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NON-STANDARD ASYMPTOTICS IN AN INHOMOGENEOUS GAMMA PROCESS

NIBEDITA BANDYOPADHYAY[1,2] AND ANANDA SEN[3]

[1] *Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, U.S.A.*
[2] *GlaxoSmithKline, One Franklin Plaza, FP 1375, Philadelphia, PA 19101, U.S.A.*
[3] *Center for Statistical Consultation and Research, University of Michigan, Ann Arbor, MI 48109, U.S.A.*

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Abstract. Nonhomogeneous Poisson process (NHPP) is a commonly used stochastic model that is utilized to describe the pattern of repeated occurrence of certain events or conditions. An *inhomogeneous gamma process* evolves as a generalization to NHPP, where the observed failure epochs correspond to every successive κ-th event of the underlying Poisson process, κ being an unknown parameter to be estimated from the data. This article focuses on a special class of inhomogeneous gamma process, called *modulated power law process* (MPLP) that assumes the Weibull form of the intensity function. The traditional power law process is a popular stochastic formulation of certain empirical relationships between the time to failure and the cumulative number of failures, often observed in industrial experiments. The MPLP retains this underlying physical basis and provides a more flexible modeling environment potentially leading to a better fit to the failure data at hand. In this paper, we investigate inference issues related to MPLP. The maximum likelihood estimators (MLE's) of the model parameters are not in closed form and enjoy the curious property that they are asymptotically normal with a singular variance-covariance matrix. Consequently, the derivation of the large-sample results requires non-standard modifications of the usual arguments. We also propose a set of simple closed-form estimators that are asymptotically equivalent to the MLE's. Extensive simulation results are carried out to supplement the theoretical findings. Finally, we implement our inference results to a failure dataset arising from a repairable system.

Key words and phrases: Asymptotics, maximum likelihood estimation, modulated power law process, power law process, recurrent event.

1. Introduction

In many scientific investigations, the event of primary interest is recurrent in the sense that it can occur repeatedly over time for each individual or system under consideration. A common example is the repeated malfunctioning of an automobile that is put back in service once the component causing the malfunction is replaced or repaired. Statistical analysis of data arising from a recurrent event has received considerable attention in the past three decades in reliability and software engineering, biomedical, actuarial and economic applications. In the reliability and software engineering applications, the event of primary interest is the *failure* of a system. As many complex and expensive systems encountered in practice are meant to be repaired rather than replaced on failure, recurrent events arise as a natural consequence. For example, during a software development process, failures (or bugs) are detected, corrected and the testing continues with the modified code. The pattern of times between successive failures is of fundamental importance in assessing whether the redesign efforts contribute to the improvement of the system. In biomedical applications, the *failures* are the occurrence of a recurrent event (e.g. carcinogenic growths in different times and locations, multiple attacks of cardiac arrest, repeated failures of a medical device inserted in patients suffering from a certain disease) in individuals.

In this paper, we focus on applications in which events are observed for a single system. The premise is natural in engineering applications where only a few prototypes of an expensive system are available for testing. Suppose the successive failures of the system occur at times $0 < T_1 < T_2 < \cdots < T_n$ and the system is observed until n events occurred (failure truncation scheme). Nonhomogeneous Poisson process (NHPP) and renewal process are often used to model recurrent events, and methods based on them are well established (e.g. Ascher and Feingold (1984), Cox and Isham (1980), Cox and Lewis (1966), Crow (1982)). NHPP assumes that after a repair the system is in the same condition as it was just before the failure. This *minimal repair* model is justified in instances where the *repair time* can be deemed negligible in comparison to the time between failures. An NHPP is characterized by an intensity function $\lambda(t)$ that represents the rate of occurrence of events. Denoting by $N(t)$, the underlying counting process enumerating the number of failures up to time t, mathematically,

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P[N(t+\Delta t) - N(t) \geqslant 1]}{\Delta t}$$

that can be interpreted as the probability of at least one failure per unit time in an infinitesimal time-interval $(t, t+\Delta t]$. If ties (in failure times) occur with probability zero, as is the case with Poisson process, $\Lambda(t) = \int_0^t \lambda(s)ds$ matches $E[N(t)]$, the mean number of failures until time t. Renewal process, on the other hand, implies that the times between successive events are independently and identically distributed i.e. after repair the system is in a *like-new* condition. As is well known, a homogeneous Poisson process (HPP), for which $\lambda(t)$ is constant, is also a renewal process in which the times between successive events are exponentially distributed.

In modeling recurrent event data, the major thrust came from certain empirical findings of Duane (1964). From the examination of time-between-failure data of several industrial systems, he observed that the empirical cumulative rate of failure typically produced a linear relationship with the cumulative operating time when plotted on a log-log scale. This phenomenon, subsequently referred to as the *Duane learning curve property*, was given a concrete stochastic basis by Crow (1974), who assumed that the failure process can be modeled by an NHPP with an intensity function of the Weibull form

$$\lambda(t) = (\beta/\theta)(t/\theta)^{\beta-1}, \qquad \theta, \beta > 0;\ t > 0 \tag{1.1}$$

where θ indicates a scale effect and β quantifies the growth or decay in the rate of failures. Due to the special structure of the intensity function, this model is often referred to as the *power law process* (PLP) in the statistical literature. Elegant mathematical properties and easily implementable diagnostic procedures have attributed to the immense popularity of the PLP as a model for recurrent event data (cf. Bain and Engelhardt (1980), Rigdon and Basu (1989)). Indeed, the defense industry has adopted the PLP model as industry standard and refers to it as the AMSAA (Army Material Systems Analysis Activity) model (MIL-HDBK-781 (1997)).

In many situations, however, neither the NHPP nor renewal process seems to portray a realistic depiction of the failure process. The underlying assumption for an NHPP that a repair returns the system to the condition as it was just before the failure, is often overly pessimistic. On the other hand, the renewal process, for which repair always brings the system to a like-new condition, is overly optimistic. In many practical situations, after repair, a system is in better condition than it was just before the failure, but may still be not in a *like-new* condition. In this article, we deal with an extension of NHPP called an *inhomogeneous gamma process* that addresses the above concern. The focus is on a specific parametric version called *modulated power law process* (MPLP) that extends the traditional PLP while retaining the physical meaning underlying it.

1.1 *Modulated power law process*

The *inhomogeneous gamma process* (IGP), introduced by Berman (1981), can be described as follows. Consider an NHPP with intensity function $\lambda(t)$. Imagine that every κ-th event of the process is observed. If the events are thought of as shocks, then the assumption entails a failure at the occurrence of every κ-th shock. If, for example, $\kappa = 4$, then every 4th shock would cause a failure. After repair a system would thus be better than it was just before the failure, since in order to cause another failure the required number of shocks must accumulate to four again. However, after repair a system would not necessarily be as good as new. The larger κ is, the larger the improvement will be. Thus, the IGP incorporates a repair-effect in the model yet allowing for a *less-than-perfect* repair. Of course, if $\kappa = 1$, the model reduces to a traditional NHPP. If $T_1 < \cdots < T_n$ are the times of the first n events observed for an IGP, then their joint density is given by

$$f(t_1,\ldots,t_n) = \left[\prod_{i=1}^{n} \lambda(t_i)\{\Lambda(t_i) - \Lambda(t_{i-1})\}^{\kappa-1}\right] e^{-\Lambda(t_n)}/\{\Gamma(\kappa)\}^n; \tag{1.2}$$

where $\Lambda(t) = \int_0^t \lambda(x)dx$ and $t_0 = 0$. The function $\Lambda(t)$ can be interpreted as the expected number of shocks before time t. Expression (1.2) defines a valid density function even for non-integer, positive values of κ. In this case, however, extra care should be taken to explain the effect of κ, since the shock model interpretation does not carry over anymore.

An alternative approach to define the IGP is as follows. Suppose that the random variables $Y_i = \Lambda(T_i) - \Lambda(T_{i-1})$ for $i = 1,\ldots,n$ are independently and identically distributed as $Gamma(\kappa, 1)$. It then follows that (1.2) is the joint density of $T_1,\ldots,T_n$. Here and in the sequel, a $Gamma(\alpha_1, \alpha_2)$ refers to the gamma distribution with shape parameter α_1 and scale parameter α_2, conforming to the pdf

$$g(x) = \frac{1}{\alpha_2^{\alpha_1}\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-x/\alpha_2}, \qquad x > 0,\ \alpha_1, \alpha_2 > 0.$$

Some distributional properties of the IGP are imminent. For example, $\Lambda(T_j) = \sum_{i=1}^{j}[\Lambda(T_i) - \Lambda(T_{i-1})]$ follows a $Gamma(j\kappa, 1)$ for every $j = 1, \ldots, n$. Also, $U_i = \Lambda(T_i)/\Lambda(T_n)$, $i = 1, \ldots, n-1$, are each distributed as beta with parameters κi and $\kappa(n-i)$ independently of T_n. If κ is an integer, the sequence of U_i's can further be viewed as the lag-κ order statistics from a random sample of size $\kappa n - 1$ from $Uniform(0,1)$. If the intensity function is of the form

$$\lambda(t) = \rho \exp\{\nu_1 z_1(t) + \cdots + \nu_p z_p(t)\}, \tag{1.3}$$

where $z_1, z_2, \ldots, z_p$ are (possibly) time-dependent covariates, then IGP is called a *modulated gamma process*. This form of $\lambda(t)$ expresses a possible dependence of the occurrence of events on the vector $z(\cdot)$ of explanatory variables. The *modulated gamma process* is an extension to the *modulated Poisson process* introduced by Cox (1972).

Recall that the premise is that of observing repeated failures of a single system. In order to make inferences about the system, a fully parametric characterization of the underlying model is warranted. Under the IGP framework, we choose to work with power-law form of (1.1) that has received substantial attention in reliability engineering literature. The resulting model was termed *modulated power law process* (MPLP) by Lakey and Rigdon (1992) as the power-law formulation conforms to the modulated structure of (1.3) with $\rho = \beta/\theta^{\beta}$, $\nu_1 = \beta - 1$ and $z_1(t) = \log t$. In this model β is a measure of the system improvement or deterioration over the course of the system's life and κ is a measure of the improvement affected by a repair. In our formulation κ will be treated as a positive real number. Since $\Lambda(T_i) = (T_i/\theta)^{\beta}$ has a $Gamma(\kappa i, 1)$ distribution, $E\{\beta(\log T_i - \log\theta)\} \sim \log\kappa + \log i$ for moderately large values of i. Thus, like PLP, even for this model $\log(i/T_i)$ is approximately linear to $\log(T_i)$, retaining the physical connection to Duane's learning curve property. Three special cases of MPLP merit special mention. When $\kappa = 1$, there is a failure at each shock and MPLP reduces to PLP. When $\beta = 1$, the times between failures are independently and identically distributed gamma random variables, so the process becomes a *gamma renewal process*. Finally, when both $\kappa = 1$ and $\beta = 1$, the process reduces to HPP.

The article is organized as follows. In Section 2, we study the large-sample properties of the maximum likelihood estimators of the parameters θ, β and κ. The earlier studies related to the inference issues of the MPLP model (e.g. Black and Rigdon (1996)) did not report the asymptotic distribution of the maximum likelihood estimators which involve non-standard manipulation due to the fact that the Hessian matrix converges in probability to a singular matrix. In Section 3, we propose simple non-iterative estimators which are asymptotically equivalent to the maximum likelihood estimators. Statistical inference on the rate of occurrence of failure (ROCOF) of the underlying process is also carried out. In Section 4, we report the results of some simulation studies and fit the MPLP model to a real-data problem. Some concluding remarks are given in Section 5.

2. Maximum likelihood estimation

Let $0 < T_1 < \cdots < T_n$ be the times of occurrence of the first n events from an MPLP. Then the log-likelihood function is

$$\log L(\theta, \beta, \kappa) = -(T_n/\theta)^{\beta} + n\log\beta - n\log\Gamma(\kappa) - n\beta\kappa\log\theta$$

$$+ (\beta - 1)\sum_{i=1}^{n} \log T_i + (\kappa - 1)\sum_{i=1}^{n} \log(T_i^\beta - T_{i-1}^\beta).$$

Writing $\boldsymbol{\mu} = (\theta, \beta, \kappa)'$ the score vector $\boldsymbol{l}_n(\boldsymbol{\mu}) = (l_{1n}(\boldsymbol{\mu}), l_{2n}(\boldsymbol{\mu}), l_{3n}(\boldsymbol{\mu}))'$ is obtained as

$$(2.1) \quad \begin{cases} l_{1n}(\boldsymbol{\mu}) \equiv \dfrac{\partial \log L}{\partial \theta} = (\beta/\theta)(T_n/\theta)^\beta - (n\beta\kappa)/\theta, \\ l_{2n}(\boldsymbol{\mu}) \equiv \dfrac{\partial \log L}{\partial \beta} = -(T_n/\theta)^\beta \log(T_n/\theta) + n/\beta - n\kappa \log\theta + \displaystyle\sum_{i=1}^{n} \log T_i \\ \qquad + (\kappa - 1)\displaystyle\sum_{i=1}^{n} \dfrac{T_i^\beta \log T_i - T_{i-1}^\beta \log T_{i-1}}{T_i^\beta - T_{i-1}^\beta}, \\ l_{3n}(\boldsymbol{\mu}) \equiv \dfrac{\partial \log L}{\partial \kappa} = -n\psi(\kappa) - n\beta \log\theta + \displaystyle\sum_{i=1}^{n} \log(T_i^\beta - T_{i-1}^\beta), \end{cases}$$

where $\psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot)$ is the di-gamma function. The components of the second-derivative matrix $\boldsymbol{A}_n(\boldsymbol{\mu}) = (a_{ij}(\boldsymbol{\mu})) = -\partial^2 \log L/\partial\boldsymbol{\mu}\partial\boldsymbol{\mu}'$:

$$(2.2) \quad \begin{cases} a_{11}(\boldsymbol{\mu}) \equiv -\dfrac{\partial^2 \log L}{\partial \theta^2} = (\beta/\theta^2)\{(\beta+1)(T_n/\theta)^\beta - n\kappa\}, \\ a_{12}(\boldsymbol{\mu}) \equiv -\dfrac{\partial^2 \log L}{\partial\theta\partial\beta} = -(\beta/\theta)(T_n/\theta)^\beta \log(T_n/\theta) - (1/\theta)(T_n/\theta)^\beta + (n\kappa)/\theta, \\ a_{12}(\boldsymbol{\mu}) \equiv -\dfrac{\partial^2 \log L}{\partial\theta\partial\beta} = -(\beta/\theta)(T_n/\theta)^\beta \log(T_n/\theta) - (1/\theta)(T_n/\theta)^\beta + (n\kappa)/\theta, \\ a_{13}(\boldsymbol{\mu}) \equiv -\dfrac{\partial^2 \log L}{\partial\theta\partial\kappa} = (n\beta)/\theta, \\ a_{22}(\boldsymbol{\mu}) \equiv -\dfrac{\partial^2 \log L}{\partial \beta^2} = (T_n/\theta)^\beta\{\log(T_n/\theta)\}^2 + \dfrac{n}{\beta^2} \\ \qquad - (\kappa - 1)\displaystyle\sum_{i=2}^{n} \dfrac{T_i^\beta(\log T_i)^2 - T_{i-1}^\beta(\log T_{i-1})^2}{T_i^\beta - T_{i-1}^\beta} \\ \qquad + (\kappa - 1)\displaystyle\sum_{i=2}^{n} \dfrac{(T_i^\beta \log T_i - T_{i-1}^\beta \log T_{i-1})^2}{(T_i^\beta - T_{i-1}^\beta)^2}, \\ a_{23}(\boldsymbol{\mu}) \equiv -\dfrac{\partial^2 \log L}{\partial\beta\partial\kappa} = n\log\theta - \displaystyle\sum_{i=1}^{n} \dfrac{T_i^\beta \log T_i - T_{i-1}^\beta \log T_{i-1}}{T_i^\beta - T_{i-1}^\beta}, \\ a_{33}(\boldsymbol{\mu}) \equiv -\dfrac{\partial^2 \log L}{\partial \kappa^2} = n\psi'(\kappa). \end{cases}$$

Here $\psi'(\cdot)$ is the tri-gamma function. Throughout this article, $\widehat{\theta}$, $\widehat{\beta}$ and $\widehat{\kappa}$ will be taken to mean a consistent sequence of roots of the likelihood equations $\boldsymbol{l}_n(\boldsymbol{\mu}) = \boldsymbol{0}$, which we will call the maximum likelihood estimates (MLE's). Later, we shall establish the existence of such a sequence. An inspection of (2.1) reveals that $l_{1n}(\boldsymbol{\mu}) = 0$ translates into the relation

$$\widehat{\theta} = \frac{T_n}{(n\kappa)^{1/\beta}}, \tag{2.3}$$

upon substituting which, both $l_{2n}(\boldsymbol{\mu})$ and $l_{3n}(\boldsymbol{\mu})$ can be expressed as functions of β and κ only. $\widehat{\beta}$ and $\widehat{\kappa}$ are thus obtained by a standard two-variable Newton-Raphson algorithm, before applying (2.3) to calculate $\widehat{\theta}$.

2.1 *Large-sample inference of the MLE's*

In this section the asymptotic properties of the MLE's are presented. We shall make use of the standard symbols $o_p(\cdot)$ and $O_p(\cdot)$ for convergence and boundedness in probability. In our subsequent discussion all limits will be taken as $n \to \infty$ unless otherwise mentioned. From now on we denote the true parameter point in the interior of the parameter space as $\mu_0 = (\theta_0, \beta_0, \kappa_0)'$ and leave μ as the argument of the various functions. We define the random variables

$$\begin{cases} U_{1n} = n^{-1/2}\left\{\beta_0 \sum_{i=1}^{n} \log(T_n/T_i) - n\right\}, \\ U_{2n} = n^{-1/2}\{(T_n/\theta_0)^{\beta_0} - n\kappa_0\}, \\ U_{3n} = n^{-1/2}\sum_{i=1}^{n}[\log\{(T_i/\theta_0)^{\beta_0} - (T_{i-1}/\theta_0)^{\beta_0}\} - \psi(\kappa_0)]. \end{cases} \tag{2.4}$$

The following result plays a pivotal role in establishing the large-sample property for the MLE's.

LEMMA 2.1. *$\boldsymbol{U}_n = (U_{1n}, U_{2n}, U_{3n})'$ converges in distribution to a multivariate normal random variable with mean vector zero and covariance matrix $\boldsymbol{\Sigma}_0$ where*

$$\boldsymbol{\Sigma}_0 = \begin{bmatrix} \kappa_0^{-1} & 0 & 0 \\ 0 & \kappa_0 & 1 \\ 0 & 1 & \psi'(\kappa_0) \end{bmatrix}.$$

To prepare the groundwork for proving Lemma 2.1 we define the random variables X_i as follows

$$X_i = \frac{(T_i/\theta_0)^{\beta_0}}{i\kappa_0}, \qquad i = 1, \ldots, n; \qquad \overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i.$$

The next lemma provides some necessary results concerning the random variables X_i which will be used in proving Lemma 2.1. Proof of this lemma is sketched in the Appendix.

LEMMA 2.2. *(a)* $n^{1/2}(X_n - 1) \to N(0, \kappa_0^{-1})$ *in distribution as* $n \to \infty$.
(b) $n^{1/2}(X_n - \overline{X}_n) \to N(0, \kappa_0^{-1})$ *in distribution as* $n \to \infty$.
(c) U_{3n} *and* $n^{1/2}(X_n - \overline{X}_n)$ *are uncorrelated.*
(d) $n^{-1/2} \sum_{i=1}^{n} \log(X_n/X_i) = n^{1/2}(X_n - \overline{X}_n) + o_p(1)$.

PROOF OF LEMMA 2.1. Noting that the random variables $Y_i = (T_i/\theta_0)^{\beta_0} - (T_{i-1}/\theta_0)^{\beta_0}$, for $i = 1, \ldots, n$ are independently and identically distributed as $Gamma(\kappa_0, 1)$, U_{2n} and U_{3n} can be re-expressed as

$$U_{2n} = n^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - \kappa_0\right)$$

$$U_{3n} = n^{1/2}\left\{\frac{1}{n}\sum_{i=1}^{n} \log Y_i - \psi(\kappa_0)\right\}.$$

Then by an application of bivariate central limit theorem, $(U_{2n}, U_{3n})'$ are asymptotically normal with zero mean vector and covariance matrix

$$\begin{bmatrix} \kappa_0 & 1 \\ 1 & \psi'(\kappa_0) \end{bmatrix}. \tag{2.5}$$

Also we can express U_{1n} in terms of X_i as

$$U_{1n} = n^{-1/2}\sum_{i=1}^{n} \log(X_n/X_i) + n^{-1/2}(n \log n - \log n! - n). \tag{2.6}$$

Using Stirling's formula, the non-random term on the right of (2.6) is $o_p(1)$. That U_{1n} is independent of U_{2n}, follows from the properties of MPLP. Lemma 2.2 entails that U_{1n} converges in distribution to $N(0, \kappa_0^{-1})$, and is independent of U_{3n}. In conjunction with (2.5), the result then follows. □

We now turn to the asymptotics of the MLE's that is the main goal of this section. Define, the scaled second derivative matrix of log-likelihood obtained from $\boldsymbol{A}_n$ as

$$\boldsymbol{C}_n(\boldsymbol{\mu}) = (c_{ij}(\boldsymbol{\mu})) = n^{-1}\begin{bmatrix} a_{11}(\boldsymbol{\mu}) & a_{12}(\boldsymbol{\mu})/\log n & a_{13}(\boldsymbol{\mu}) \\ a_{12}(\boldsymbol{\mu})/\log n & a_{22}(\boldsymbol{\mu})/(\log n)^2 & a_{23}(\boldsymbol{\mu})/\log n \\ a_{13}(\boldsymbol{\mu}) & a_{23}(\boldsymbol{\mu})/\log n & a_{33}(\boldsymbol{\mu}) \end{bmatrix}. \tag{2.7}$$

Evidently, the nonuniform scalings used here are quite unlike those involved in the treatment of ML asymptotics in a standard situation. The scalings have important bearings on the convergence results as will be transparent later in this section.

Denoting $\boldsymbol{C}_n(\boldsymbol{\mu}_0) = \boldsymbol{C}_n = (c_{ij})_{i,j=1,2,3}$ and referring to (2.2), (2.7) and the representation (2.4), we obtain

$$
(2.8)\quad
\begin{cases}
c_{11} = \dfrac{\beta_0^2\kappa_0}{\theta_0^2} + \dfrac{\beta_0(\beta_0+1)}{\theta_0^2}\dfrac{U_{2n}}{n^{1/2}}, \\
c_{12} = -\dfrac{\kappa_0}{\theta_0} - \dfrac{U_{2n}}{\theta_0 n^{1/2}} - \dfrac{\kappa_0}{\theta_0 \log n}\left\{\log\kappa_0 + \log\left(1+\dfrac{U_{2n}}{\kappa_0 n^{1/2}}\right)\right\} \\
\qquad - \dfrac{U_{2n}}{\theta_0 n^{1/2}\log n}\left\{\log\kappa_0 + \log\left(1+\dfrac{U_{2n}}{\kappa_0 n^{1/2}}\right)\right\}, \\
c_{13} = \dfrac{\beta_0}{\theta_0}, \\
c_{22} = \dfrac{\kappa_0}{\beta_0^2} + \dfrac{U_{2n}}{\beta_0^2 n^{1/2}} + \dfrac{2}{\beta_0^2\log n}\left\{\log\kappa_0 + \log\left(1+\dfrac{U_{2n}}{\kappa_0 n^{1/2}}\right)\right\}\left(\dfrac{U_{2n}}{n^{1/2}} + \kappa_0\right) \\
\qquad + \dfrac{1}{(\beta_0\log n)^2}\left(\dfrac{U_{2n}}{n^{1/2}} + \kappa_0\right)\left\{\log\kappa_0 + \log\left(1+\dfrac{U_{2n}}{\kappa_0 n^{1/2}}\right)\right\}^2 \\
\qquad - \dfrac{\kappa_0 - 1}{n(\beta_0\log n)^2} \\
\qquad + \dfrac{1}{(\beta_0\log n)^2}\left[\kappa_0 + \dfrac{\kappa_0-1}{n}\sum_{i=2}^{n}\left\{\dfrac{T_i^{\beta_0}T_{i-1}^{\beta_0}}{(T_i^{\beta_0}-T_{i-1}^{\beta_0})^2}\left(\log\dfrac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}}\right)^2 - 1\right\}\right], \\
c_{23} = -\dfrac{1}{\beta_0} - \dfrac{1}{\beta_0\log n}\left\{\log\kappa_0 + \log\left(1+\dfrac{U_{2n}}{\kappa_0 n^{1/2}}\right)\right\} \\
\qquad + \dfrac{1}{\beta_0 n^{1/2}\log n}\left\{U_{1n} + \dfrac{1}{n^{1/2}}\right. \\
\qquad\qquad \left. - \dfrac{1}{n^{1/2}}\sum_{i=2}^{n}\left(\dfrac{T_{i-1}^{\beta_0}}{T_i^{\beta_0}-T_{i-1}^{\beta_0}}\log\dfrac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}} - 1\right)\right\}, \\
c_{33} = \psi'(\kappa_0).
\end{cases}
$$

In deriving the probability limit of $\boldsymbol{C}_n$ as well as other derivations throughout the article, the following convergence result involving T_i's is used. To avoid any significant digression from the main goal, we relegate its proof to the Appendix.

LEMMA 2.3. (a) $n^{-1/2}\sum_{i=2}^{n}\left(\frac{T_{i-1}^{\beta_0}}{T_i^{\beta_0}-T_{i-1}^{\beta_0}}\log\frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}} - 1\right) = o_p(1)$.
(b) $n^{-1}\sum_{i=2}^{n}\left\{\frac{T_i^{\beta_0}T_{i-1}^{\beta_0}}{(T_i^{\beta_0}-T_{i-1}^{\beta_0})^2}\left(\log\frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}}\right)^2 - 1\right\} = o_p(1)$.

Note from the expression in (2.8) that $\boldsymbol{C}_n$ involves $\boldsymbol{U}_n = (U_{1n}, U_{2n}, U_{3n})'$ as well as the quantities appearing in Lemma 2.3. Using Lemma 2.3 and the fact that $\boldsymbol{U}_n = O_p(1)$, it follows that $\boldsymbol{C}_n$ converges in probability to $\boldsymbol{\Sigma}$, where

$$
\boldsymbol{\Sigma} = \begin{bmatrix} (\beta_0^2\kappa_0)/\theta_0^2 & -\kappa_0/\theta_0 & \beta_0/\theta_0 \\ -\kappa_0/\theta_0 & \kappa_0/\beta_0^2 & -1/\beta_0 \\ \beta_0/\theta_0 & -1/\beta_0 & \psi'(\kappa_0) \end{bmatrix}.
$$

Note that $\boldsymbol{\Sigma}$ is a singular matrix with rank 2. This clearly makes the premise quite distinct from the classical ML asymptotics, and warrants nonstandard modification of the Taylor-series argument that is used in practice. Some such modifications have been adopted by Bhattacharyya and Ghosh (1991) and Sen and Fries (1997) in the context of some discrete reliability growth models. These methods either do not directly apply in our context or involve unnecessarily complex manipulations. Instead, we proceed by reducing the problem to two-dimensions and appealing to the distributional properties of the MPLP.

Let us define the centered and scaled MLE's as $W_{1n} = n^{1/2}(\log n)^{-1}(\hat{\theta}_n - \theta_0)$, $W_{2n} = n^{1/2}(\hat{\beta}_n - \beta_0)$, $W_{3n} = n^{1/2}(\hat{\kappa}_n - \kappa_0)$. We partition the parameters as $\boldsymbol{\mu}' = (\theta, \boldsymbol{\alpha}') = (\theta, \beta, \kappa)$ and $\boldsymbol{W}'_n = (W_{1n}, \boldsymbol{W}^{*'}_n) = (W_{1n}, W_{2n}, W_{3n})$. Substituting $\widehat{\theta}$ from (2.3) into (2.1), the score functions $l_{2n}(\boldsymbol{\mu})$ and $l_{3n}(\boldsymbol{\mu})$ reduces to $\boldsymbol{l}^*_n(\boldsymbol{\alpha}) = (l^*_{1n}(\boldsymbol{\alpha}), l^*_{2n}(\boldsymbol{\alpha}))'$, where

$$(2.9) \quad \begin{cases} l^*_{1n}(\boldsymbol{\alpha}) = -n\kappa \log T_n + n/\beta + \sum_{i=1}^{n} \log T_i \\ \qquad\qquad + (\kappa - 1) \sum_{i=1}^{n} \dfrac{T_i^\beta \log T_i - T_{i-1}^\beta \log T_{i-1}}{T_i^\beta - T_{i-1}^\beta}, \\ l^*_{2n}(\boldsymbol{\alpha}) = -n\psi(\kappa) + n \log \kappa - n\beta \log T_n + n \log n + \sum_{i=1}^{n} \log(T_i^\beta - T_{i-1}^\beta). \end{cases}$$

The problem now is two dimensional. Differentiating the functions $\boldsymbol{l}^*_n(\boldsymbol{\alpha})$ given in (2.9) we obtain the elements of the matrix $\boldsymbol{A}^*_n(\boldsymbol{\alpha}) = (a^*_{ij}(\boldsymbol{\alpha})) = -\partial \boldsymbol{l}^*_n(\boldsymbol{\alpha})/\partial \boldsymbol{\alpha}$:

$$(2.10) \quad \begin{cases} a^*_{11}(\boldsymbol{\alpha}) = (n/\beta^2) - (\kappa - 1) \sum_{i=2}^{n} \dfrac{T_i^\beta (\log T_i)^2 - T_{i-1}^\beta (\log T_{i-1})^2}{T_i^\beta - T_{i-1}^\beta} \\ \qquad\qquad + (\kappa - 1) \sum_{i=2}^{n} \left(\dfrac{T_i^\beta \log T_i - T_{i-1}^\beta \log T_{i-1}}{T_i^\beta - T_{i-1}^\beta} \right)^2, \\ a^*_{12}(\boldsymbol{\alpha}) = a^*_{21}(\boldsymbol{\alpha}) = n \log T_n - \sum_{i=1}^{n} \dfrac{T_i^\beta \log T_i - T_{i-1}^\beta \log T_{i-1}}{T_i^\beta - T_{i-1}^\beta}, \\ a^*_{22}(\boldsymbol{\alpha}) = n\psi'(\kappa) - (n/\kappa). \end{cases}$$

Assume for the time being that $\boldsymbol{l}^*_n(\boldsymbol{\alpha}) = 0$ has a solution $\widehat{\boldsymbol{\alpha}}_n = (\widehat{\beta}_n, \widehat{\kappa}_n)'$. The appropriate neighborhood of $\boldsymbol{\alpha}_0 = (\beta_0, \kappa_0)'$ in which the solution exists is specified in Lemma 2.4. Expanding $\boldsymbol{l}^*_n(\widehat{\boldsymbol{\alpha}}_n)$ around $\boldsymbol{\alpha}_0$ we obtain

$$(2.11) \qquad \boldsymbol{l}^*_n(\boldsymbol{\alpha}_0) = \boldsymbol{A}^*_n(\zeta_n)(\widehat{\boldsymbol{\alpha}}_n - \boldsymbol{\alpha}_0),$$

where ζ_n is on the line segment joining $\widehat{\boldsymbol{\alpha}}_n$ and $\boldsymbol{\alpha}_0$. Defining $\boldsymbol{Z}^*_n = n^{-1/2}\boldsymbol{l}^*_n(\boldsymbol{\alpha}_0)$ and $\boldsymbol{C}^*_n(\cdot) = n^{-1}\boldsymbol{A}^*_n(\cdot)$, we have from (2.11),

$$(2.12) \qquad \boldsymbol{Z}^*_n = \boldsymbol{C}^*_n(\zeta_n)\, \boldsymbol{W}^*_n.$$

The asymptotic normality of $(\widehat{\beta}_n, \widehat{\kappa}_n)'$ rest on convergence results for $\boldsymbol{Z}^*_n$ and $\boldsymbol{C}^*_n$, which we demonstrate next.

LEMMA 2.4. (a) *$\boldsymbol{Z}^*_n$ converges in distribution to a bivariate normal random variable with mean vector zero and covariance matrix $\boldsymbol{\Sigma}^*$ where*

$$\boldsymbol{\Sigma}^* = \begin{bmatrix} \kappa_0/\beta_0^2 & 0 \\ 0 & \psi'(\kappa_0) - 1/\kappa_0 \end{bmatrix}.$$

(b) $C_n^*(\boldsymbol{\alpha}_0)$ *converges in probability to* $\boldsymbol{\Sigma}^*$.

(c) *Define a sequence of neighborhoods of* $\boldsymbol{\alpha}_0$ *by*

$$M_n(\boldsymbol{\alpha}_0) = \{\boldsymbol{\alpha} : \beta = \beta_0 + \tau_1 n^{-\delta}, \kappa = \kappa_0 + \tau_2 n^{-\delta}, \|\tau\| \leq h\},$$

where δ *and* h *are fixed numbers with* $0 < \delta < 1/2$, $0 < h < \infty$. *Then,* $[C_n^*(\boldsymbol{\alpha}) - C_n^*(\boldsymbol{\alpha}_0)] \to 0$ *in probability uniformly in* $\boldsymbol{\alpha} \in M_n(\boldsymbol{\alpha}_0)$.

PROOF OF LEMMA 2.4. (a) Observe that using (2.4) we can reexpress $l_{1n}^*(\boldsymbol{\alpha}_0)$ and $l_{2n}^*(\boldsymbol{\alpha}_0)$ as

$$l_{1n}^*(\boldsymbol{\alpha}_0) = -\frac{\kappa_0}{\beta_0} n^{1/2} U_{1n} - \frac{\kappa_0 - 1}{\beta_0} + \frac{\kappa_0 - 1}{\beta_0} \sum_{i=2}^{n} \left(\frac{T_{i-1}^{\beta_0}}{T_i^{\beta_0} - T_{i-1}^{\beta_0}} \log \frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}} - 1 \right),$$

$$l_{2n}^*(\boldsymbol{\alpha}_0) = -n \log \left(1 + \frac{U_{2n}}{n^{1/2}\kappa_0} \right) + n^{1/2} U_{3n}.$$

An application of Lemma 2.3 yields

$$Z_{1n}^* = -(\kappa_0/\beta_0) U_{1n} + o_p(1),$$
$$Z_{2n}^* = -(U_{2n}/\kappa_0) + U_{3n} + o_p(1).$$

Using Lemma 2.1 it follows that $\boldsymbol{Z}_n^*$ is asymptotically normal with zero mean vector and covariance matrix $\boldsymbol{\Sigma}^*$.

(b) Using (2.4) again, it is possible to express the elements of $C_n^*(\boldsymbol{\alpha}_0)$ as

$$c_{11}^*(\boldsymbol{\alpha}_0) = \frac{\kappa_0}{\beta_0^2} - \frac{\kappa_0 - 1}{n\beta_0^2} + \frac{\kappa_0 - 1}{n\beta_0^2} \sum_{i=2}^{n} \left\{ \frac{T_i^{\beta_0} T_{i-1}^{\beta_0}}{(T_i^{\beta_0} - T_{i-1}^{\beta_0})^2} \left(\log \frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}} \right)^2 - 1 \right\}$$

$$c_{12}^*(\boldsymbol{\alpha}_0) = \frac{U_{1n}}{n^{1/2}\beta_0} + \frac{1}{n\beta_0} - \frac{1}{n\beta_0} \sum_{i=2}^{n} \left(\frac{T_{i-1}^{\beta_0}}{T_i^{\beta_0} - T_{i-1}^{\beta_0}} \log \frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}} - 1 \right)$$

$$c_{22}^*(\boldsymbol{\alpha}_0) = \psi'(\kappa_0) - \frac{1}{\kappa_0}$$

which implies that $C_n^*(\boldsymbol{\alpha}_0)$ converges to the non-singular matrix $\boldsymbol{\Sigma}^*$ in probability.

(c) The proof rests on showing the uniform convergence of relevant functions. Details of the proof are sketched in the Appendix. □

In the development thus far, we have assumed the availability of a consistent sequence of roots of the likelihood equations. Theorem 2.1 given below demonstrates the existence of such a sequence and establishes its asymptotic normality.

THEOREM 2.1. (a) *With probability tending to* 1 *as* $n \to \infty$, *there exists a sequence of roots* $\widehat{\boldsymbol{\alpha}}_n \in M_n(\boldsymbol{\alpha}_0)$ *of the likelihood equations. Furthermore, such* $\widehat{\boldsymbol{\alpha}}_n$*'s correspond to local maxima of the likelihood function.*

(b) $\boldsymbol{W}_n$ *is asymptotically* (*singular*) *normal with mean vector zero and covariance matrix* $\boldsymbol{\Sigma}_1$, *where*

$$\boldsymbol{\Sigma}_1 = \begin{bmatrix} \theta_0^2/(\beta_0^2\kappa_0) & \theta_0/\kappa_0 & 0 \\ \theta_0/\kappa_0 & \beta_0^2/\kappa_0 & 0 \\ 0 & 0 & \kappa_0/\{\kappa_0\psi'(\kappa_0) - 1\} \end{bmatrix}.$$

PROOF. (a) The existence proof follows along lines very similar to a corresponding proof given in Sen and Bhattacharyya (1993) and we omit the details here.

(b) Premultiplying both sides of (2.12) by ${\mathbf{\Sigma}^*}^{-1}$ we have

$$\begin{aligned} {\mathbf{\Sigma}^*}^{-1}\boldsymbol{Z}_n^* &= {\mathbf{\Sigma}^*}^{-1}[\boldsymbol{C}_n^*(\zeta_n) - \boldsymbol{C}_n^*(\boldsymbol{\alpha}_0) + \boldsymbol{C}_n^*(\boldsymbol{\alpha}_0)]\,\boldsymbol{W}_n^* \\ &= \boldsymbol{W}_n^* + o_p(1). \end{aligned}$$

For $\zeta_n \in M_n(\boldsymbol{\alpha}_0)$, the last equality follows from Lemma 2.4 and the fact that $\boldsymbol{C}_n^*(\boldsymbol{\alpha}_0)$ converges to $\mathbf{\Sigma}^*$ in probability. This implies $\boldsymbol{W}_n^*$ is asymptotically normal with zero mean and covariance matrix ${\mathbf{\Sigma}^*}^{-1}$. Furthermore, note that

$$\begin{aligned} \beta_0(\log\hat{\theta} - \log\theta_0) &= (\log n)(1 - \beta_0/\hat{\beta}) + \log\{1 + U_{2n}/(n^{1/2}\kappa_0)\} \\ &\quad - (\beta_0/\hat{\beta})(\log\hat{\kappa} - \log\kappa_0) + (\log\kappa_0)(1 - \beta_0/\hat{\beta}). \end{aligned}$$

Using the asymptotic normality of both U_{2n} and $\boldsymbol{W}_n^*$, it follows that

$$n^{1/2}(\log n)^{-1}(\log\hat{\theta} - \log\theta_0) = \beta_0^{-2}W_{2n} + o_p(1).$$

and the result follows. □

Remarks. 1. The asymptotic result of Theorem 2.1 provides some curious insights into the behavior of the MLE's of the MPLP parameters. Apart from the singularity and nonuniform scalings of the MLE's, also note that Theorem 2.1 entails that $\widehat{\kappa}$ is asymptotically independent of $\widehat{\theta}$ and $\widehat{\beta}$. The asymptotic result for the MLE's of the usual PLP is also recoverable from Theorem 2.1 by simply substituting $\kappa_0 = 1$ in the 2×2 top left submatrix of $\mathbf{\Sigma}_1$. This is also the part of the matrix that makes it singular.

2. Not all submodel results, however, can be derived as special cases of Theorem 2.1. For instance, for the gamma renewal process with $\beta_0 = 1$, the asymptotic result of the corresponding MLE's are quite different. Specifically, denoting by θ^*, κ^* the MLE's for θ, κ in the gamma renewal process, it follows from standard results concerning gamma distribution, that the vector $(\sqrt{n}(\theta^* - \theta_0), \sqrt{n}(\kappa^* - \kappa_0))'$ converges in distribution to a bivariate normal with mean $\mathbf{0}$ and variance-covariance matrix

$$\mathbf{\Sigma}_1^* = (\kappa_0\psi'(\kappa_0) - 1)^{-1}\begin{bmatrix} \theta_0^2\psi'(\kappa_0) & -\theta_0 \\ -\theta_0 & \kappa_0 \end{bmatrix}.$$

Evidently, the inclusion of the unknown growth parameter β in the MPLP model has crucial impact on the rate of convergence as well as the dependence structure.

3. An interesting observation regarding the large-sample behavior possibly lies at the root of the pathology we observe here. The joint asymptotics of the three parameter model rests on the crucial Taylor-expansion step, that can be expressed (after suitable scaling) as

$$\begin{pmatrix} n^{-1/2}l_{1n}(\boldsymbol{\mu}_0) \\ n^{-1/2}(\log n)^{-1}l_{2n}(\boldsymbol{\mu}_0) \\ n^{-1/2}l_{3n}(\boldsymbol{\mu}_0) \end{pmatrix} = \begin{bmatrix} \log n & 0 & 0 \\ 0 & \log n & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{C}_n(\zeta^*)\,\boldsymbol{W}_n, \tag{2.13}$$

where ζ^* is an intermediate random point between $\widehat{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}_0$. While $\boldsymbol{W}_n = O_p(1)$, $\boldsymbol{C}_n(\cdot)$ converges to a nonstochastic singular matrix uniformly in a neighborhood around $\boldsymbol{\mu}_0$. In spite of the presence of the multiplier matrix of growing constants on the right of (2.13) it can be shown that the left hand side converges to a (singular) normal, thereby making it a $O_p(1)$ term. This is in stark contrast to classical i.i.d. asymptotics.

2.2 *Inference for rate of occurrence of failure (ROCOF)*

Often people are interested in the inference concerning the intensity function of a point process. There has been considerable work relating to the estimation of intensity function of a PLP. For PLP, or for that matter, for any NHPP, the intensity function matches the rate of occurrence of failure (ROCOF) defined as the instantaneous rate of change of the mean function. As we show next, this is not the case for an IGP.

Let $N(t)$ and $H(t)$ denote the counting process and the mean function of an IGP. Then it follows from the properties of IGP that $N_0(x) \equiv N(\Lambda^{-1}(x))$, $\Lambda(x) = t$, is the counting process associated with a gamma renewal process where $\Lambda(x_i) - \Lambda(x_{i-1})$ follows a $Gamma(\kappa, 1)$ distribution. Moreover, if we define $H_0(x) = E[N_0(x)]$ then $H_0(x) = H(\Lambda^{-1}(x))$. From standard theory of renewal processes (see Cox (1967), p. 48–53) we have

$$H(\Lambda^{-1}(x)) = \begin{cases} \frac{x}{\kappa} + \frac{1-\kappa}{2\kappa} + O(\exp(-ax)) & \text{if } \kappa \text{ is integer;} \\ \frac{x}{\kappa} + \frac{1-\kappa}{2\kappa} + O(e^{-x}x^{-\kappa-1}) & \text{if } \kappa \text{ is non-integer,} \end{cases} \tag{2.14}$$

where 'a' is a constant with $0 < a \leqslant 2$. Infact 'a' evolves as a bound for the real part of the non-zero roots of $(1+s)^\kappa = 1$. Substituting $t = \Lambda^{-1}(x)$ in (2.14) we thus have,

$$H(t) = \begin{cases} \frac{\Lambda(t)}{\kappa} + \frac{1-\kappa}{2\kappa} + O(\exp(-a\Lambda(t))) & \text{if } \kappa \text{ is integer;} \\ \frac{\Lambda(t)}{\kappa} + \frac{1-\kappa}{2\kappa} + O(e^{-\Lambda(t)}(\Lambda(t))^{-\kappa-1}) & \text{if } \kappa \text{ is non-integer.} \end{cases}$$

Since in IGP the probability of simultaneous failures is zero, the ROCOF for an IGP is

$$h(t) = \frac{d}{dt}H(t) = \begin{cases} \frac{\lambda(t)}{\kappa} + O(\lambda(t)\exp(-a\Lambda(t))) & \text{if } \kappa \text{ is integer;} \\ \frac{\lambda(t)}{\kappa} + O(\lambda(t)e^{-\Lambda(t)}(\Lambda(t))^{-\kappa-1}) & \text{if } \kappa \text{ is non-integer.} \end{cases} \tag{2.15}$$

Unlike NHPP, the ROCOF of IGP is not equal to the "complete" intensity function. Defining $F_t = \{N(s) : 0 \leq s < t\}$ as the history of the process up to time t, the complete intensity function of IGP is

$$\begin{aligned} \mu(t, F_t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\Pr\{\text{a failure in the interval } [t, t+\Delta t) \mid F_t\}] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\Pr\{\text{a failure in Gamma renewal process in the interval} \\ &\qquad\qquad [\Lambda(t), \Lambda(t+\Delta t)) \mid F_t\}] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\Pr\{\text{a failure in Gamma renewal process in} \\ &\qquad\qquad [\Lambda(t), \Lambda(t) + \lambda(t)\Delta t) \mid F_t\}] \\ &= Z(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t), \end{aligned}$$

where $T_{N(t-)}$ is the time of the last event prior to t and $Z(\cdot)$ is the hazard function of $Gamma(\kappa, 1)$. It is thus seen that the complete intensity function of IGP is a product of $\lambda(t)$, which depends on the age t of the system and another factor which depends on the (transformed) time from the last event. This is in contrast to Cox's modulated renewal process (Cox (1972)) and the model of Lawless and Thiagarajah (1996), which instead use complete intensities of the form $Z(t - T_{N(t-)})\lambda(t)$.

In this section we study the current ROCOF of an IGP. The current ROCOF refers to the achieved value of the intensity function at the current time, usually the termination point of the developmental testing phase. For IGP, this quantity can be thought of as an initial estimate of the *rate of occurence of failure* (ROCOF) at the subsequent operational testing phase in which the system failures are assumed to be governed by a Gamma Renewal Process (at least initially). An objective evaluation of the current ROCOF is extremely crucial to reliability engineers in deciding the extent of effectiveness of a developmental program in achieving a planned reliability goal.

From (2.15), the intensity h_n at the n-th failure of MPLP can be expressed as

$$h_n = h(T_n) = \begin{cases} \frac{\beta(T_n/\theta)^{\beta-1}}{\theta\kappa} + O_p(\frac{\beta}{\theta}(T_n/\theta)^{\beta-1}\exp(-a(T_n/\theta)^{\beta})) & \text{if } \kappa \text{ is integer;} \\ \frac{\beta(T_n/\theta)^{\beta-1}}{\theta\kappa} + O_p(\frac{\beta}{\theta}(T_n/\theta)^{-(\beta\kappa+1)}\exp(-(T_n/\theta)^{\beta})) & \text{if } \kappa \text{ is non-integer.} \end{cases}$$

The above quantity is random since T_n, the time of the n-th failure is random. An estimate $\hat{h}_n$ of h_n can be obtained by replacing the parameters by their MLEs in the leading term of h_n, i.e.

$$\widehat{h}_n = \frac{\widehat{\beta}(T_n/\widehat{\theta})^{\widehat{\beta}-1}}{\widehat{\theta}\widehat{\kappa}} = \frac{n\widehat{\beta}}{T_n}.$$

THEOREM 2.2. *The quantity* $n^{1/2}(h_n/\widehat{h}_n - 1)$ *converges to a normal distribution with mean zero and variance* $2/\kappa_0$.

PROOF. First notice that

$$n^{1/2}\left(\frac{h_n}{\widehat{h}_n} - 1\right) = \begin{cases} n^{1/2}\left\{\frac{\beta_0(T_n/\theta_0)^{\beta_0}}{n\hat{\beta}\kappa_0} - 1\right\} + \frac{T_n}{n^{1/2}\hat{\beta}}O_p\left(\frac{\beta_0}{\theta_0}(T_n/\theta_0)^{\beta_0-1}\exp(-a(T_n/\theta_0)^{\beta_0})\right) \\ \qquad \text{if } \kappa \text{ is integer;} \\ n^{1/2}\left\{\frac{\beta_0(T_n/\theta_0)^{\beta_0}}{n\hat{\beta}\kappa_0} - 1\right\} \\ \quad + \frac{T_n}{n^{1/2}\hat{\beta}}O_p\left(\frac{\beta_0}{\theta_0}(T_n/\theta_0)^{-(\beta_0\kappa_0+1)}\exp(-(T_n/\theta_0)^{\beta_0})\right) \\ \qquad \text{if } \kappa \text{ is non-integer.} \end{cases}$$

Using (2.4) and the fact that $\beta_0/\hat{\beta} \to 1$ in probability as $n \to \infty$, we arrive at

$$n^{1/2}\left\{\frac{\beta_0(T_n/\theta_0)^{\beta_0}}{n\hat{\beta}\kappa_0} - 1\right\} = \frac{U_{2n}}{\kappa_0} - \frac{W_{2n}}{\beta_0} + o_p(1).$$

Further,

$$\frac{T_n}{n^{1/2}\hat{\beta}}O_p\left(\frac{\beta_0}{\theta_0}(T_n/\theta_0)^{\beta_0-1}\exp(-a(T_n/\theta_0)^{\beta_0})\right) \tag{2.16}$$

$$= O_p\left(n^{1/2}\exp\left(-an\kappa_0\left(1+\frac{U_{2n}}{n^{1/2}\kappa_0}\right)\right)\right),$$

$$\frac{T_n}{n^{1/2}\hat{\beta}}O_p\left(\frac{\beta_0}{\theta_0}(T_n/\theta_0)^{-(\beta_0\kappa_0+1)}\exp(-(T_n/\theta_0)^{\beta_0})\right) \tag{2.17}$$
$$= O_p\left(n^{-\kappa_0-1/2}\exp\left(-n\kappa_0\left(1+\frac{U_{2n}}{n^{1/2}\kappa_0}\right)\right)\right).$$

A quick reflection reveals that the terms in (2.16) and (2.17) are, infact $o_p(1)$. Consequently, irrespective of the value of κ_0,

$$n^{1/2}\left(\frac{h_n}{\widehat{h}_n}-1\right) = \frac{U_{2n}}{\kappa_0} - \frac{W_{2n}}{\beta_0} + o_p(1).$$

Further from Theorem 2.1(b) we have $W_{2n} = -\beta_0 U_{1n} + o_p(1)$. Hence using Lemma 2.1 the result follows. □

3. Simple estimators

The maximum likelihood estimators for the parameters in the MPLP are not available in closed form. In this section, as an alternative to the MLE's, we construct simple estimators of the parameters that are easy to compute and are motivated by moment-type considerations. Noting that $(T_i/\theta)^\beta$ follows $Gamma(\kappa i, 1)$ distribution, we have

$$E[\beta(\log T_n - \log T_i)] = \psi(n\kappa) - \psi(\kappa i) \approx \log n - \log i.$$

This expression is true for large values of n or κ. Thus $\beta\log(T_n/T_i)$ would correspond to $-\log(i/n)$. Taking summation over i we have

$$\frac{\beta}{n}\sum_{i=1}^{n}\log(T_n/T_i) \leftrightarrow -\frac{1}{n}\sum_{i=1}^{n}\log(i/n) \stackrel{n\to\infty}{\longrightarrow} 1$$

which motivates an estimator for β as

$$\tilde{\beta} = n/\sum_{i=1}^{n}\log(T_n/T_i).$$

There is yet another motivation to use $\tilde{\beta}$ as an estimator of β. Note that $\tilde{\beta}$ is the maximum likelihood estimator of β under the non-modulated power law process. So, the behavior of $\tilde{\beta}$ as an estimator of β for MPLP, in essence, assesses the effect of misspecifying κ.

To motivate our new estimator of κ, we first observe that $V_i \equiv (T_i^\beta - T_{i-1}^\beta)$, $i = 1,\ldots,n$, are independently and identically distributed $Gamma(\kappa, \theta^\beta)$ random variables. If β were known, maximum likelihood estimator of κ would satisfy,

$$h(\widehat{\kappa}) = \log\bar{V} - \frac{1}{n}\sum_{i=1}^{n}\log V_i \tag{3.1}$$
$$= \beta\log T_n - \log n - \frac{1}{n}\sum_{i=1}^{n}\log(T_i^\beta - T_{i-1}^\beta),$$

where $h(\kappa) = \log\kappa - \Psi(\kappa)$. Replacing β by $\tilde{\beta}$ in (3.1) an alternative estimator of κ is obtained by

$$\widetilde{\kappa} = h^{-1}\left(\tilde{\beta}\log T_n - \log n - \frac{1}{n}\sum_{i-1}^{n}\log(T_i^{\tilde{\beta}} - T_{i-1}^{\tilde{\beta}})\right).$$

Substituting $\tilde{\beta}$ and $\widetilde{\kappa}$ in (2.3), we have a non-maximum likelihood estimator of θ as

$$\widetilde{\theta} = \frac{T_n}{(n\widetilde{\kappa})^{1/\tilde{\beta}}}.$$

The simple estimators of β and θ are in closed form. Though $\widetilde{\kappa}$ is not in closed form, it can be determined by inverse interpolation in a table of the function $h(\kappa)$. Such tables have been published by Masuyama and Kuroiwa (1952) and Chapman (1956).

We now turn to the large-sample behavior of the simple estimators. Let $\widetilde{W}_{1n} = n^{1/2}(\log n)^{-1}(\widetilde{\theta} - \theta_0)$, $\widetilde{W}_{2n} = n^{1/2}(\tilde{\beta} - \beta_0)$. $\widetilde{W}_{3n} = n^{1/2}(\widetilde{\kappa} - \kappa_0)$. The next theorem estabilishes that the simple estimators are asymptotically equivalent to the maximum likelihood estimators.

THEOREM 3.1. *$\widetilde{\boldsymbol{W}}_n = (\widetilde{W}_{1n}, \widetilde{W}_{2n}, \widetilde{W}_{3n})'$ is asymptotically (singular) normal with mean vector zero and covaiance matrix $\boldsymbol{\Sigma}_1$.*

PROOF. Note that using (2.4),

$$\widetilde{W}_{2n} = n^{1/2}(\tilde{\beta} - \beta_0) = -\frac{\beta_0 U_{1n}}{1 + n^{-1/2}U_{1n}}. \tag{3.2}$$

Further

$$\begin{aligned} n^{1/2}(h(\widetilde{\kappa}) - h(\kappa_0)) &= n^{1/2}\left\{\frac{\tilde{\beta}}{\beta_0}\log\left(\frac{T_n}{\theta_0}\right)^{\beta_0} - \log n - \log\kappa_0\right\} \\ &\quad - n^{-1/2}\sum_{i=1}^{n}\left[\log\left\{\left(\frac{T_i}{\theta_0}\right)^{\beta_0} - \left(\frac{T_{i-1}}{\theta_0}\right)^{\beta_0}\right\} - \Psi(\kappa_0)\right] \\ &\quad - n^{-1/2}\sum_{i=1}^{n}\left[\log\left\{\left(\frac{T_i}{\theta_0}\right)^{\tilde{\beta}} - \left(\frac{T_{i-1}}{\theta_0}\right)^{\tilde{\beta}}\right\}\right. \\ &\qquad \left. - \log\left\{\left(\frac{T_i}{\theta_0}\right)^{\beta_0} - \left(\frac{T_{i-1}}{\theta_0}\right)^{\beta_0}\right\}\right]. \end{aligned} \tag{3.3}$$

Applying Taylor's expansion about β_0 to $\log\{(T_i/\theta_0)^{\tilde{\beta}_0} - (T_{i-1}/\theta_0)^{\tilde{\beta}_0}\}$ in (3.3) and following some algebraic manipulations, we have

$$\begin{aligned} n^{1/2}(h(\widetilde{\kappa}) - h(\kappa_0)) &= n^{1/2}\log\left(1 + \frac{U_{2n}}{n^{1/2}\kappa_0}\right) - U_{3n} + n^{1/2}\frac{(\tilde{\beta} - \beta_0)}{\beta_0}\frac{U_{1n}}{n^{1/2}} \\ &\quad + n^{-1/2}\frac{(\tilde{\beta} - \beta_0)}{\beta_0} \\ &\quad - \frac{(\tilde{\beta} - \beta_0)}{\beta_0}\frac{1}{n^{1/2}}\sum_{i=2}^{n}\left[\frac{\left(\frac{T_{i-1}}{\theta_0}\right)^{\beta^*}}{\left(\frac{T_i}{\theta_0}\right)^{\beta^*} - \left(\frac{T_{i-1}}{\theta_0}\right)^{\beta^*}}\log\left(\frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}}\right) - 1\right], \end{aligned}$$

where β^* is an intermediate random point between $\tilde{\beta}$ and β_0. Since β^* converges to β_0 in probability, using Lemma 2.3 and the facts that $\boldsymbol{U}_n = O_p(1)$ and $\widetilde{W}_{2n} = O_p(1)$, it follows,

$$n^{1/2}(h(\widetilde{\kappa}) - h(\kappa_0)) = \frac{U_{2n}}{\kappa_0} - U_{3n} + o_p(1),$$

which implies

$$\widetilde{W}_{3n} = n^{1/2}(\widetilde{\kappa} - \kappa_0) = -\{\psi'(\kappa_0) - 1/\kappa_0\}^{-1}\left(\frac{U_{2n}}{\kappa_0} - U_{3n}\right) + o_p(1).$$

Now, appealing to Lemma 2.1, we have

$$\begin{pmatrix}\widetilde{W}_{2n}\\ \widetilde{W}_{3n}\end{pmatrix} \xrightarrow{d} N\left(\mathbf{0}, \begin{bmatrix}\beta_0^2/\kappa_0 & 0\\ 0 & \kappa_0/\{\kappa_0\psi'(\kappa_0) - 1\}\end{bmatrix}\right) \tag{3.4}$$

as $n \to \infty$, where $\xrightarrow{d}$ refers to convergence in distribution. Also

$$\begin{aligned}\beta_0(\log\widetilde{\theta} - \log\theta_0) &= \log n\left(1 - \frac{\beta_0}{\tilde{\beta}}\right) + \log\{1 + U_{2n}/(n^{1/2}\kappa_0)\}\\ &\quad - \frac{\beta_0}{\tilde{\beta}}(\log\widetilde{\kappa} - \log\kappa_0) + \log\kappa_0\left(1 - \frac{\beta_0}{\tilde{\beta}}\right).\end{aligned} \tag{3.5}$$

Note that U_{2n}, $\widetilde{W}_{2n}$ and $\widetilde{W}_{3n}$ are bounded in probability. Consequently, we can reexpress (3.5) as

$$n^{1/2}(\log n)^{-1}(\log\widetilde{\theta} - \log\theta_0) = \beta_0^{-2}\widetilde{W}_{2n} + o_p(1),$$

which implies

$$\widetilde{W}_{1n} = n^{1/2}(\log n)^{-1}(\widetilde{\theta} - \theta_0) = \theta_0\beta_0^{-2}\widetilde{W}_{2n} + o_p(1).$$

The result now follows from (3.4). □

Since the simple estimators are asymptotically equivalent to the MLE's, Remarks following Theorem 2.1 in Section 2 also apply here. Further, asymptotic results for functions of parameters such as the ROCOF h_n of Subsection 2.2 will be identical to that in Theorem 2.2, if we replace the MLE's in $\widehat{h}_n$ by the corresponding simple estimators. The attraction of using the simple estimators is clearly the ease of computation. Their finite sample performance with their ML counterparts, however, needs to be investigated, which we pursue next.

4. Simulation and application

4.1 *Comparative study of estimator performance*

Monte Carlo simulation techniques are employed to study the performances of the maximum likelihood estimators $(\widehat{\theta}, \widehat{\beta}, \widehat{\kappa})$ and the simple estimators $(\widetilde{\theta}, \widetilde{\beta}, \widetilde{\kappa})$ developed in Sections 2 and 3, respectively. We compare the bias and the mean squared error of the estimators for both small and large sample sizes. To obtain confidence limits for parameters, we rely on the approximate normality of the estimators in large samples. It is, of course, desirable to check the adequacy of such approximations for the small to medium sample sizes encountered in practice.

We considered the MPLP for various combinations of $(\theta_0, \beta_0, \kappa_0)$ values. For each case one thousand realizations of the two sets of estimators were obtained with the sample sizes $n = 10$, 25, 50 and 100. The MLEs $\widehat{\beta}$ and $\widehat{\kappa}$ were computed using the two-variable Newton-Raphson procedure with the simple estimators $\widetilde{\beta}$ and $\widetilde{\kappa}$ as the initial value for the iteration. Table 1 gives the estimated bias and mean squared error of the MLEs as well as the simple estimators.

As is evident from Table 1, all the estimators have a tendency to overestimate for small sample sizes over all ranges of $(\theta_0, \beta_0, \kappa_0)$ values considered. Overall the MLEs for θ and β perform better than the corresponding simple estimators. Both the estimators for θ show a substantial variability. By contrast, the MLE and the simple estimator for β appear to be quite stable. The simple estimator for κ performs better than the corresponding MLE in small sample sizes, whereas for large sample sizes they are comparable. The asymptotic distributions of $\widehat{\kappa}$ and $\widetilde{\kappa}$ are inaccurate unless sample size is very large. Moreover, with moderate and large values of κ, the Newton-Raphson method for MLEs of β and κ fails to converge frequently even with samples of size three hundred or four hundred.

The above phenomenon is not surprising at all for our model. Note that T_i follows the generalized gamma distribution with pdf

$$\frac{\beta}{\Gamma(i\kappa)}\left(\frac{t_i}{\theta}\right)^{\beta\kappa i-1}\exp\left[-\left(\frac{t_i}{\theta}\right)^{\beta}\right], \qquad t_i > 0$$

where $\beta > 0$, $\theta > 0$ and $\kappa > 0$ are parameters. Several authors (e.g. Hager and Bain (1970), Parr and Webster (1965), Harter (1967), Stacy and Mihram (1965)) encountered problems analogous to that discussed in the previous paragraph with maximum likelihood estimation when T_i's, $i = 1, \ldots, n$; are independently and identically distributed generalized gamma variables. Prentice (1974) studied the generalized gamma distribution in a different but equivalent form, which makes the properties and potential difficulties with estimation in the model much more transparent. He then suggested a reparameterization that tends to alleviate some of these problems. In our case, with non i.i.d. T_i's, the reparameterization similar to Prentice (1974) does not work well. An alternative reparameterization similar to that employed by Lawless (1980) for generalized gamma distribution can be used here. This entails considering $Y_i = \log T_i$ instead of T_i and reparameterizeing the model as

$$V_i = \frac{Y_i - \mu}{\sigma},$$

where $\mu = \log\theta + \log\kappa/\beta$ and $\sigma = 1/(\beta\sqrt{\kappa})$. With this reparameterization the MLEs of μ and σ are very stable. However, $\widehat{\kappa}$ suffers from convergence and stability difficulties which are similar to those without the reparameterization. Moreover the convergence problem is worse for moderate and large values of κ. For instance, in one case with $\theta = 2.0$, $\beta = 1.5$, $\kappa = 7.0$ and $n = 400$, without reparameterization the Newton-Raphson method fails to converge 18.49% times while with reparametarization it fails 29.98% times.

For practical applications of the asymptotic results, it is important to examine how the normal approximation improves with increasing sample sizes. An investigation in that direction is made through the normal scores plots of the estimates for θ, β and κ. Plots for both $\widehat{\beta}$ and $\widetilde{\beta}$ indicate a fairly linear pattern. However, the corresponding

Table 1. Estimated bias and mean squared error of the MLEs and the simple estimators.

Sample Size	Maximum Likelihood Estimators						Simple Estimators					
(n)	Bias($\hat{\theta}$)	MSE($\hat{\theta}$)	Bias($\hat{\beta}$)	MSE($\hat{\beta}$)	Bias($\hat{\kappa}$)	MSE($\hat{\kappa}$)	Bias($\tilde{\theta}$)	MSE($\tilde{\theta}$)	Bias($\tilde{\beta}$)	MSE($\tilde{\beta}$)	Bias($\tilde{\kappa}$)	MSE($\tilde{\kappa}$)
	$\theta_0 = 2.0$, $\beta_0 = 0.75$, $\kappa_0 = 0.9$											
10	1.334	17.871	0.182	0.213	0.426	0.794	1.228	13.959	0.189	0.202	0.370	0.545
25	1.077	12.648	0.057	0.036	0.125	0.108	0.940	10.916	0.052	0.034	0.124	0.104
50	0.749	5.942	0.030	0.015	0.064	0.034	0.676	5.457	0.027	0.015	0.064	0.034
100	0.613	3.486	0.019	0.007	0.023	0.015	0.555	3.224	0.016	0.007	0.023	0.015
	$\theta_0 = 2.0$, $\beta_0 = 0.75$, $\kappa_0 = 1.0$											
10	1.429	18.722	0.167	0.155	0.472	0.901	1.506	16.416	0.197	0.170	0.394	0.601
25	1.304	13.217	0.074	0.037	0.147	0.129	1.309	12.202	0.079	0.037	0.143	0.123
50	0.801	6.468	0.032	0.014	0.076	0.049	0.800	6.148	0.033	0.014	0.075	0.048
100	0.474	3.599	0.012	0.006	0.037	0.019	0.475	3.516	0.013	0.006	0.037	0.019
	$\theta_0 = 2.0$, $\beta_0 = 1.0$, $\kappa_0 = 1.0$											
10	0.799	7.263	0.228	0.302	0.426	0.859	0.845	6.163	0.255	0.299	0.356	0.569
25	0.577	3.905	0.082	0.062	0.145	0.132	0.591	3.646	0.088	0.062	0.141	0.125
50	0.355	2.331	0.032	0.024	0.061	0.042	0.356	2.217	0.034	0.024	0.061	0.041
100	0.310	1.401	0.022	0.011	0.033	0.018	0.312	1.381	0.023	0.011	0.033	0.018
	$\theta_0 = 4.0$, $\beta_0 = 1.0$, $\kappa_0 = 2.5$											
10	0.780	19.598	0.081	0.073	1.067	3.573	3.627	41.658	0.261	0.158	0.513	1.409
25	0.619	8.951	0.036	0.020	0.426	1.028	2.004	15.646	0.108	0.034	0.342	0.846
50	0.449	4.835	0.017	0.009	0.171	0.317	1.270	7.424	0.055	0.012	0.151	0.300
100	0.370	3.246	0.013	0.005	0.098	0.129	0.870	4.404	0.033	0.006	0.092	0.127
	$\theta_0 = 4.0$, $\beta_0 = 1.5$, $\kappa_0 = 2.5$											
10	0.133	5.660	0.109	0.145	1.013	3.495	1.791	9.827	0.374	0.316	0.473	1.401
25	0.226	3.233	0.053	0.045	0.446	1.066	1.077	4.873	0.162	0.074	0.360	0.871
50	0.133	1.830	0.222	0.020	0.189	0.309	0.645	2.466	0.079	0.027	0.168	0.291
100	0.072	1.073	0.008	0.009	0.088	0.129	0.381	1.316	0.038	0.011	0.082	0.127

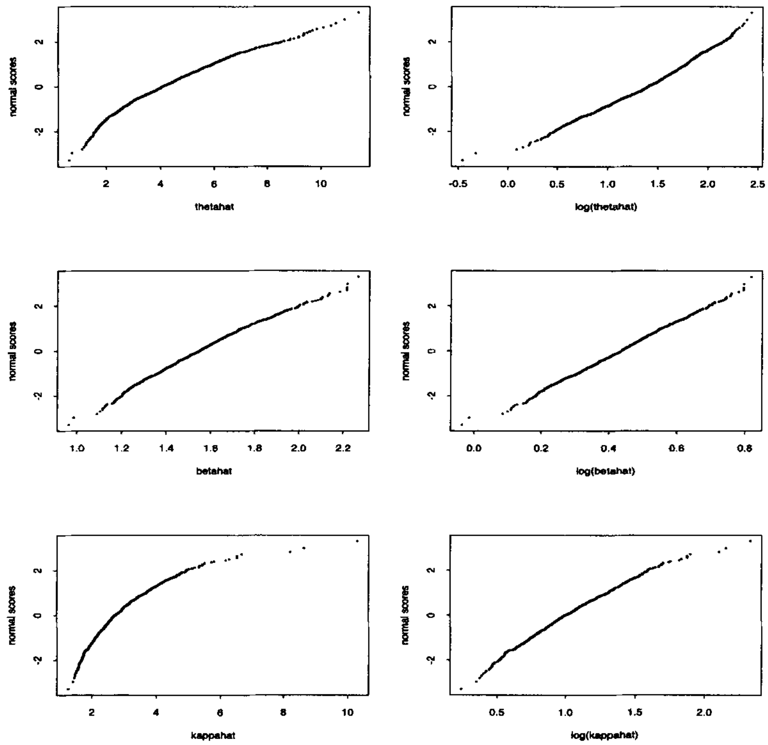


Fig. 1. Normal scores plot for the maximum likelihood estimates of the parameters and their logarithms. ($\theta = 4.0$, $\beta = 1.5$, $\kappa = 2.5$, $n = 25$)

plots for $\widehat{\theta}$, $\widetilde{\theta}$ and $\widehat{\kappa}$, $\widetilde{\kappa}$ show a substantial departure from a straight line pattern. This appears to be due to a considerable fluctuation in the estimated values. Also $\widehat{\theta}$ and $\widetilde{\theta}$ have the slow rate ($\sqrt{n}/\log n$) of convergence. For parameters that are constrained to be positive, the normal approximation is sometimes unsatisfactory because the distribution of the estimates can be highly skewed. In these cases one alternative is to use logarithmic transformation on the estimates. Logarithmic transformation on the estimates of θ and κ is found to stabilize their variations substantially, and the agreement with the normal scores is also considerably improved. In logarithmic scale the estimates of β also perform very well. Figures 1 and 2 exhibit these features for the case $n = 25$. These indicate that when setting large-sample confidence interval for a parameter, say θ, one should first construct a confidence interval for $\log\theta$ using the asymptotic normality result and then exponentiate the limits to construct a confidence interval for θ.

We further supplemented these findings with joint normality checks of $(\hat{\theta}, \hat{\beta})$. Figure 3 demonstrates the bivariate scatter plot of $(\log\hat{\theta}, \log\hat{\beta})$ for a typical parameter combination. The elliptical scatter conform to the asymptotic normality with a positive correlation. As expected, with an increase in sample size, the ellipticity is more prominent, and the contour is narrower, indicating an increase in correlation. Table 2 demonstrates the simulated empirical correlation of $\hat{\theta}$, $\hat{\beta}$ for a few typical parameter combinations, which are reasonably close to the asymptotic value of unity even for a moderate sample size. Note that a more formal check of bivariate normality based on chi-square probability plots requires existence of invertible variance-covariance matrix

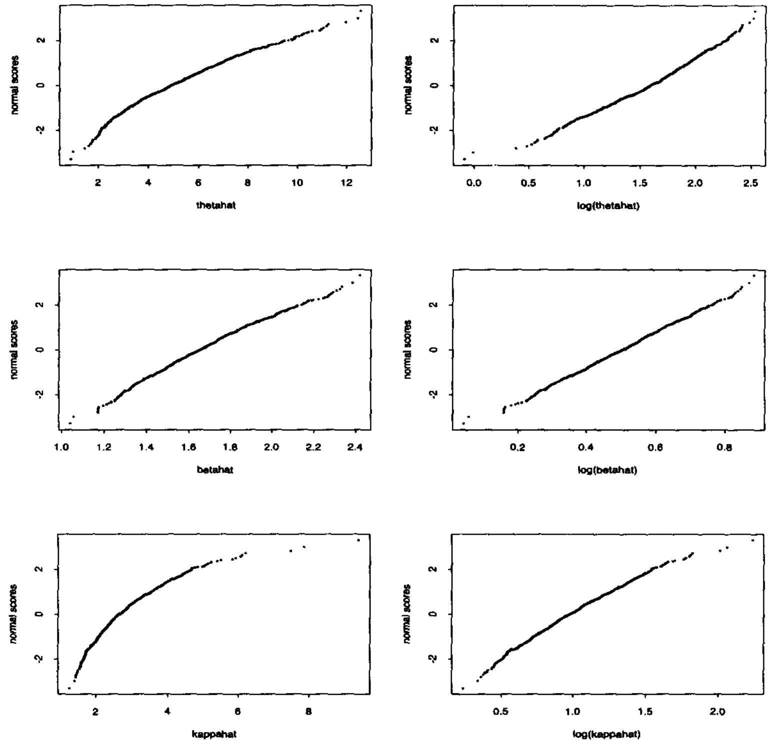


Fig. 2. Normal scores plot for the simple estimates of the parameters and their logarithms. ($\theta = 4.0$, $\beta = 1.5$, $\kappa = 2.5$, $n = 25$)

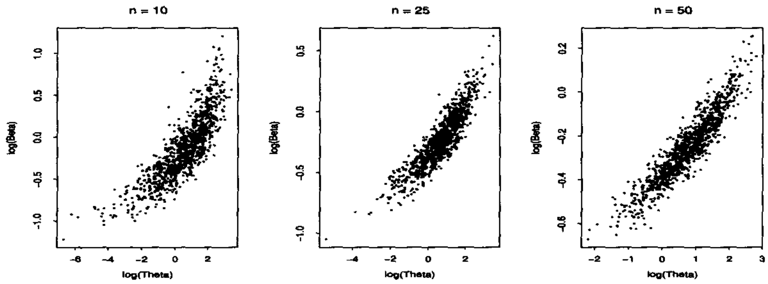


Fig. 3. Scatter plot for logarithms of the maximum likelihood estimates of θ and β. ($\theta = 2.0$, $\beta = 0.75$, $\kappa = 0.9$)

and hence is not directly applicable here.

4.2 *An example*

To illustrate inference procedures associated with an MPLP fit, we consider the failure times of an aircraft generator described in the original article by Duane (1964). The data, read off the plots presented by Duane (1964), were tabulated in Table 5 of Black and Rigdon (1996). There were 14 failures occurring at the cumulative hours of 10, 55, 166, 205, 341, 488, 567, 731, 1308, 2050, 2453, 3115, 4017 and 4596 respectively.

Table 2. Simulated correlation of $\hat{\theta}$ and $\hat{\beta}$.

Parameters	Sample Size (n) 10	25	50	100
$\theta_0 = 2.0,\ \beta_0 = 0.75,\ \kappa_0 = 0.9$	0.774	0.842	0.879	0.909
$\theta_0 = 2.0,\ \beta_0 = 0.75,\ \kappa_0 = 1.0$	0.775	0.843	0.885	0.893
$\theta_0 = 2.0,\ \beta_0 = 1.00,\ \kappa_0 = 1.0$	0.823	0.873	0.912	0.933
$\theta_0 = 4.0,\ \beta_0 = 1.00,\ \kappa_0 = 2.5$	0.811	0.855	0.876	0.909
$\theta_0 = 4.0,\ \beta_0 = 1.50,\ \kappa_0 = 2.5$	0.850	0.877	0.894	0.919

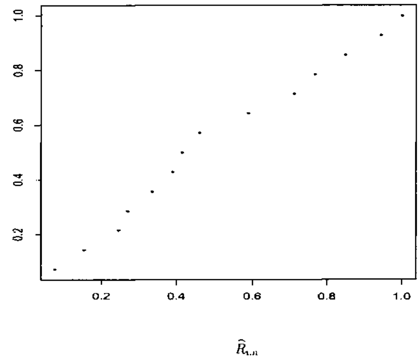


Fig. 4. Graphical check for the aircraft generator failure data.

For a quick graphical check of goodness of fit, we consider

$$\widehat{R}_{i,n} = (t_i/t_n)^{\hat{\beta}}, \qquad i = 1, \ldots, n-1.$$

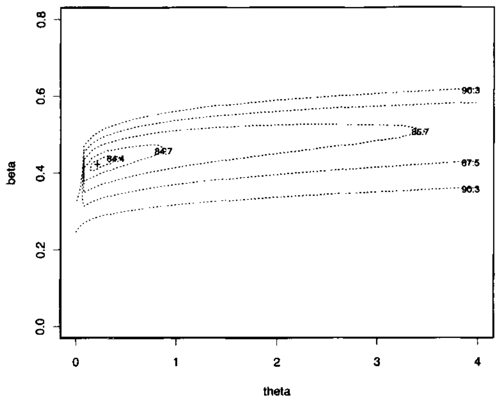
Note that under MPLP $R_{i,n}$'s are beta variates with parameters κi and $\kappa(n-i)$. So a plot of the points $(\widehat{R}_{i,n}, \alpha_i)$, where $\alpha_i = i/n$ is the expected value of $R_{i,n}$, will serve as a diagnostic tool for model validation. This plot, however, fails to distinguish between MPLP and the power law process. So if the plot is reasonably close to a straight line with unit slope and κ significantly different from unity, then the MPLP model can be deemed satisfactory. The corresponding plot for this example, presented in Fig. 4, does not show any substantial departure from linearity. Following the numerical procedures described in Sections 2 and 3 we obtained the maximum likelihood estimates and simple estimates of the parameters. Table 3 exhibits the estimates and associated 95% confidence intervals. Note that either maximum likelihood method or the methods of Section 3 yield a negative lower confidence limit for θ. Approximate interval estimation for positive parameters often suffers from this difficulty. In order to circumvent this problem, we recommend applying the asymptotic approximation in the logarithmic scale, and subsequently transforming the results back to the original unit of measurement. Confidence limits obtained via this route for all the parameters are reported in Table 3. The profile log-likelihood plot of θ, β in Fig. 5 demonstrates that the likelihood value changes little for a wide range of θ values, which aptly explains the uncertainty in the estimate of

Table 3. Parameter estimates and 95% confidence intervals from the aircraft generator failure data.

			Confidence Limits Without Transformation		Confidence Limits With Log Transformation	
Method	Parameter	Estimate	Lower	Upper	Lower	Upper
Maximum	θ	0.218	−0.107	0.543	0.048	0.969
Likelihood	β	0.423	0.322	0.524	0.333	0.537
Estimator	κ	4.800	1.361	8.241	2.345	9.829
	θ	0.958	−0.366	2.283	0.241	3.816
Simple	β	0.483	0.361	0.605	0.375	0.622
Estimator	κ	4.288	1.227	7.348	2.100	8.755

Table 4. Bootstrap estimates of standard error and percentiles for the distribution of the MLE's from the aircraft generator failure data.

			Percentiles		
Parameter	Mean	Standard Error	2.5%	50%	97.5%
θ	0.496	0.907	0.005	0.160	2.709
β	0.433	0.056	0.341	0.427	0.546
κ	6.221	2.697	2.941	5.666	13.280



$+ : (\hat{\theta}, \hat{\beta})$

Fig. 5. Contour plot of the profile log likelihood for θ and β for the aircraft generator failure data.

θ. Note from Table 3, that there is strong evidence that κ parameter is significantly different from 1. In conjunction with the plot described earlier, this supports a MPLP fit to the dataset.

For this example, we further carried out the estimation procedure using parametric bootstrap. The results shown in Table 4 are based on 1000 bootstrap replications and indicate some differences with the one-shot ML estimates provided in Table 3. Some key features of the analysis, however, are reconfirmed by the bootstrap calculations. For instance, the bootstrap estimate of κ is substantially larger than 1, reaffirming the de-

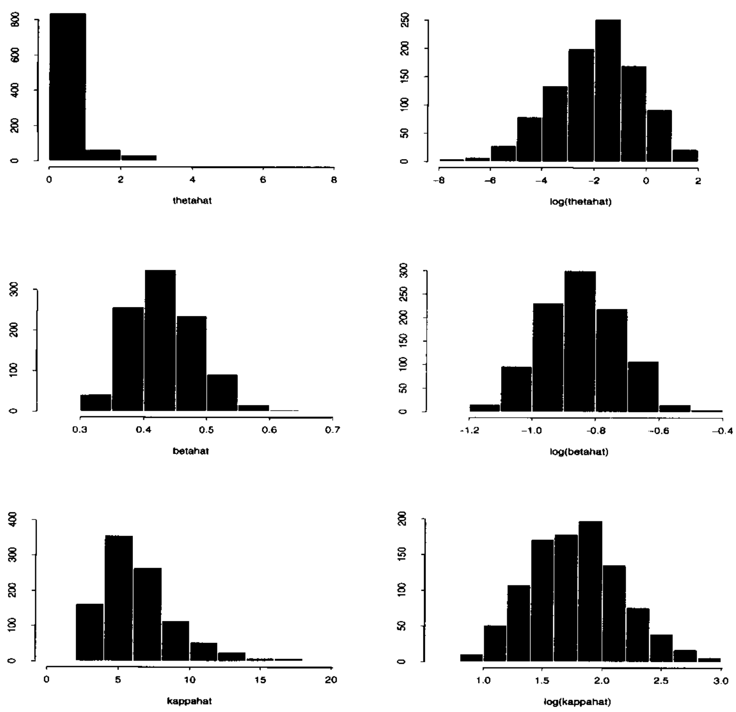


Fig. 6. Histogram of 1000 parametric bootstrap replications of the maximum likelihood estimates of the parameters and their logarithms for the aircraft generator failure data.

parture from a standard PLP fit. The histograms in Fig. 6 indicate heavily right-skewed distributions for the MLE's of both θ and κ. For each of the parameters, the median of the bootstrap distribution is closer than the mean to the estimates in Table 3. As shown in Fig. 6, transformation to the log-scale symmetrize the distributions substantially. The normal-scores plots of Fig. 7 further confirm that the proximity to normality of the MLE's is greatly improved by the log transformation. In view of the smallness of the dataset, bootstrap offers a reliable method of inference for this example.

5. Summary and conclusions

In this article, we investigate a parametric model for analysis of a single repairable system. The model generalizes some existing formulation used in practice in the context of observing a recurrent event. The parametric specification stems from a physical connection with certain phenomenon observed in the failure pattern of a variety of complex industrial devices. The main thrust of the present article lies in the study of inference procedures for the model parameters. The likelihood-based inference presents itself with a very interesting case of asymptotics that is quite nonstandard and is in stark contrast with what one typically encounters in ML estimation theory. A set of competing estimators, that are asymptotically equivalent to the MLE's, are developed that are extremely attractive from a computational viewpoint. Comparison of the estimators are also drawn from extensive finite-sample simulation studies. Hypothesis testing, which has not been discussed here, is currently under study.

Clearly, the model described here is a simple extension of power law process, the

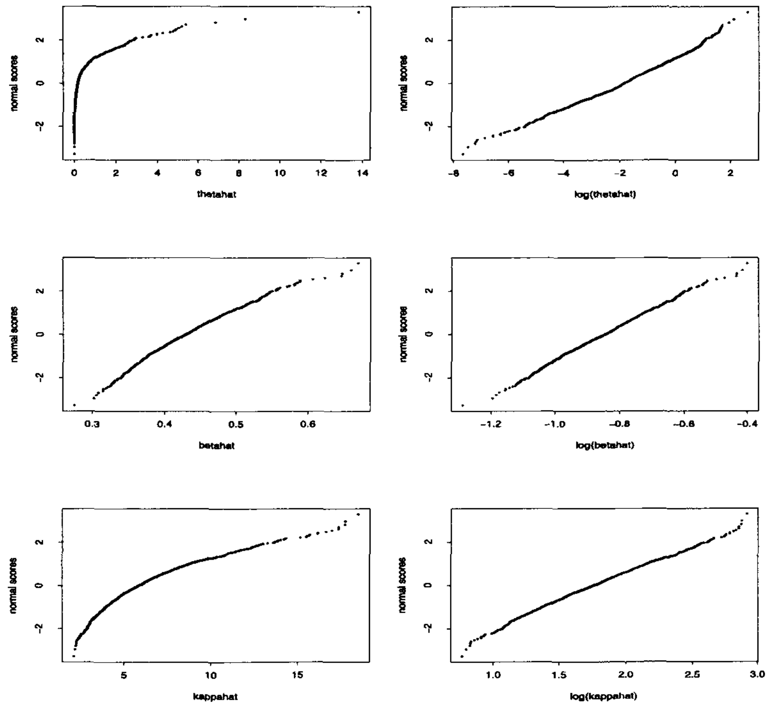


Fig. 7. Normal scores plot of 1000 parametric bootstrap replications of the maximum likelihood estimates of the parameters and their logarithms for the aircraft generator failure data.

simple most popular parametric model for a repairable system. Further modifications such as assuming κ to change between failures are certainly worth consideration. In the context of IGP, other parametric formulations are also strong contenders to MPLP. Some of these alternatives have been discussed in Berman (1981).

When multiple systems (or individuals) that are experiencing recurrence of certain events are under investigation, a nonparametric approach may be a more appropriate alternative to a fully parametric analysis. In the presence of covariates, a semiparametric regression model is often used to provide a robust and flexible framework for inference. Lawless *et al.* (2001) provides an excellent review of models and associated methodologies in this context.

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Appendix

PROOF OF LEMMA 2.2. (a) Noting that X_n is a linear function of the independent gamma random variables $Y_i = (T_i/\theta_0)^{\beta_0} - (T_{i-1}/\theta_0)^{\beta_0}$, $i = 1, \ldots, n$ with shape parameter κ_0 and scale parameter 1, the result follows from an application of the Lindeberg-Feller central limit theorem.

(b) It is possible to write $\overline{X}_n - X_n$ as a linear function of Y_i's. Using the interchange of summation formula

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j = \sum_{i=1}^{n}\sum_{j=i}^{n} b_i a_j, \tag{A.1}$$

we write

$$\sum_{i=1}^{n} X_i = \kappa_0^{-1}\sum_{i=1}^{n} i^{-1}\sum_{j=1}^{i} Y_j = \kappa_0^{-1}\sum_{i=1}^{n} Y_i \sum_{j=i}^{n} j^{-1}.$$

Also, $nX_n = \kappa_0^{-1}\sum_{i=1}^{n} Y_i = \kappa_0^{-1}\sum_{i=1}^{n} Y_i \sum_{j=i}^{n}(n-i+1)^{-1}$. Thus, we have

$$K_n \equiv n^{-1/2}\sum_{i=1}^{n}(X_i - X_n) = \kappa_0^{-1} n^{-1/2}\sum_{i=1}^{n} Y_i e_{in},$$

where $e_{in} = \sum_{j=i}^{n}\{j^{-1} - (n-i+1)^{-1}\}$.

Using (A.1) we have

$$E(K_n) = n^{-1/2}\sum_{i=1}^{n} e_{in} = n^{-1/2}\left(\sum_{i=1}^{n} i^{-1}\sum_{j=1}^{i} 1 - \sum_{i=1}^{n} 1\right) = 0.$$

Also, from the expression of K_n we readily obtain

$$\mathrm{Var}(K_n) = \frac{1}{n\kappa_0}\sum_{i=1}^{n} e_{in}^2$$

$$\kappa_0^{-4} n^{-2}\sum_{i=1}^{n} E(Y_i)^4 e_{in}^4 = \kappa_0^{-3}(\kappa_0^3 + 6\kappa_0^2 + 11\kappa_0 + 6) n^{-2}\sum_{i=1}^{n} e_{in}^4.$$

Setting a correspondence of e_{in} with a Riemann sum, we observe that as $n \to \infty$,

$$\mathrm{Var}(K_n) \to \kappa_0^{-1}\int_0^1\left\{\int_u^1\left(\frac{1}{v} - \frac{1}{1-u}\right)dv\right\}^2 du = \frac{1}{\kappa_0}$$

$$\kappa_0^{-4} n^{-2}\sum_{i=1}^{n} E(Y_i)^4 e_{in}^4$$

$$\sim n^{-1}\kappa_0^{-3}(\kappa_0^3 + 6\kappa_0^2 + 11\kappa_0 + 6)\int_0^1\left\{\int_u^1\left(\frac{1}{v} - \frac{1}{1-u}\right)dv\right\}^4 du \to 0.$$

These facts in conjunction with the result that $E(K_n) = 0$, enable us to use the central limit theorem to conclude part (b) of the lemma.

(c) Note that

$$\begin{aligned}\mathrm{Cov}[U_{3n}, n^{1/2}(X_n - \overline{X}_n)] &= \mathrm{Cov}\left[n^{-1/2}\sum_{i=1}^{n}(\log Y_i - \psi(\kappa_0)), n^{-1/2}\kappa_0^{-1}\sum_{i=1}^{n} Y_i e_{in}\right]\\ &= -(n\kappa_0)^{-1}\sum_{i=1}^{n}\mathrm{Cov}[(Y_i - \kappa_0)(\log Y_i - \psi(\kappa_0))]e_{in}\\ &= -(n\kappa_0)^{-1}\sum_{i=1}^{n} e_{in} = 0.\end{aligned}$$

(d) Let

$$G_{in} \equiv \log(X_n/X_i) - (X_n - X_i).$$

Since $(x-1)/x \le \log x \le x-1$ for $x > 0$, we have

$$\text{(A.2)} \quad \left(\frac{1}{X_n} - 1\right) n^{-1/2} \sum_{i=1}^{n} (X_n - X_i) \le n^{-1/2} \sum_{i=1}^{n} G_{in}$$
$$\le n^{-1/2} \sum_{i=1}^{n} (X_n - X_i) \left(\frac{1}{X_i} - 1\right).$$

We use Slutsky's theorem in conjunction with the results in parts (a) and (b) to conclude that the lower bound in expression (A.2) is $o_p(1)$. The upper bound equals

$$n^{-1/2} \sum_{i=1}^{n} \{(X_n - 1) - (X_i - 1)\} \left(\frac{1}{X_i} - 1\right)$$
$$= n^{1/2}(X_n - 1) \left\{ n^{-1} \sum_{i=1}^{n} \left(\frac{1}{X_i} - 1\right) \right\} - n^{-1/2} \sum_{i=1}^{n} (X_i - 1) \left(\frac{1}{X_i} - 1\right).$$

Denote the first and second terms on the right hand side by B_1 and B_2, respectively. By part (a) of the lemma, we have $B_1 = o_p(1)$. Now, by the Cauchy-Schwarz inequality,

$$B_2^2 \le (\log n)^{-1} \sum_{i=1}^{n} (X_i - 1)^2 \left\{ \frac{\log n}{n} \sum_{i=1}^{n} \left(\frac{1}{X_i} - 1\right)^2 \right\}.$$

Note that

$$E\left[\left|(\log n/n) \sum_{i=1}^{n} (1/X_i - 1)^2\right|\right] \leqslant (\log n/n) \sum_{i=1}^{n} E(1/X_i - 1)^2$$
$$= (\log n/n) \sum_{i=1}^{n} \frac{i\kappa_0 + 2}{(i\kappa_0 - 1)(i\kappa_0 - 2)}$$
$$\leqslant (\log n/n) \sum_{i=1}^{n} \frac{\kappa_0 + 2}{i(\kappa_0 - 1)(\kappa_0 - 2)} \to 0.$$

Hence using Markov inequality we have $n^{-1} \log n \sum_{i=1}^{n} (1/X_i - 1)^2 = o_p(1)$. To show that $(\log n)^{-1} \sum_{i=1}^{n} (X_i - 1)^2 = O_p(1)$ we note that

$$E\left\{ \frac{1}{\log n} \sum_{i=1}^{n} (X_i - 1)^2 \right\} = \frac{1}{\log n} \sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i\kappa_0} \to \frac{1}{\kappa_0}.$$

Thus, $(\log n)^{-1} \sum_{i=1}^{n} (X_i - 1)^2$ is bounded in expectation and hence is $O_p(1)$ which implies $B_2 = o_p(1)$. □

PROOF OF LEMMA 2.3. (a) Since $T_i^{\beta_0} > T_{i-1}^{\beta_0} > 0$ using the relation $(x-1)/x \le \log x \le x - 1$ for $x > 0$ we have

$$\frac{T_i^{\beta_0} - T_{i-1}^{\beta_0}}{T_i^{\beta_0}} \le \log \frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}} \le \frac{T_i^{\beta_0} - T_{i-1}^{\beta_0}}{T_{i-1}^{\beta_0}}$$

which implies

$$n^{-1/2}\sum_{i=2}^{n}\left(\frac{T_{i-1}^{\beta_0}}{T_i^{\beta_0}}-1\right)\le n^{-1/2}\sum_{i=2}^{n}\left(\frac{T_{i-1}^{\beta_0}}{T_i^{\beta_0}-T_{i-1}^{\beta_0}}\log\frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}}-1\right)\le 0.$$

Denote the lower bound by B_3. The proof is completed once we establish that $B_3=o_p(1)$. It is easy to check that the random variables

$$\text{(A.3)}\qquad Y_i^*=\frac{(T_{i-1}/\theta_0)^{\beta_0}}{(T_i/\theta_0)^{\beta_0}};\qquad i=2,\dots,n$$

are independently distributed as $Beta(i\kappa_0-\kappa_0,\kappa_0)$. So

$$\begin{aligned}E(B_3^2)&=\frac{1}{n}\sum_{i=2}^{n}\mathrm{Var}(Y_i^*)+\frac{1}{n}\left[\sum_{i=2}^{n}\{E(Y_i^*)-1\}\right]^2\\&=\frac{1}{n}\sum_{i=2}^{n}\frac{(i-1)}{i^2(i\kappa_0+1)}+\frac{(\log n)^2}{n}\left(\frac{1}{\log n}\sum_{i=2}^{n}\frac{1}{i}\right)^2\to 0.\end{aligned}$$

Thus $B_3=o_p(1)$.

(b) Since $0<T_{i-1}^{\beta_0}<T_i^{\beta_0}$, from part (a) of the lemma we have

$$\left(\frac{T_i^{\beta_0}-T_{i-1}^{\beta_0}}{T_i^{\beta_0}}\right)^2\le\left(\log\frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}}\right)^2\le\left(\frac{T_i^{\beta_0}-T_{i-1}^{\beta_0}}{T_{i-1}^{\beta_0}}\right)^2$$

which implies

$$\begin{aligned}n^{-1}\sum_{i=2}^{n}\left(\frac{T_{i-1}^{\beta_0}}{T_i^{\beta_0}}-1\right)&\le n^{-1}\sum_{i=2}^{n}\left\{\frac{T_{i-1}^{\beta_0}}{T_i^{\beta_0}-T_{i-1}^{\beta_0}}\left(\log\frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}}\right)^2-1\right\}\\&\le n^{-1}\sum_{i=2}^{n}\left(\frac{T_i^{\beta_0}}{T_{i-1}^{\beta_0}}-1\right).\end{aligned}$$

By part (a) of the lemma lower bound of the above inequality is $o_p(1)$. Denote the upper bound of the inequality by B_4. The proof is completed once we establish that $B_4=o_p(1)$. Note that

$$\begin{aligned}E(B_4^2)&=\frac{1}{n^2}\sum_{i=2}^{n}\mathrm{Var}(Y_i^{*^{-1}})+\left[\frac{1}{n}\sum_{i=2}^{n}\{E(Y_i^{*^{-1}})-1\}\right]^2\\&=\frac{1}{n^2}\sum_{i=2}^{n}\frac{(i\kappa_0-1)\kappa_0}{(i\kappa_0-\kappa_0-1)^2(i\kappa_0-\kappa_0-2)}+\left(\frac{1}{n}\sum_{i=2}^{n}\frac{\kappa_0}{i\kappa_0-\kappa_0-1}\right)^2\to 0.\end{aligned}$$

Hence $B_4=o_p(1)$. □

Proof of Lemma 2.4. (c) The result will follow by an application of Markov inequality once we show that $E_{\boldsymbol{\alpha}_0}(|c_{ij}^*(\boldsymbol{\alpha})-c_{ij}^*(\boldsymbol{\alpha}_0)|)\to 0$ uniformly in $\boldsymbol{\alpha}\in M_n(\boldsymbol{\alpha}_0)$. By Taylor series expansion we have

$$c_{22}^*(\boldsymbol{\alpha})-c_{22}^*(\boldsymbol{\alpha}_0)=(\kappa-\kappa_0)(\psi''(\kappa^*)+1/\kappa^{*2})$$

from which we obtain

$$E_{\boldsymbol{\alpha}_0}(|c_{22}^*(\boldsymbol{\alpha}) - c_{22}^*(\boldsymbol{\alpha}_0)|) \leqslant |\kappa - \kappa_0|(|\psi''(\kappa^*)| + 1/\kappa^{*2}) \\ \leqslant 2hn^{-\delta}\{\psi''(\kappa_0 + hn^{-\delta}) + 1/(\kappa_0 - hn^{-\delta})^2\}$$

for $\boldsymbol{\alpha} \in M_n(\boldsymbol{\alpha}_0)$. Since $\delta > 0$, we have the required uniform convergence for $E_{\boldsymbol{\alpha}_0}(|c_{22}^*(\boldsymbol{\alpha}) - c_{22}^*(\boldsymbol{\alpha}_0)|)$.

From (2.10) and (2.12) note that

$$c_{12}^*(\boldsymbol{\alpha}) = \log T_n - \frac{1}{n}\sum_{i=1}^{n} \log T_i - \frac{1}{n}\sum_{i=2}^{n} \frac{T_{i-1}^{\beta}}{T_i^{\beta} - T_{i-1}^{\beta}} \log \frac{T_i}{T_{i-1}}.$$

Then by Taylor series expansion we have

$$c_{12}^*(\boldsymbol{\alpha}) - c_{12}^*(\boldsymbol{\alpha}_0) = (\beta - \beta_0)\frac{1}{n}\sum_{i=2}^{n}\left\{\frac{T_i^{\beta^*}T_{i-1}^{\beta^*}}{(T_i^{\beta^*} - T_{i-1}^{\beta^*})^2}\left(\log\frac{T_i}{T_{i-1}}\right)^2\right\}.$$

Thus

$$E_{\boldsymbol{\alpha}_0}(|c_{12}^*(\boldsymbol{\alpha}) - c_{12}^*(\boldsymbol{\alpha}_0)|) \leqslant |\beta - \beta_0|\frac{1}{n}\sum_{i=2}^{n} E\left[\frac{T_i^{\beta^*}T_{i-1}^{\beta^*}}{(T_i^{\beta^*} - T_{i-1}^{\beta^*})^2}\left(\log\frac{T_i}{T_{i-1}}\right)^2\right].$$

Note that $\log x \leqslant x - 1$ for any real number $x > 0$. Since $0 < T_{i-1} < T_i$ we have

$$\log(T_i^{\beta^*}/T_{i-1}^{\beta^*}) \leqslant (T_i^{\beta^*} - T_{i-1}^{\beta^*})/T_{i-1}^{\beta^*}$$

which implies

$$\frac{T_i^{\beta^*}T_{i-1}^{\beta^*}}{(T_i^{\beta^*} - T_{i-1}^{\beta^*})^2}\left(\log\frac{T_i^{\beta^*}}{T_{i-1}^{\beta^*}}\right)^2 \leqslant \frac{T_i^{\beta^*}}{T_{i-1}^{\beta^*}} = Y_i^{*^{-\beta^*/\beta_0}},$$

where Y_i^* is defined in (A.3). Thus, we arrive at the inequality

$$E_{\boldsymbol{\alpha}_0}(|c_{12}^*(\boldsymbol{\alpha}) - c_{12}^*(\boldsymbol{\alpha}_0)|) \leqslant |\beta - \beta_0|\frac{1}{\beta^{*2}}\frac{1}{n}\sum_{i=2}^{n}\frac{\Gamma\{(i-1)\kappa_0 - \beta^*/\beta_0\}}{\Gamma\{(i-1)\kappa_0\}}\frac{\Gamma(i\kappa_0)}{\Gamma\{i\kappa_0 - \beta^*/\beta_0\}}.$$

From Abramowitz and Stegun ((1974), p. 257), we have

$$Z^{b-a}\frac{\Gamma(Z+a)}{\Gamma(Z+b)} \sim 1 + \frac{(a-b)(a+b-1)}{2Z} + \cdots.$$

Thus for $\boldsymbol{\alpha}$ in $M_n(\boldsymbol{\alpha}_0)$, we have

$$E_{\boldsymbol{\alpha}_0}(|c_{12}^*(\boldsymbol{\alpha}) - c_{12}^*(\boldsymbol{\alpha}_0)|) \leqslant 2hn^{-\delta}\frac{1}{(\beta_0 - hn^{-\delta})^2}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{i}{i-1}\right)^{1+hn^{-\delta}} \to 0.$$

Similarly using multivariate Taylor series expansion we can show that $E_{\boldsymbol{\alpha}_0}(|c_{11}^*(\boldsymbol{\alpha}) - c_{11}^*(\boldsymbol{\alpha}_0)|) \to 0$ uniformly in $\boldsymbol{\alpha} \in M_n(\boldsymbol{\alpha}_0)$. □

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