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Confidence estimation for tolerance intervals

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Abstract The post-data performances of normal tolerance intervals are studied. Under a robust Bayesian predictive scheme, we establish the ordering and bounds of the confidence estimators. It is found that the nominal confidence coefficient tends to be extreme yet coincides with the limiting Bayes estimators in some scenarios. A remark on the choice of beta priors is also given.

Keywords Tolerance intervals · Infinite exchangeability · De Finetti's representation theorem · Robust Bayesian analysis

1 Introduction

We investigate the post-data performance of tolerance intervals through an estimated confidence approach under normal settings. Proposed in Berger (1988), the estimated confidence approach has been employed to evaluate the post-data performance of the usual pivot-based confidence intervals and leads to a few interesting results, for example, Hwang and Brown (1991), Goutis and Casella (1992), Tsao and Hwang (1999) and references therein. Roughly speaking, these studies scrutinize the appropriateness of the confidence coefficient (usually denoted as $1 - \alpha$, say 95%) as an estimator for $1_{[\theta \in C(X)]}$ where 1_A is the indicator function of event A, θ is the parameter of interest and $C(X)$ is the $(1 - \alpha)$ confidence interval. See Goutis and Casella (1995) for a review and related approaches for frequentist post-data inference. Although intensive studies have been done for confidence intervals via the estimated confidence approach, we are not aware of parallel works for tolerance intervals. In addition, instead of focusing on the decision theoretical properties of the estimator such as admissibility, domination, minimaxity, etc., we

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compare the confidence coefficient with other Bayes estimators in a robust Bayesian predictive framework. This study hints the possibility and potential difficulty along this direction. The interested readers are referred to Berger (1984, 1994) for reviews on robust Bayesian analyses.

The tolerance interval, while popular in some application domains, seldom lends itself to introductory statistics textbook. Introduced in Wilks (1941), Guttman (1970) and Aitchison and Dunsmore (1975) provide book-length coverages. See, for example, Jílek (1981), Jílek and Ackermann (1989) and Patel (1986) for comprehensive bibliography. Di Bucchianico, Einmahl and Mushkudiani (2001) is a recent advancement in the theory of nonparametric tolerance intervals and contains good updated references. As indicated in Carrol and Ruppert (1999), the terminology for the tolerance interval is not standardized yet. Here the definition of Guttman (1970) is adopted. In addition, we also assume normality for simplicity and practicality. Precisely, let $X_1, \dots, X_n$ be iid random variables from a normal distribution with mean θ and a known variance $\sigma^2 > 0$. The usual one-sided β content/$(1-\alpha)$ confidence tolerance interval is [query 1, 2]

$$C(\mathbf{X}) = (-\infty, \bar{X} + \sigma c] \tag{1}$$

where $c = c(\alpha, \beta, n) = z_{1-\beta} + \frac{1}{\sqrt{n}} z_\alpha$, $\Phi(z_\alpha) = 1-\alpha$, Φ is the standard normal distribution function and $\mathbf{X} = (X_1, \dots, X_n)'$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. By definition, for all θ

$$P_{\mathbf{X}|\theta}\left[P_{X'}(X' \in C(\mathbf{X})) \geq \beta\right] = 1-\alpha$$

for $X' \sim N(\theta, \sigma^2)$, a new independent sample from the same population as Xs. It has been noted, for example, in Green (1969) and Patel (1986) that one-sided β content/$(1-\alpha)$ confidence tolerance interval can also be considered as a confidence interval for quantiles and lends itself to important practical applications.

In this study, we investigate the validity of the confidence coefficient $1-\alpha$ as a post-data measure of performance of the β content/$(1-\alpha)$ confidence tolerance intervals. Specifically, we want to know whether the confidence coefficient $1-\alpha$ is a reasonable assessment of the event

$$P_{X'}(X' \in C(\mathbf{x})) \geq \beta$$

for a realized $\mathbf{x}$. The left-hand side of the expression above is an approximation of a more practical yet less tractable

$$\frac{1}{m}\sum_{i=1}^{m} 1_{[X'_{n+i} \in C(\mathbf{x})]} \tag{2}$$

where X'_{n+i} denotes the ith future observation. Therefore, estimators of Eq. (2) provide information about post-data performance of the tolerance interval.

In a Bayesian predictive scheme, for example Geisser (1993), the predictive probability of future $X_{n+1}, \dots, X_{n+m}$ given $x_1, \dots, x_n$

$$p(x_{n+1}, \dots, x_{n+m}|x_1, \dots, x_n) = \int_\Theta p(x_{n+1}, \dots, x_{n+m}|\theta)\pi(\theta|\mathbf{x})\mathrm{d}\theta \tag{3}$$

where $p(.|\theta)$ is the sampling density and $\pi(\theta|\mathbf{x})$ is the posterior density given $\mathbf{x} = (x_1, \ldots, x_n)'$. This conditional probability density can then be used in the ensuing calculations. This formulation Eq. (3) can also be understood as if these Xs are assumed to be infinitely exchangeable. Specifically, under our normality assumption (in the sampling distribution), it is equivalent to assume $X_1, \ldots$ be random variables and for any n, $X_1, \ldots, X_n$ has the joint probability density

$$p(x_1, \ldots, x_n) = \int_R \prod_{i=1}^{n} \frac{1}{\sigma}\phi\left(\frac{x_i - \theta}{\sigma}\right)\pi(\theta)\mathrm{d}\theta, \tag{4}$$

for some unknown prior π, and σ^2 is known and ϕ denotes the standard normal probability density function (pdf). This expression is closely related to the de Finetti's representation of joint pdf of infinitely exchangeable random variables, see, for example, de Finetti (1991) and Bernardo and Smith (1993). For simplicity, we consider $\pi(\theta)$ as a normal density $N(\mu, \tau^2)$. Two quantities of interest, from a predictive viewpoint, for evaluating tolerance intervals are

$$v(Y_1, \ldots, Y_m) = 1_{[\sum_{i=1}^{m} Y_i \geq m\beta]} \tag{5}$$

and

$$w(t) = P_{Y|t}\left(\sum_{i=1}^{m} Y_i \geq m\beta\right) \tag{6}$$

where $Y_i = 1_{[X'_{n+i} \in C(\mathbf{x})]}$, X'_{n+i} denotes the ith future observation and $t = P[Y_i = 1]$.

Note that $Y_i = 1$ when the ith future observation belongs to $C(\mathbf{x})$ and Eq. (5) equals 1 if the interval does have correct content coverage. By interchanging the integrals, the problem of assessing Eq. (5) is equivalent to estimate Eq. (6) since

$$E_{X'|x,\pi} v(Y(X')) = E_{\pi(\theta|x)} w(t(X')). \tag{7}$$

The expression Eq. (7) is a Bayes estimate of Eq. (5) with respect to prior π. If the prior is known to be true then Eq. (7) constitutes a post-data measure of confidence. However, the choice of prior can be rather arbitrary and subjective. To alleviate the drawback, we perform a robust Bayesian analysis by allowing π to vary within some reasonable class Γ and derive

$$\sup_{\pi \in \Gamma} E_{\pi(\theta|x)} w(t(X')) \quad \text{and} \quad \inf_{\pi \in \Gamma} E_{\pi(\theta|x)} w(t(X')). \tag{8}$$

If Γ is indeed reasonable then a good estimate of $w(t(X'))$ should lie within the bounds Eq. (8). Particularly, the supremum should be close to or greater than $1 - \alpha$ for most $\mathbf{x}$, from a frequentist viewpoint, since $C(\mathbf{X})$ is a β content/$(1 - \alpha)$ confidence tolerance interval. If otherwise, it indicates the confidence coefficient $(1-\alpha)$ might be too liberal as a post-data measure of confidence. Under this robust Bayesian predictive scheme, we establish the ordering and bounds of the Bayes estimators with respect to normal prior families. We find the confidence coefficient tends to be more extreme than other proper Bayes estimators yet coincides with the limiting Bayes estimators in some cases. It is also noted that the derived prior can be well approximated by a beta density which has better analytical tractability.

2 Main results

In light of de Finetti representation theorem (see de Finetti 1991; Bernardo and Smith 1993), Eq. (4) implies that $X_1, \dots$ are infinitely exchangeable normal random variables. Furthermore, the induced Ys are in turn infinitely exchangeable taking values on 0–1. Again by the representation theorem, their joint pdf is

$$p(y_1, \dots, y_m) = \int_0^1 t^{\sum_i^m y_i} (1-t)^{m-\sum y_i} \, \mathrm{d}\lambda(t) \tag{9}$$

for some mixing distribution λ. However, because of our assumptions on the marginal joint pdf of X given in Eq. (4), we can further derive

$$\begin{aligned}\lambda(t) &= \lim_{k\to\infty} E_{\pi(\theta)} P_{Y|\theta}\left[\frac{\sum_{i=1}^k Y_i - kt_\theta}{\sqrt{kt_\theta(1-t_\theta)}} \le \sqrt{k}\left(\frac{t-t_\theta}{t_\theta(1-t_\theta)}\right)\right] \\ &= E_{\pi(\theta)} 1_{[t>t_\theta]},\end{aligned}$$

where $t_\theta = P_{Y|\theta}[Y=1] = P_{X'|\theta}\big[X' - \bar{x} < \sigma c\big] = \Phi\Big(c - \Big(\frac{\theta-\bar{x}}{\sigma}\Big)\Big)$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$. In fact, the observed $\mathbf{x}$ can be utilized to update the predictive probability and we have

$$\begin{aligned}\lambda(t|\mu, \tau^2; \bar{x}) &= E_{\pi(\theta|\mathbf{x})} 1_{[t>t_\theta]} \\ &= 1 - \Phi\Big(\rho^{1/2}[\bar{x} - \mu_{\bar{x}} + \sigma(c - \Phi^{-1}(t))]\Big)\end{aligned} \tag{10}$$

where $\rho = (\sigma_n^2)^{-1} + (\tau^2)^{-1}$, $\quad \mu_{\bar{x}} = \frac{\sigma_n^2\mu + \tau^2\bar{x}}{\sigma_n^2 + \tau^2}$ $\quad$ and $\sigma_n^2 = \sigma^2/n$.

Note that, under our framework, $\bar{x}$ is considered as an observed fixed quantity unless specified otherwise. We then pursue two objectives:

(1) Assess the bounds of $E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t)$ allowing π to vary within some class of normal priors Γ.

(2) Compare $\lambda(t|\mu, \tau^2; \bar{x})$ with beta distributions which are the widely used conjugate priors in the binary problems.

We exploit the ordering structure in the posterior distributions to derive bounds for

$$E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t) = \int w(t) \mathrm{d}\lambda(t|\mu, \tau^2; \bar{x})$$

for $\pi \in \Gamma_{\tau^2}$ and Γ_μ where

$$\begin{aligned}\Gamma_{\tau^2} &= \{\pi : \pi(\theta|\mu, \tau^2) \sim N(\mu, \tau^2), \mu \in (-\infty, \infty)\}, \\ \Gamma_\mu &= \{\pi : \pi(\theta|\mu, \tau^2) \sim N(\mu, \tau^2), \tau^2 > 0\}.\end{aligned}$$

These families are simply location and scale families of normal priors. Recall:

Lemma 2.1 *Let F and G be two distributions and $\bar{F}(x) = 1 - F(x)$. If $\bar{F}(x) \le \bar{G}(x)$ for all x then*

$$\int \psi(x)\mathrm{d}F(x) \le \int \psi(x)\mathrm{d}G(x)$$

for all increasing ψ, provided both integrals exist.

The following theorem establishes the (pointwise) ordering of the posterior expectations of $w(t)$ with respect to μ.

Theorem 2.1 *For any $\tau^2 > 0$, if $\pi \in \Gamma_{\tau^2}$ then*

$$E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t)$$

is decreasing in μ.

Proof Since

$$\frac{\partial}{\partial \mu}\lambda(t|\mu, \tau^2; \bar{x}) = \frac{\sigma_n^2 \rho^{1/2}}{\sigma_n^2 + \tau^2}\phi\left\{\rho^{1/2}[\bar{x} - \mu_{\bar{x}} + \sigma(c - \Phi^{-1}(t))]\right\} \ge 0$$

thus for any $\tau^2 > 0$ and $\mu_1 < \mu_2$, there exists some constant ξ in (μ_1, μ_2)

$$\lambda(t|\mu_2, \tau^2; \bar{x}) - \lambda(t|\mu_1, \tau^2; \bar{x}) = \left(\frac{\partial}{\partial \mu}\lambda(t|\mu, \tau^2; \bar{x})\bigg|_{\mu=\xi}\right)(\mu_2 - \mu_1) \ge 0.$$

Hence $\lambda(t|\mu_2, \tau^2; \bar{x}) \ge \lambda(t|\mu_1, \tau^2; \bar{x})$ for all t. By Lemma 2.1 and $w(t)$ is increasing, the result immediately follows. □

The (pointwise) ordering of the posterior expectation of $w(t)$ is harder to establish and requires involved calculations. Now we rephrase the *Karlin–Norvikoff cut-criterion* (cf. Theorem E, Szekli, 1995, p. 17) in our context.

Theorem 2.2 *Suppose that for distribution functions F, G with means m_F, m_G respectively. If $m_F \le m_G$ and for some $\xi \in \mathcal{R} = (-\infty, \infty)$*

$$F(x) \le G(x), \text{ for } x \le \xi \quad \text{and} \quad F(x) \ge G(x), \text{ for } x > \xi.$$

Then

$$\int \psi(x)\mathrm{d}F(x) \le \int \psi(x)\mathrm{d}G(x)$$

for all increasingly convex ψ, provided both integrals exist.

The following lemma establishes ordering of integrals of increasingly convex functions with respect to τ^2.

Lemma 2.2 *If $\mu, \bar{x}$ satisfy $\mu - \bar{x} \ge \mu_*(\tau_1^2)$ where $\mu_*(\tau^2) = \sigma c(\sigma_n^2 + \tau^2/\sigma_n^2 + 2\tau^2)$. Then for all $\tau_1^2 < \tau_2^2$ and ψ is increasingly convex*

$$\int \psi(t)\mathrm{d}\lambda(t|\mu, \tau_1^2; \bar{x}) \le \int \psi(t)\mathrm{d}\lambda(t|\mu, \tau_2^2; \bar{x}),$$

provided both integrals exist.

Proof If we can show that for some t_*

$$\begin{aligned} \lambda(t|\mu, \tau_1^2; \bar{x}) &\le \lambda(t|\mu, \tau_2^2; \bar{x}) \quad \text{for } t \le t_*, \\ \lambda(t|\mu, \tau_1^2; \bar{x}) &\ge \lambda(t|\mu, \tau_2^2; \bar{x}) \quad \text{for } t > t_*. \end{aligned} \tag{11}$$

and $m_{\tau_1^2} \le m_{\tau_2^2}$ then by Theorem 2.2 the proof is complete. Here m_{τ^2} denotes the mean with respect to $\lambda(t|\mu, \tau^2)$.

For ease of notation, we denote $\lambda(t|\tau^2) = \lambda(t|\mu, \tau^2; \bar{x})$ which should raise no confusion here. Firstly, we will show that

$$\frac{\partial}{\partial \tau^2}\lambda(t|\tau^2) \begin{cases} \ge 0 & \text{if } t < t_* = \Phi\left(c + \left(\frac{\sigma_n^2+2\tau^2}{\sigma_n^2+\tau^2}\right)\left(\frac{\bar{x}-\mu}{\sigma}\right)\right), \\ \le 0 & \text{if } t \ge t_*. \end{cases} \tag{12}$$

Thus for some τ_*^2 lies in (τ_1^2, τ_2^2)

$$\lambda(t|\tau_2^2) - \lambda(t|\tau_1^2) = \left(\frac{\partial}{\partial \tau^2}\lambda(t|\tau^2)\bigg|_{\tau^2=\tau_*^2} \right)(\tau_2^2 - \tau_1^2).$$

Therefore Eq. (12) implies Eq. (11). Now

$$\frac{\partial}{\partial \tau^2}\lambda(t|\tau^2) = -\left(\frac{\partial}{\partial \tau^2}\rho^{1/2}[\bar{x} - \mu_{\bar{x}} + \sigma a(t)] \right)\phi\left(\rho^{1/2}[\bar{x} - \mu_{\bar{x}} + \sigma a(t)] \right)$$

where $a(t) = c - \Phi^{-1}(t)$. Note that

$$\frac{\partial}{\partial \tau^2}\rho = -(\tau^2)^{-2} \quad \text{and} \quad \frac{\partial}{\partial \tau^2}\mu_{\bar{x}} = \sigma_n^2(\bar{x} - \mu)/(\sigma_n^2 + \tau^2)^2.$$

In addition, $(\rho\tau^2)^{-1} = (\sigma_n^2)/(\sigma_n^2 + \tau^2)$. Thus

$$\frac{\partial}{\partial \tau^2}\rho^{1/2}[\bar{x} - \mu(\bar{x}) + \sigma a(t)] = \frac{-1}{2\rho^{1/2}\tau^4}\left(\left(\frac{\sigma_n^2 + 2\tau^2}{\sigma_n^2 + \tau^2} \right)(\bar{x} - \mu) + \sigma a(t) \right).$$

Because the sign of $\frac{\partial}{\partial \tau^2}\lambda(t|\tau^2)$ is the same as that of

$$\left(\frac{\sigma_n^2 + 2\tau^2}{\sigma_n^2 + \tau^2} \right)(\bar{x} - \mu) + \sigma a(t) \tag{13}$$

which changes from positive to negative at t_*. This proves Eq. (12).

Next, we will show that $m_{\tau_1^2} \le m_{\tau_2^2}$. Applying integration by parts, we can rewrite

$$m_{\tau^2} = \int_0^1 t \mathrm{d}\lambda(t|\tau^2) = 1 - \int_0^1 \lambda(t|\tau^2)\mathrm{d}t.$$

Hence

$$\frac{\partial}{\partial \tau^2} m_{\tau^2} = \frac{-1}{2\rho^{1/2}\tau^4} \int_0^1 \left[\left(\frac{\sigma_n^2 + 2\tau^2}{\sigma_n^2 + \tau^2} \right) (\bar{x} - \mu) + \sigma a(t) \right]$$
$$\times \phi\Big(\rho^{1/2} [\bar{x} - \mu_{\bar{x}} + \sigma a(t)] \Big) \mathrm{d}t.$$

Note that Eq. (13) can be expressed as

$$s + \frac{2\tau^2}{\sigma_n^2 + \tau^2} (\bar{x} - \mu),$$

where $s = \bar{x} - \mu_{\bar{x}} + \sigma a(t)$. By a change of variable,

$$\mathrm{d}t = \frac{-1}{\sigma} \phi \left(\frac{s - s_0}{\sigma} \right) \mathrm{d}s \quad \text{with} \;\; s_0 = \bar{x} - \mu_{\bar{x}} + \sigma c.$$

We have

$$\frac{\partial}{\partial \tau^2} m_{\tau^2} = \frac{-1}{2\tau^4 \rho} \int_{-\infty}^{\infty} \left[(s - s_0) + s_0 + \frac{2\tau^2}{\sigma_n^2 + \tau^2} (\bar{x} - \mu) \right]$$
$$\times \frac{1}{\sigma_0} \phi \left(\frac{s - s_0}{\sigma_0} \right) \mathrm{d}s,$$

where $\sigma_0 = \sqrt{\sigma^2 + \rho^{-1}}$. Consequently, if $\mu - \bar{x} \geq \mu_*(\tau_1^2)$ then for any $\tau^2 > \tau_1^2$

$$\frac{\partial}{\partial \tau^2} m_{\tau^2} = \frac{-1}{2\tau^4 \rho} \left[\left(\frac{\sigma_n^2 + 2\tau^2}{\sigma_n^2 + \tau^2} \right) (\bar{x} - \mu) + \sigma c \right] \geq 0.$$

This completes the proof. □

For the case when μ is fixed and varying τ^2, if $w(t)$ is increasingly convex then the ordering of its Bayes estimators immediately follows. Unfortunately, it is not. Yet we note that $w(t)$ is increasing and "nearly" convex. It is readily seen that $w(t)$ is increasing because

$$w(t) = P_t \left(\sum_{i=1}^{m} Y_i \geq m\beta \right) = P[V \leq t]$$

where $V \sim \text{Beta}(m\beta, m(1-\beta)+1)$ and

$$w'(t) = \frac{1}{B(m\beta, m(1-\beta)+1)} t^{m\beta - 1} (1-t)^{m(1-\beta)} > 0.$$

We shall explain what we mean by "near convexity". Define

$$w_v(t) = w_v(t | a_0, b_0, t_0) = a_0 + b_0 (t - t_0)_+ \quad \text{where } (y)_+ = \max(0, y).$$

This function is increasingly convex for any $b_0 > 0, t_0, a_0 \in \mathcal{R}$. Therefore

Theorem 2.3 *For $\mu - \bar{x} \ \ge \ \mu_*$, if $\pi \in \Gamma_\mu$ then*

$$E_{\lambda(t|\mu,\tau^2;\bar{x})} w_v(t) \tag{14}$$

is increasing in τ^2.

Proof Since $w_v(t)$ is increasingly convex by our construction, the inequalities readily follow by Lemma 2.2 and Theorem 2.2. □

Furthermore, the numerical calculation indicates that $E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t)$ can be approximated quite well by $E_{\lambda(t|\mu,\tau^2;\bar{x})} w_v(t)$ for some chosen a_0, b_0, t_0 over large range of τ^2. Details are deferred to the coming section. This suggests for $\mu - \bar{x} \ge \mu_*$, $E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t)$ is increasing in τ^2.

In addition, if more information about the hyperparameter is available, the bounds can be further sharpened. Define

$$\Gamma_{\tau^2}(\mu_0, \mu_1) = \{\pi : \pi(\theta|\mu, \tau^2), \mu \in (\mu_0, \mu_1), \mu_0 < \mu_1\},$$
$$\Gamma_\mu(\tau_0^2, \tau_1^2) = \{\pi : \pi(\theta|\mu, \tau^2), \tau^2 \in (\tau_0^2, \tau_1^2), \tau_0^2 < \tau_1^2\}.$$

We have

Corollary 2.1 *For any $\tau^2 \ > \ 0$ and $\Gamma \ = \ \Gamma_{\tau^2}(\mu_0, \mu_1)$ then*

$$\inf_{\pi\in\Gamma} E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t) = E_{\lambda(t|\mu_0,\tau^2;\bar{x})} w(t)$$
$$\textit{and} \ \sup_{\pi\in\Gamma} E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t) = E_{\lambda(t|\mu_1,\tau^2;\bar{x})} w(t).$$

Corollary 2.2 *For any $\mu - \bar{x} \ge \mu_*$ and $\Gamma = \Gamma_\mu(\tau_0^2, \tau_1^2)$ then*

$$\inf_{\pi\in\Gamma} E_{\lambda(t|\mu,\tau^2;\bar{x})} w_v(t) = E_{\lambda(t|\mu,\tau_0^2;\bar{x})} w_v(t)$$
$$\textit{and} \ \sup_{\pi\in\Gamma} E_{\lambda(t|\mu,\tau^2;\bar{x})} w_v(t) = E_{\lambda(t|\mu,\tau_1^2;\bar{x})} w_v(t).$$

We have so far dealt with finite m, the limiting cases for $m \to \infty$ are given below

Proposition 2.1 *If π is a normal prior and let $w_\infty(t) = \lim_{m\to\infty} w(t)$, then*

$$\lim_{\tau^2\to\infty} E_{\lambda(t|\mu,\tau^2;\bar{x})} w_\infty(t) = 1 - \alpha, \tag{15}$$

$$\lim_{\mu\to 0} E_{\lambda(t|\mu,\tau^2;\bar{x})} w_\infty(t) = \Phi\left[\rho^{1/2}\left(\left(\frac{\sigma_n^2}{\sigma_n^2+\tau^2}\right)\bar{x} + \sigma_n z_\alpha\right)\right] \tag{16}$$

Proof It follows directly from the Central Limit Theorem that

$$w_\infty(t) = \lim_{m\to\infty} \Phi\left(\frac{\sqrt{m}(t-\beta)}{\sqrt{t(1-t)}}\right) = 1_{[t>\beta]}.$$

Note

$$\lim_{\tau^2\to\infty} \lambda(t|\mu, \tau^2; \bar{x}) = 1 - \Phi\left[\sqrt{n}\left(c - \Phi^{-1}(t)\right)\right]$$
$$\lim_{\mu\to 0} \lambda(t|\mu, \tau^2; \bar{x}) = 1 - \Phi\left[\rho^{1/2}\left(\left(\frac{\sigma_n^2}{\sigma_n^2+\tau^2}\right)\bar{x} + \sigma_n\left(c - \Phi^{-1}(t)\right)\right)\right].$$

Since

$$\lim_{\tau^2\to\infty} E_{\lambda(t|\mu,\tau^2;\bar{x})} w_\infty(t) = \lambda(t|\mu,\tau^2;\bar{x})\Big|_\beta^1 = (1-\alpha).$$

The proof for $\mu \to 0$ is similar. □

To summarize, with numerical calculation, Theorems 2.1 and 2.3 clarify the orders of Bayes confidence estimators with respect to hyperparameters. Proposition 2.1 characterizes the limiting cases. The ordering in posterior distributions is shown to be an important structure to derive the ordering and bounds in robust Bayesian analysis.

3 Numerical calculation

Now we provide numerical results of bounds of $E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t)$ for π varying in $\Gamma_{\tau^2}(\mu_0, \mu_1)$ and $\Gamma_\mu(\tau_0^2, \tau_1^2)$, respectively. By numerical integration, we calculate the results for some configurations of $n = 10, 100$ and $\alpha = 0.05, 0.1$; $\beta = 0.95, 0.9$ and $\bar{x} = 0$. Recall also $c = z_{1-\beta} + 1/\sqrt{n} z_\alpha$. The computation is done using R on an AMD Athlon 1200 PC. Note that the sign of $\mu - \bar{x}$ is an important factor (recalling Theorem 2.3 assumes $\mu - \bar{x} \geq \mu_*$). Figures 1, 2, 3, and 4 are the plots of the Bayes estimators of various μs: the solid line, dashed line and dotted line correspond to $\tau^2 = 1, 10, 100$. The horizontal lines are $y = 1-\alpha$, the confidence coefficients. For Figures 5 and 6, the solid line and the dashed line denote the Bayes estimate $E_{\lambda(t|\mu,\tau_0^2;\bar{x})} w(t)$ and its approximation $E_{\lambda(t|\mu,\tau_0^2;\bar{x})} w_v(t)$, respectively. The results and figures under other configurations are similar and thus omitted.

Overall, the numerical computation agrees with our theoretical anticipation. Figures 1, 2, 3, 4 and 5 are graphical displays for Theorems 2.1, 2.2 and 2.3. Moreover, note that the horizontal $y = 1 - \alpha$ lies above other lines means that the confidence coefficient overstates the post-data performance of the tolerance intervals. This undesirable phenomenon tends to occur for large n (in our case $n = 100$). It is also interesting to note that when m is small, say 10, the tolerance intervals tend to perform better than the confidence coefficient $1-\alpha$ suggested. In Fig. 5, $E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t)$ is also increasing in τ^2 and smaller than the confidence coefficient. That is, the confidence coefficient is too liberal. Unfortunately, we are not able to give a theoretical proof for this property. For Figs. 5 and 6, $E_{\lambda(t|\mu,\tau^2;\bar{x})} w_v(t)$ provide good approximation to $E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t)$ with suitable tuning. For Fig. 6 (relating to $\mu - \bar{x} \leq \mu_*$), the monotonicity of $E_{\lambda(t|\mu,\tau^2;\bar{x})} w(t)$ is suggestive and interesting. It might be the case that $\lambda(t|\mu,\tau^2;\bar{x})$ is "nearly" decreasing in τ^2 for any fixed t, unfortunately, we do not have a formal proof yet.

We conclude this section with a remark on the choice of priors. Our posterior distribution $\lambda(t|\mu,\tau;\bar{x})$ has a density function of the form

$$f(t|a,b) = \frac{\frac{1}{b}\phi\left(\frac{1}{b}[\Phi^{-1}(t) - a]\right)}{\phi\left(\Phi^{-1}(t)\right)} \tag{17}$$

where $a = \frac{1}{\sigma}(\bar{x} - \mu_{\bar{x}}) + c$ and $b = (\sigma^2\rho)^{-1/2}$. It has been noted by Pearson (1939) and Koziol and Tuckwell (1999) that Eq. (17) can be well approximated by

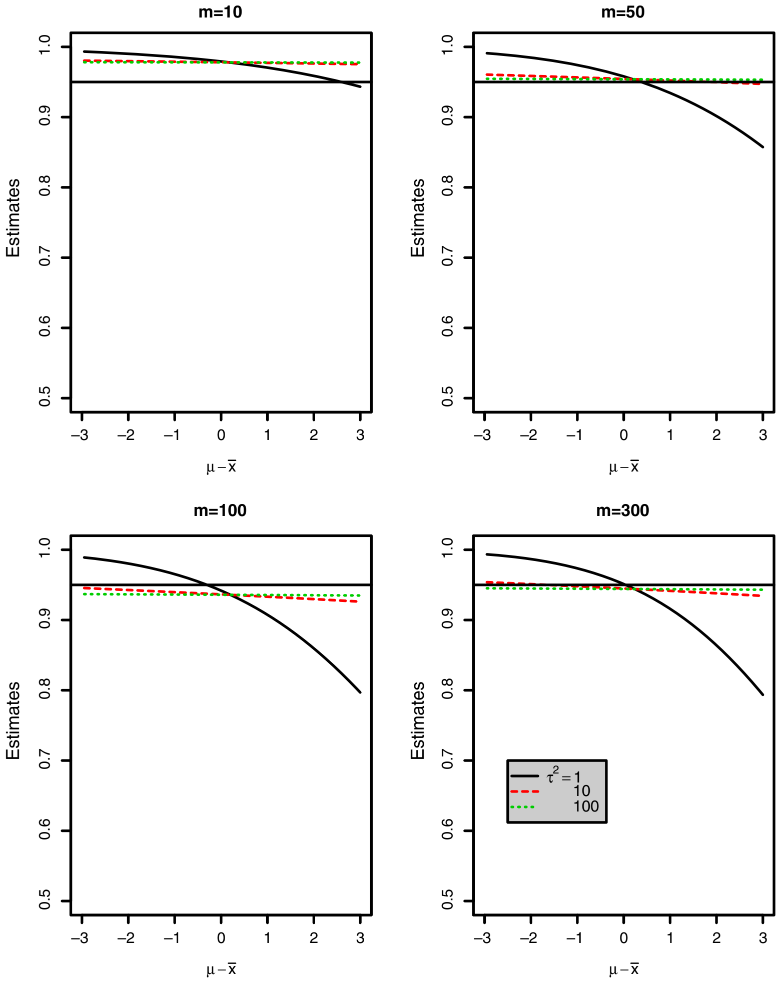

Fig. 1 Bayes confidence estimators: $(n, \alpha, \beta) = (10, 0.05, 0.95)$

a beta distribution. However, no rigorous proof has been given so far nor detailed study will be pursued here. Rather, we derive a moment-matching beta density of $f(t|a, b)$ and examine its property. Specifically, Let m_1, m_2 be the first two moments of density function $\lambda(t|\mu, \tau^2; \bar{x})$. By change of variables,

$$m_1 = \int_0^1 t \mathrm{d}\lambda(t|\mu, \tau; \bar{x}) = \int_0^1 \Phi\left[b^{-1}(a - \Phi^{-1}(t))\right]\mathrm{d}t$$

and

$$m_2 = \int_0^1 t^2 \mathrm{d}\lambda(t|\mu, \tau; \bar{x}) = 2\int_0^1 t\Phi\left[b^{-1}(a - \Phi^{-1}(t))\right]\mathrm{d}t.$$

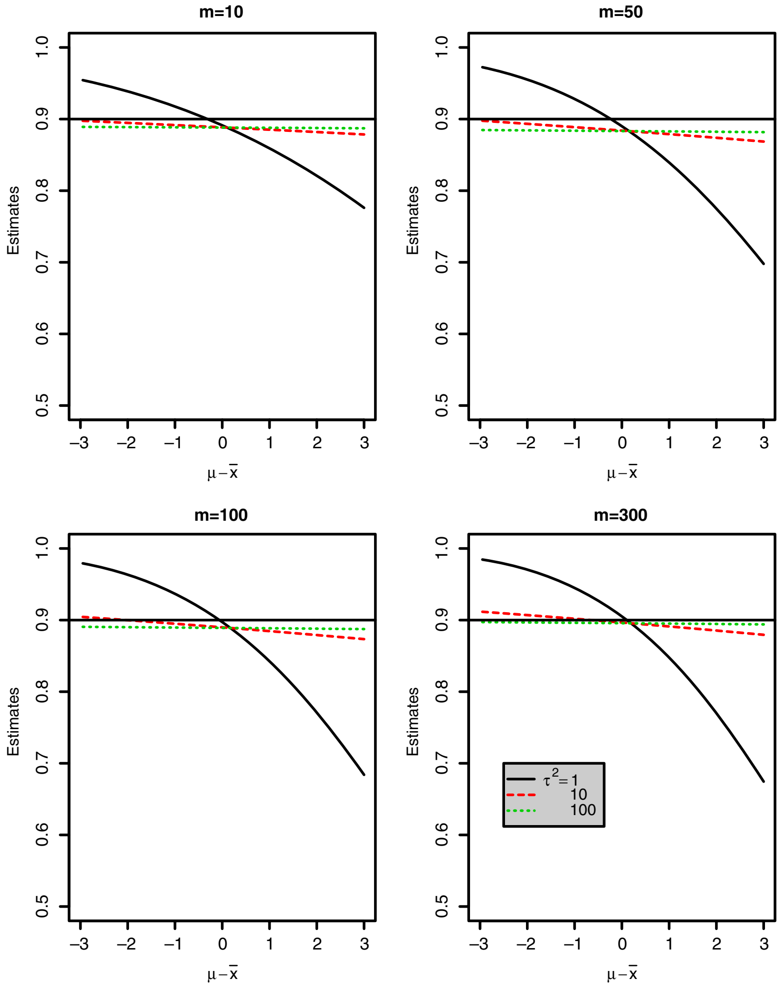

Fig. 2 Bayes confidence estimators: $(n, \alpha, \beta) = (10, 0.1, 0.9)$

Both m_1, m_2 can be easily calculated via numerical integration. A (first two) moment-matching beta density is a Beta$(cp_0, c(1-p_0))$ where

$$p_0 = m_1 \quad \text{and} \quad c = \frac{m_1 - m_2}{m_2 - m_1^2}.$$

We found very good moment-matching beta approximation to Eq. (17). For configurations examined, the figures are similar to those shown in Koziol and Tuckwell (1999) and omitted. In contrast to modeling through beta prior, we derive $\frac{\partial}{\partial t}\lambda(t|\mu, \tau; \bar{x})$ by taking advantage of the underlying normal–normal structure. This approach enables more detailed theoretical analysis. In our case, it clarifies orders of the Bayes predictive probabilities with respect to hyperparameters. This would be difficult if beta priors are assumed instead.

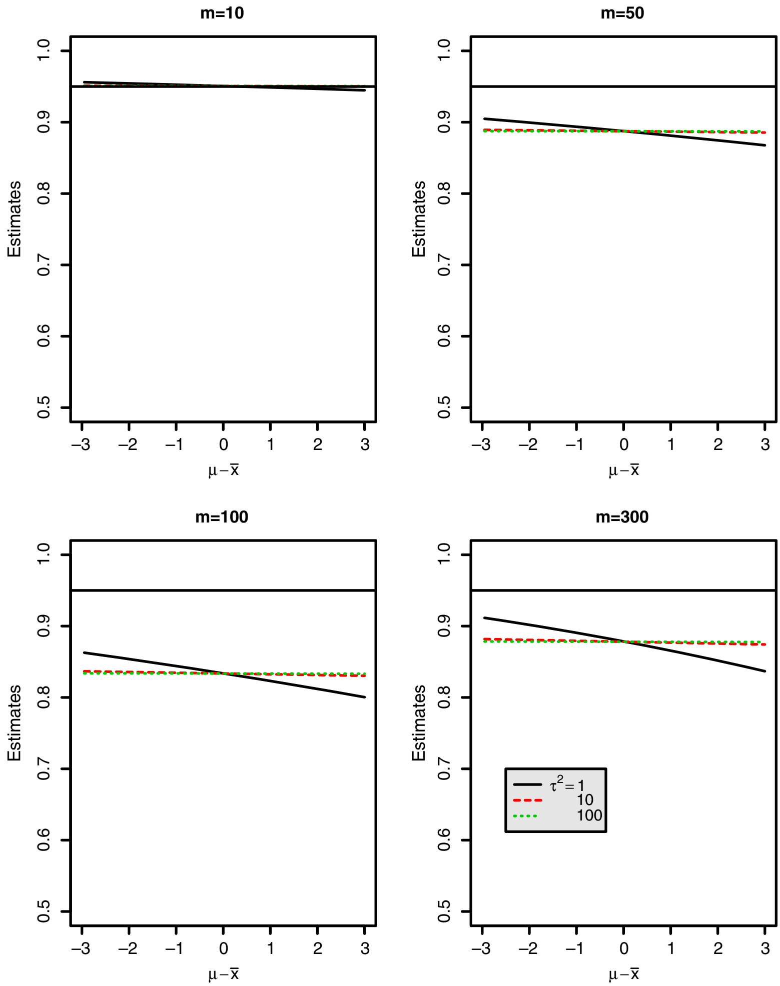

Fig. 3 Bayes confidence estimators: $(n, \alpha, \beta) = (100, 0.05, 0.95)$

4 Discussion and conclusion

We study the post-data performance of one-sided normal tolerance intervals via an estimated confidence approach. The traditional confidence coefficient $(1-\alpha)$ tends to be more extreme than Bayes estimators: it is too conservative when $\mu - \bar{x} > \mu_*$ and too liberal when $\mu - \bar{x} \leq \mu_*$. The discrepancy between the confidence coefficient and Bayes estimators is more marked as the sample size n increases. The results are based on analytic derivation along with numerical approximation.

From the practical viewpoint, the differences between the confidence coefficient and Bayes estimators are mild when the sample size is small. Therefore, the confidence coefficient $1-\alpha$ is an acceptable assessment for post-data performance of a one-sided β content/$(1-\alpha)$ confidence tolerance interval for small n. However,

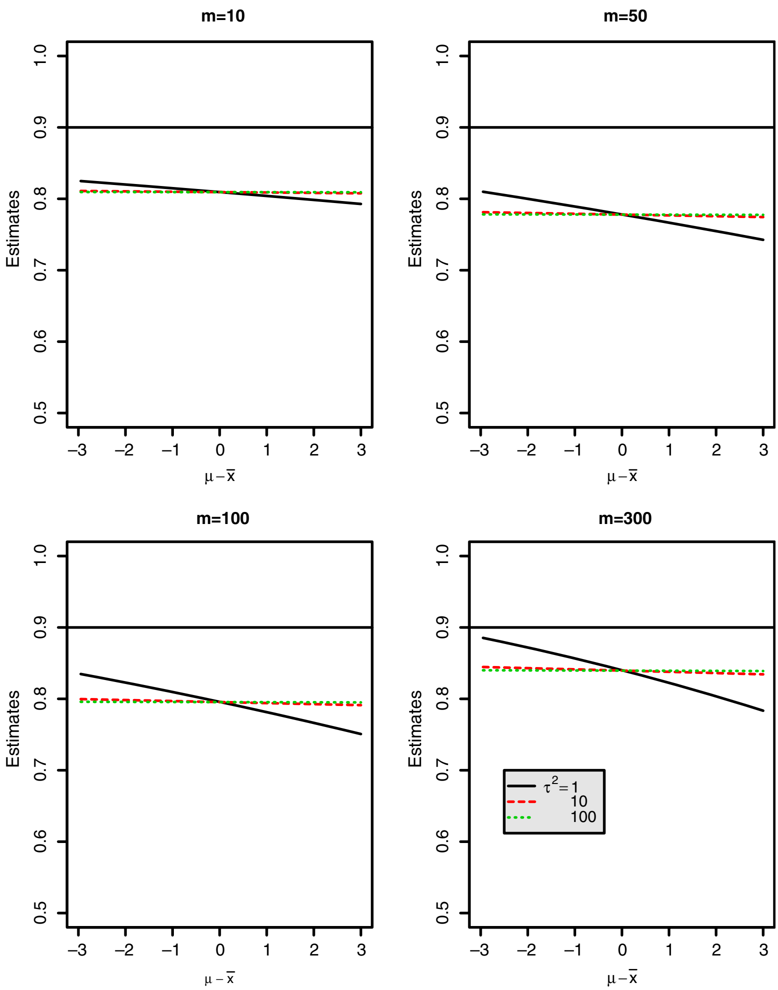

Fig. 4 Bayes confidence estimators: $(n, \alpha, \beta) = (100, 0.1, 0.9)$

for large n, moderate m and $\mu - \bar{x} \le \mu_*$, confidence coefficient is much larger than other estimators. That is, the confidence coefficient overstates the post-data correct rate of the tolerance intervals in these scenarios.

On the theoretical front, the discrepancy between confidence coefficient and the Bayes estimators can be understood as another example of Frequentist–Bayesian conflict, see Casella and Berger (1987), Tsao (2002) and Oh and DasGupta (1999) for more discussion. Similar phenomenon occurs in estimated confidence problems for confidence intervals; see Hwang and Brown (1991), Tsao and Hwang (1999) for example.

Our result also hints a way to choose/construct the prior or mixing distribution in the de Finetti's representation theorem. In its simplest form, the representation theorem characterizes the joint pdf of infinitely exchangeable 0–1 random variables as in Eq. (9). Unfortunately, it gives no guidance on how the prior should be

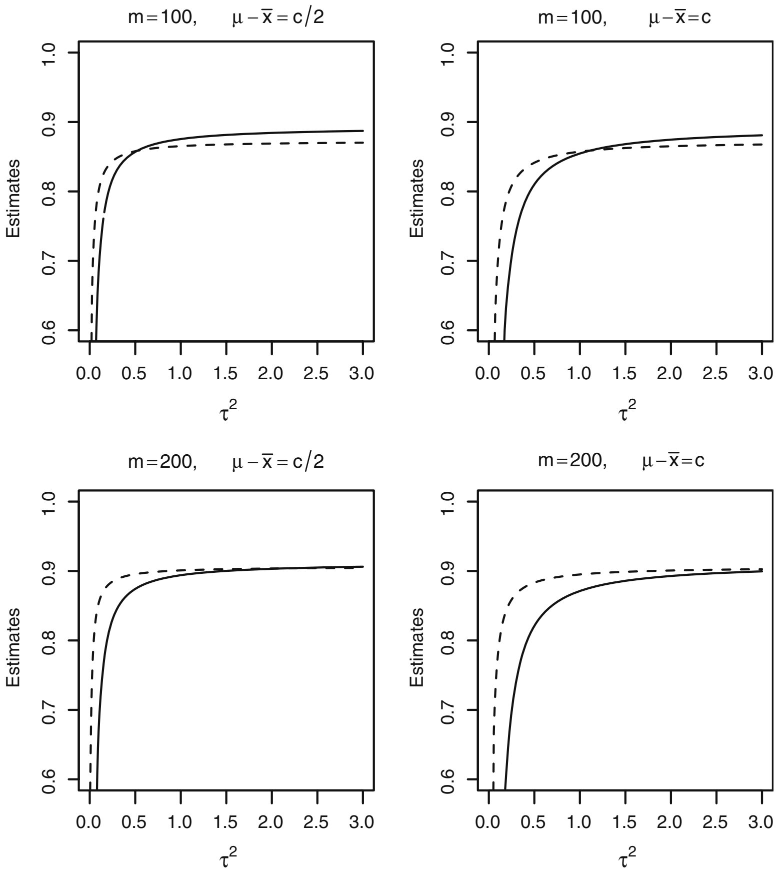

Fig. 5 Bayes confidence estimators: $\mu - \bar{x} > 0$, $\tau^2 \in (0, 3)$

chosen. Usually, the beta prior has been employed widely as the "natural" priors. In this context, this practice is justifiable since the derived $\lambda(t|\mu, \tau^2; \bar{x})$ can be well approximated by a beta distribution. Nonetheless, $\lambda(t|\mu, \tau^2; \bar{x})$ has better analytical tractability.

A final technical note Theorem 2.1 and Theorem 2.3 characterize the orders of Bayes predictive probabilities. They greatly simplify the task of numerical computation of bounds of Bayes estimators. In contrast, many earlier results based on the monotone likelihood ratio property often call for detailed analysis on the integrand. Utilizing ordering in posterior distributions directs the focus on the structures and often leads to clearer pictures, see, for example, Tsay and Tsao (2003) and Tsao and Tseng (2004). The interested readers are referred to Shaked and Shanthikumar (1994) and Szekli (1995) for discussion on stochastic orderings.

This study deals with only one-sided tolerance interval with known variance. Further research in unknown variance and two-sided tolerance intervals are of importance yet demands more involved calculations; for example, the analytic computation of the so-called k factor which is usually implicitly defined and calls for nontrivial approximation. We leave them for future investigation.

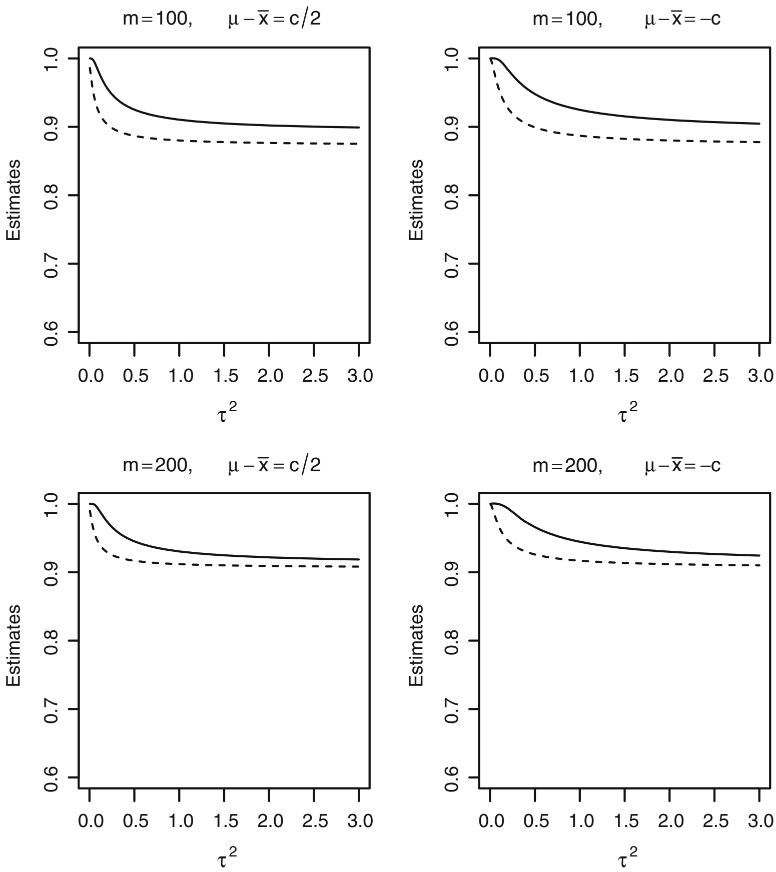

Fig. 6 Bayes confidence estimators: $\mu - \bar{x} < 0$, $\tau^2 \in (0, 3)$

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