(n^{-1/2}\sum_{t=1}^{n}Y_{m,t}\right) \to \mathcal{N}(0, \sigma_m^2), \tag{10}$$

as $n \to \infty$ *where* $\sigma_m^2 = \sum EY_{m,1}Y_{m,1+v}$ *and the summation is over all* v *such that* $|v| \le m-1$.

Proof Note that $\{Y_{m,t}\}$ is a $(m-1)$ dependent SSP which implies that (10) holds by a result due to Hoeffding and Robbins (1948). □

Lemma 3.3 *Let* m *be an arbitrary but fixed integer* ≥ 1 *and let conditions of Lemma 3.1 hold. Then*

$$E\ V_{m,n}^2 \le M_m, \tag{11}$$

for every $n \ge 1$, *where* M_m *depends strictly on* m *but independent of* n *and* $M_m \to 0$ *as* $m \to \infty$.

Proof Observe that $E\ V_{m,n}^2 < 2\sum_{v=0}^{n-1}(1-v/n)E\ Z_{m,1}\ Z_{m,1+v}$. By using an argument similar to that leading to (6) we, therefore, have that

$$EZ_{m,1}^2 < MH_m \tag{12}$$

and

$$|EZ_{m,1}Z_{m,1+v}| \le MH_m.$$

If $v \ge m+1$ we can, in a similar manner, using the facts that $X_{m,1+v}$ and $X_{v,1+v}$ are both independent of $Z_{m,1}$ and $EZ_{m,1} = 0$ conclude that

$$\begin{aligned} |EZ_{m,1}Z_{m,1+v}| &= |EZ_{m,1}(g(X_{1+v}) - g(X_{v,1+v}))| \\ &\le ME^{1/2}Z_{m,1}^2E^{1/2}|X_{1+v}-X_{v,1+v}|^k \\ &\quad (1+|X_{v,1+v}-\varepsilon_{1+v}|^\ell + |X_{1+v}-X_{v,1+v}|^\ell) \\ &< MH_m^{1/2}(H_v^{k/2}H_1^{\ell/2} + H_v^{(k+\ell)/2}) \\ &< MH_m^{1/2}H_v^{1/2}(\because k \ge 1). \end{aligned} \tag{13}$$

Therefore,

$$|\sum_{v=0}^{\infty} EZ_{m,1}Z_{m,1+v}| \leq M\left(\sum_{v=0}^{m-1} EZ_{m,1}^2 + \sum_{v=m}^{\infty} |EZ_{m,1}Z_{m,1+v}|\right)$$
$$\leq M\left(mH_m + H_m^{1/2}\sum_{v=m}^{\infty} H_v^{1/2}\right) = M_m \text{ (say).} \quad (14)$$

(14) implies that (11) holds and the proof of Lemma 3.3 is complete. □

We now prove

Theorem 3.1 *Let* $\{X_t\}$ *satisfy the a.s. representation in (1). Assume that (5) holds,* $Eg^2(\varepsilon_1) < \infty$, $E|\varepsilon_1^{k+\ell}| < \infty$, $E|W_{r,1}|^{k+\ell} < Mh_r^{k+\ell}(r \geq 1)$ *for some sequence* $\{h_r, r \geq 1\}$ *of positive numbers such that* $\sum_{r=1}^{\infty} rh_r^{1/2} < \infty$. *Set* $\sigma^2 = \sum_{v=-\infty}^{\infty} EY_1Y_{1+v}$. *Then* $\sigma^2 < \infty$. *If* $\sigma > 0$ *then*

$$\mathcal{L}(Z_n) \to \mathcal{N}(0, \sigma^2), \quad \text{as } n \to \infty. \quad (15)$$

Proof The result of Theorem 4 follows from relations (6), (10), (11) and Theorem 7.7.1 in Anderson (1971). □

Remark 1 Sometimes we can deal with functions which do not satisfy (5). For example let $\{X_t\}$ be an LP and let $g(x) = \exp(cx)$ where c is a finite constant. Assume that $E\exp(u\varepsilon_1) < \exp(\gamma|u|^\theta)$ for all u and some $\gamma > 0$ and $\theta \in (0, 2]$. Then $Eg^2(X_1) < M\exp(\gamma(2c)^\theta \sum_{r=0}^{\infty} |g_r|^\theta)$ and $J_1(\alpha, \beta) < M(\beta - \alpha)^2 \exp(2|c|(|\alpha| + |\beta|))$. We can then show that the results of Lemmas 3.1–3.3 will hold and hence the result of Theorem 4 obtains.

Now consider a U-statistics of order a defined by

$$U_n = \binom{n}{a}^{-1} \sum g(X_{i_1}, X_{i_2}, \ldots, X_{i_a}), \quad (16)$$

where $g(x_1, x_2, \ldots, x_a)$ is a symmetric function of its arguments, and the summation is over all integer values of $i_1, i_2, \ldots, i_a$ such that $1 \leq i_1 < i_2 < \cdots < i_a \leq n$. Write

$$\theta = \int g(x_1, x_2, \ldots x_a) \prod_{j=1}^{a} \mathrm{d}F(x_j), \quad (17)$$

where F is the DF of X_1. We shall establish the asymptotic normality of $Z_n = n^{\frac{1}{2}}(U_n - \theta)$ under some regularity conditions on g and F when X_t satisfies the a.s. representation (1) with $W_{r,t}$ being a function of $\varepsilon_{t-1}, \ldots, \varepsilon_{t-r}$.

In order to simplify the details of derivation relating to an arbitrary but fixed value of a we limit our discussion to the case $a = 2$. Only some tedious but routine extension of this particular case will be needed in order to establish the general case.

Set

$$Z_n := n^{\frac{1}{2}}(U_n - \theta),$$
$$T_n := 2n^{-\frac{1}{2}}\sum_{i=1}^{n}(g_1(X_i) - \theta),$$
$$g_1(x) := \int_{-\infty}^{\infty} g(x, y)\mathrm{d}F(y) \tag{18}$$

and

$$J_2(a, b, \alpha, \beta) := E(g(a + b, \varepsilon_1 + \beta) - g(a, \varepsilon_1 + \alpha))^2. \tag{19}$$

Assume that

$$J_2(a, b, \alpha, \beta) < M(|b|^k + |\beta - \alpha|^k)(1 + |a|^\ell + |b|^\ell + |\alpha|^\ell + |\beta|^\ell) \tag{20}$$

for some $k \geq 1$ and $\ell \geq 0$.

As examples of situations where (20) holds we may cite the following cases. (i) Let $g(x, y)$ be a step function of $x + y$ satisfying some mild regularity conditions and assume that F_0' is uniformly bounded everywhere. Then (20) holds. (ii) Let $g(x, y) = P(x, y)/Q(x, y)$ where P and Q are polynomials of finite order and $|Q(x, y)| \geq C$ for all x, y and for some $C > 0$. Then condition (20) is satisfied for suitable choice of k and ℓ provided some moment condition on ε_1 holds.

Lemma 3.4 *Let the symmetric function $g(x, y)$ be such that $Eg^2(\varepsilon_1, \varepsilon_2) < \infty$. Assume that (20) holds, $E|\varepsilon_1|^{k+\ell} < \infty$, $E|W_{r,1}|^{k+\ell} \leq Mh_r^{k+\ell}$ for some sequence $\{h_r, r \geq 1\}$ of positive numbers for which $\sum_{r=1}^{\infty} rh_r^{1/2} < \infty$. Then*

$$\mathcal{L}(T_n) \to \mathcal{N}(0, 4\sigma^2), \tag{21}$$

where $\sigma^2 = \sum_{v=-\infty}^{\infty} \gamma_v$, and $\gamma_v = E(g_1(X_1) - \theta)(g_1(X_{1+v}) - \theta))$.

Proof Note that we can write $g_1(x) = Eg(x, X_1)$. Let $\{\hat{\varepsilon_t}\}$ be an independent copy of $\{\varepsilon_t\}$ and let $\{\hat{X}_t\}$ be the correspondingly defined independent version of $\{X_t\}$. Then by (20) and the conditions of Lemma 3.3 we conclude that

$$\begin{aligned}
Eg_1^2(\varepsilon_1) &= EE^2(g(\varepsilon_1, \hat{X}_1)|\varepsilon_1) \\
&\leq Eg^2(\varepsilon_1, \hat{X}_1) \\
&\leq M(Eg^2(\varepsilon_1, \hat{\varepsilon}_1) + E(g(\varepsilon_1, \hat{X}_1) - g(\varepsilon_1, \hat{\varepsilon}_1))^2 \\
&\leq M(1 + EJ_2(\varepsilon_1, 0, 0, \hat{X}_1 - \hat{\varepsilon}_1)) \\
&< M(1 + E|\hat{X}_1 - \hat{\varepsilon}_1|^k(1 + |\varepsilon_1|^\ell + |\hat{X}_1 - \hat{\varepsilon}_1|^\ell) \\
&< MH_1^{k+\ell} < \infty.
\end{aligned} \tag{22}$$

Similarly we can show that relation (5) is satisfied with k and ℓ as defined in (20). Therefore, all conditions of Theorem 3.1 hold and we conclude that

$$\mathcal{L}(T_n) \to \mathcal{N}(0, 4\sigma^2), \tag{23}$$

as $n \to \infty$. This completes the proof of Lemma 3.4.

Routine computation leads to the relation

$$EZ_n^2 = \sum_{\alpha=1}^{3} Q_{\alpha,n}, \tag{24}$$

where

$$\begin{aligned} Q_{1,n} &= C_n^2 \sum_{r=1}^{n-1} (n-r) EY_{1,1+r}, \\ Q_{2,n} &= 2C_n^2 \sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} (n-r-s) \\ &\quad \times E(Y_{1,1+r}Y_{1,1+r+s} + Y_{1,1+r}Y_{1+r,1+r+s} + Y_{1,1+r+s}Y_{1+r,1+r+s}), \\ Q_{3,n} &= 2C_n^2 \sum_{t=1}^{n-3} \sum_{s=1}^{n-t-2} \sum_{r=1}^{n-s-t-1} (n-r-s-t) \\ &\quad \times E(Y_{1,1+r}Y_{1+r+s,1+r+s+t} + Y_{1,1+r+s}Y_{1+r,1+r+s+t} + Y_{1,1+r+s+t}Y_{1+r,1+r+s}), \end{aligned}$$

$Y_{ij} = g(X_i, X_j) - \theta$ and $C_n = 2n^{1/2}/(n(n-1))$. Similarly we can establish that

$$ET_nZ_n = \sum_{\alpha=4}^{5} Q_{\alpha,n}, \tag{25}$$

where

$$\begin{aligned} Q_{4,n} &= D_n \sum_{r=1}^{n-1} (n-r) E(R_1 Y_{1,1+r} + R_{1+r} Y_{1,1+r}), \\ Q_{5,n} &= D_n \sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} (n-r-s) \\ &\quad \times E(R_1 Y_{1+r,1+r+s} + R_{1+r} Y_{1,1+r+s} + R_{1+r+s} Y_{1,1+r}), \end{aligned}$$

$R_i = g_1(X_i) - \theta$ and $D_n = 4/(n(n-1))$.

Our main objective at this stage is to show that $E(T_n - Z_n) \to 0$ as $n \to \infty$, so that we can be sure that T_n and Z_n have the same asymptotic distribution. An appeal to Lemma 3.4 will then establish the asymptotic normality of Z_n. The following Lemma will be useful in proving this asymptotic result. □

Lemma 3.5 *Let the conditions of Lemma 3.1 hold. Then for any positive integers* a, b, *and* r

$$E(g(X_1, X_{1+r}) - g(X_{a,1}, X_{b,1+r}))^2 \le M(H_a + H_b). \tag{26}$$

Also if $Eg^2(\varepsilon_1, \varepsilon_2) < \infty$ *then*

$$E(g(X_1, X_{1+r}) - \theta)^2 = \gamma_0 + O(H_r^{1/2}). \tag{27}$$

Proof Note that since X_1, $X_{1+r} - \varepsilon_{1+r}$, $X_{a,1}$ and $X_{b,1+r} - \varepsilon_{1+r}$ are independent of ε_{1+r} we have by (20) that the expectation on the left side of (26) is equal to

$$\begin{aligned}
&EJ_2(X_{a,1}, X_1 - X_{a,1}, X_{b,1+r} - \varepsilon_{1+r}, X_{1+r} - \varepsilon_{1+r})\\
&\le ME(|X_1 - X_{a,1}|^k + |X_{1+r} - X_{b,1+r}|^k)\\
&\times(1 + |X_{a,1}|^\ell + |X_1 - X_{a,1}|^\ell + |X_{b,1+r} - \varepsilon_{1+r}|^\ell + |X_{1+r} - \varepsilon_{1+r}|^\ell)\\
&\le M(H_a + H_b).
\end{aligned}$$

By using similar arguments we can show that the expectation on the left side of (27) is less than or equal to

$$M(E(g(X_1, X_{r,1+r}) - \theta)^2 + E(g(X_1, X_{1+r} - g(X_1, X_{r,1+r}))^2. \tag{28}$$

It is easy to see from (26) that the second expectation in (28) is less than or equal to MH_r. Also since X_1 is independent of $X_{r,1+r}$ the first expectation in (28) is equal to

$$E(g(X_1, \hat{X}_{r,1+r}) - \theta)^2 = \gamma_0 + O(H_r),$$

where $\gamma_0 = E(g_1(X_1) - \theta)^2$, and $\hat{X}_t$ has been defined earlier in the proof of Lemma 3.4 with the corresponding formulation of $\hat{X}_{a,t}$. This completes the proof of Lemma 3.5. □

Let us know go back to $Q_{\alpha,n}(1 \le \alpha \le 5)$ and show how to deal with the components of these Q's. The computations are tedious but routine and uses the methodology described in Lemma 3.5. As an example we prove the following

Lemma 3.6 *Let the conditions of Lemma 3.1 hold. Then for all positive integers r and s we have the following relation.*

$$EY_{1,1+r}Y_{1,1+r+s} = \gamma_0 + O(H_r^{1/2}) + O(H_s^{1/2}). \tag{29}$$

Proof We have, by Lemma 3.5 and the fact that X_1, $X_{r,1+r}$ and $X_{s,1+r+s}$ are mutually independent,

$$\begin{aligned}
EY_{1,1+r}Y_{1,1+r+s} &= E[g(X_1, X_{r,1+r}) - \theta + g(X_1, X_{1+r})\\
&\quad -g(X_1, X_{r,1+r})][g(X_{r,1+r}, X_{s,1+r+s}) - \theta\\
&\quad +g(X_{1+r}, X_{1+r+s}) - g(X_{r,1+r}, X_{s,1+r+s})]\\
&= E(g(X_1, X_{r,1+r}) - \theta)(g(X_{r,1+r}, X_{s,1+r+s}) - \theta)\\
&\quad +O(H_r^{1/2}) + O(H_s^{1/2})\\
&= E(g(X_1, \hat{X}_{r,1+r}) - \theta)(g(\hat{X}_{r,1+r}, \tilde{X}_{s,1+r+s}) - \theta)\\
&\quad +O(H_r^{1/2}) + O(H_s^{1/2}),
\end{aligned} \tag{30}$$

where $\{\hat{\varepsilon}_t\}$ and $\{\tilde{\varepsilon}_t\}$ are two independent copies of $\{\varepsilon_t\}$ and $\{\hat{X}\}$ and $\{\tilde{X}_t\}$ are the correspondingly defined independent versions of $\{X_t\}$. Another application of Lemma 3.5 will now lead to the fact that the last expression on the right side of (30) can be written as

$$\gamma_0 + O(H_r^{1/2}) + O(H_s^{1/2}).$$

This completes the proof of Lemma 3.6. □

By using arguments similar to those in Lemma 3.6 we can establish the following results

$$EY_{1,1+r}Y_{1+r,1+r+s} = \gamma_0 + O(H_r^{1/2}) + O(H_s^{1/2}), \tag{31}$$

$$EY_{1,1+r+s}Y_{1+r,1+r+s} = \gamma_0 + O(H_r^{1/2}) + O(H_s^{1/2}). \tag{32}$$

We now use relations (24), (29), (31), (32) and the fact that $\sum_{r=1}^{\infty} H_r^{1/2} \leq \sum_{u=1}^{\infty} uh_u^{1/2} < \infty$ and conclude that

$$Q_{1,n} \to 0 \text{ and } Q_{2,n} \to 4\gamma_0 \quad \text{as } n \to \infty. \tag{33}$$

Consider now the components of $Q_{3,n}$.

Lemma 3.7 *Let the conditions of Lemma 3.1 hold. Then for all positive integers* r, s,*and* t *we have that*

$$\begin{aligned} EY_{1,1+r}Y_{1+r+s,1+r+s+t} &= \gamma_s + O(H_r^{1/2}) + O(H_s^{1/2}) + O(H_t^{1/2}) \\ EY_{1,1+r+s}Y_{1+r,1+r+s+t} &= \gamma_r + \gamma_s + \gamma_t + O(H_r^{1/2}) + O(H_s^{1/2}) + O(H_t^{1/2}), \\ EY_{1,1+r+s+t}Y_{1+r,1+r+s} &= \gamma_r + \gamma_s + O(H_r^{1/2}) + O(H_s^{1/2}) + O(H_t^{1/2}). \end{aligned} \tag{34}$$

Proof We can establish the results of Lemma 3.7 by repeated applications of arguments similar to those in Lemmas 3.5 and 3.6. The details, however, are tedious to derive but entirely routine and are, therefore, withheld. □

Lemma 3.8

$$Q_{3,n} \to 8\sum_{t=1}^{\infty} \gamma_t \quad \textit{as } n \to \infty. \tag{35}$$

Proof By Lemma 3.7 we conclude that

$$Q_{3,n} = 48/(n(n-1)^2)\sum_{t=1}^{n-3} \binom{n-t}{3} \gamma_t + R_n, \tag{36}$$

where R_n is the contribution to $Q_{3,n}$ due to terms $O(H_r^{1/2}) + O(H_s^{1/2}) + O(H_t^{1/2})$. The first term on the right side of (36) $\to 8\sum_{t=1}^{\infty} \gamma_t$ as $n \to \infty$. In order to estimate R_n we first choose an integer w_n such that $w_n \to \infty$ but $w_n^3/n^2 \to 0$ as $n \to \infty$. Then for any one of r, s or t exceeding w_n the contribution to R_n due to $O(H_r^{1/2}) + O(H_s^{1/2}) + O(H_t^{1/2})$ is less than $[M/(n(n-1)^2)]n^3 \sum_{u=w_n}^{\infty} H_u^{1/2} \to 0$ as $n \to \infty$, whereas if none of the integers r, s, and t exceeds w_n then the contribution to R_n due $O(H_r^{1/2}) + O(H_s^{1/2}) + O(H_t^{1/2})$ will be less than or equal to $[M/(n(n-1)^2)]nw_n^3 \to 0$ as $n \to \infty$ (because each of the expectations on the left side of (34) is bounded (by M)). This implies that $R_n \to 0$ as $n \to \infty$ and, therefore, Lemma 3.8 is proved. □

From relations (33), (34) and (35) we conclude immediately that

$$EZ_n^2 \to 4\gamma_0 + 8\sum_{t=1}^{\infty} \gamma_t = 4\sigma^2. \tag{37}$$

Lemma 3.9

$$ET_nZ_n \to 4\sigma^2 \quad as\ n \to \infty. \tag{38}$$

Proof Consider the relation (25) which shows that $ET_nZ_n = Q_{4,n} + Q_{5,n}$. By routine although somewhat long and tedious computation using arguments similar to those leading to Lemmas 3.5 and 3.6 we conclude that $Q_{4,n} \to 4\gamma_0$ and $Q_{5,n} \to 8\sum_{t=1}^{\infty}\gamma_t$ as $n \to \infty$. The result of Lemma 3.9 will, therefore, follow easily from (25). Since $ET_n^2 \to 4\sigma^2$ as $n \to \infty$,we have proved Lemma 3.9. □

Theorem 3.2 *Let the conditions of Lemma 3.4 hold. Then*

$$\mathcal{L}(n^{1/2}(U_n - \theta)) \to \mathcal{N}(0, 4\sigma^2) \quad as\ n \to \infty, \tag{39}$$

where $\sigma^2 = \sum_{r=-\infty}^{\infty}\gamma_r$ *and* $\gamma_r = E(g_1(X_1) - \theta)(g_1(X_{1+r}) - \theta),\ -\infty < r < \infty.$

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