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EXACT ASYMPTOTICS FOR BOUNDARY CROSSINGS OF THE BROWNIAN BRIDGE WITH TREND WITH APPLICATION TO THE KOLMOGOROV TEST

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Abstract. We consider a boundary crossing probability of a Brownian bridge B_0 and a piecewise linear boundary function $u(t) - \gamma h(t)$. The main result of this paper is an asymptotic expansion for $\gamma \to \infty$ of the boundary crossing probability that $B_0(t)$ is larger than the piecewise linear boundary function $u(t) - \gamma h(t)$ for some t. Such probabilities occur for instance in the context of change point problems when the Kolmogorov test is used. Examples are discussed showing that the approximation is rather accurate even for small positive γ values.

Key words and phrases: Brownian bridge with trend, boundary crossing probability, exact asymptotics, extreme values, large deviations, Kolmogorov test.

1. Introduction

We consider boundary crossing probabilities of the Brownian bridge $B_0(t)$ and a piecewise linear boundary function $u(t) - \gamma h(t)$. For an arbitrary positive function $u(t)$, $t \in [0, 1]$, we are interested in

$$p(\gamma; h) := \boldsymbol{P}\{B_0(t) > u(t) - \gamma h(t) \text{ for some } t \in [0, 1]\}, \qquad \gamma > 0. \tag{1.1}$$

Such functions $p(\gamma; h)$ occur as asymptotic power functions of weighted Kolmogorov tests for testing the hypothesis that a regression function is constant against the alternative that an arbitrary regression function appears. Note that for this interpretation h corresponds to a fixed normalized alternative. For more details see Section 3.

There is a lot of literature dealing with the computation of boundary crossing probabilities $\boldsymbol{P}\{B_0(t) > \Gamma(t) \text{ for some } t \in [0, 1]\}$. A nice survey on boundary crossing probabilities is given in Siegmund (1986). If Γ is a straight line, the probability is well known, see for example Karatzas and Shreve (1991), pp. 264–265. Scheike (1992) gave an expression for the probability if Γ consists of two straight lines. Wang and Pötzelberger (1997) and Janssen and Kunz (2000) dealt with the case of a piecewise linear Γ and the Brownian motion instead of the Brownian bridge. These results can be easily transformed to the

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case of piecewise linear Γ and the Brownian bridge. Wang and Pötzelberger (1997) gave an integral expression for (1.1) which can be numerically determined by Monte Carlo simulations, see also Pötzelberger and Wang (2001) for two-sided boundaries. Novikov *et al.* (1999) and Pötzelberger and Wang (2001) obtained bounds for the approximation error of the integral expression if an arbitrary function is approximated by a piecewise linear function. Janssen and Kunz (2000) expanded the integral expression for piecewice linear functions in a sum of multivariate normal distribution functions. This sum becomes more and more complicated for an increasing number of linear pieces. Thus, (1.1) can be calculated numerically but not analytically by these formulas. In order to compare the weighted Kolmogorov tests with other tests, however, one needs some analytic expression for (1.1).

In this paper we consider the problem of accurately approximating this boundary crossing probability (1.1) for a rather large γ. Often the large deviations principle or extreme value theory can be applied to derive the leading term of such an approximation. Similar estimates using the large deviations principle are accurate usually up to the order $o(\gamma^2)$ for the log of the probability (1.1). Such results can be found e.g. in Varadhan (1984) and Ledoux (1996). By a first accurate estimate, Bischoff *et al.* (2003*a*, 2003*b*) derived an estimate for the log of the probability (1.1) up to the $o(\gamma)$-term where the leading term of this expression is of order γ^2. We are interested, however, in a more accurate estimate to obtain an approximation of the power function $p(\gamma; h)$. For that we need an approximation which is asymptotically equivalent to (1.1).

In Section 2 we derive such an exact asymptotic approximation of (1.1) for $\gamma h(t) - u(t)$ piecewise linear. In Section 3 we apply our result to some particular examples. Numerical calculations show that the approximation formula is rather accurate even for small values of γ.

2. Main results

Recently, Bischoff *et al.* (2003*a*, 2003*b*) have shown the following boundary crossing probability for the Brownian bridge and the boundary function $b_\gamma(t) := u(t) - \gamma h(t)$, $t \in [0,1]$. This probability occurs in the context of weighted Kolmogorov tests in a natural way, see Section 3 for details. Let $\tilde{h}$ be the smallest concave majorant of h and denote the right-hand derivative of $\tilde{h}$ on $[0,1)$ by $\tilde{h}'$. Let $\tilde{h}'(1)$ be defined as the left-hand derivative of $\tilde{h}$ in 1. Then under certain conditions (for details see Bischoff *et al.* (2003*a*, 2003*b*)) it is true that

$$\begin{aligned} (2.1) \qquad & \boldsymbol{P}\{\forall t \in [0,1] : B_0(t) < b_\gamma(t)\} \\ &= \exp\left(-\gamma^2\|\tilde{h}\|^2/2 - \gamma \int u d\tilde{h}' + o(\gamma)\right), \qquad \gamma \to \infty, \end{aligned}$$

where $\|\cdot\|$ denotes the norm of the reproducing kernel Hilbert space belonging to the Brownian bridge $B_0(t)$, $t \in [0,1]$. For results for the behavior of $\boldsymbol{P}\{\forall t \in [0,1] : B_0(t) < b_\gamma(t)\}$ for $\gamma \to 0$ see e.g. Janssen (1995) and Janssen and Kunz (2002).

In the following we derive the exact asymptotics ($\gamma \to \infty$) for the special case of a piecewise linear boundary function b_γ. For stating our main result we need the following notation. Let $\boldsymbol{x} = (x_1, \ldots, x_k)^\top \in I\!R^k$ and let $M \subset \{1, \ldots, k\}$. Then the vector that consists of the components whose indices belong to M is denoted by $\boldsymbol{x}_M := (x_i)_{i\in M}^\top \in I\!R^{|M|}$. Accordingly, as for vectors, A_{JI} denotes the $|J| \times |I|$ submatrix of the $k \times k$ matrix

A obtained by deleting all rows and all columns with indices in $\{1,\dots,k\}\setminus J$ and in $\{1,\dots,k\}\setminus I$, respectively. If $J=I$, then we write A_J instead of A_{JI}. In the last case we drop the index J if $J=\{1,\dots,k\}$. Further for two sets $L,M\subseteq\{0,1,\dots,k+1\}$, the set LM is defined by $LM:=\{i\in\{0,1,\dots,k\}: i\in L, i+1\in M\}$. The latter notation is not used when indexing matrices.

THEOREM 2.1. *Let $h:[0,1]\to\mathbb{R}$ be a continuous and piecewise linear function with $h(0)=h(1)=0$ and $h(t)>0$ for some $t\in(0,1)$, let $0=t_0<t_1<\cdots<t_k<t_{k+1}=1$, $k\geq 1$, be the points such that $\{(0,h(0)),(t_1,h(t_1)),\dots,(1,h(1))\}$ are all nodes of the polygon h, let $\tilde h\geq h$ be the smallest upper concave polygon of h, and let $I\subseteq\{1,\dots,k\}$ be the set of indices such that $\{(0,\tilde h(0)),(t_i,\tilde h(t_i)),(1,\tilde h(1))\mid i\in I\}$ are all nodes of $\tilde h$. Further we put*

$$J:=\{j\in\{1,\dots,k\}\setminus I: h(t_j)=\tilde h(t_j)\},\qquad \hat J:=J\cup\{0,k+1\},$$
$$K:=\{1,\dots,k\}\setminus(I\cup J)=\{j\in\{1,\dots,k\}: h(t_j)\neq\tilde h(t_j)\}.$$

Let $u:[0,1]\to\mathbb{R}$ be a continuous and piecewise linear function which is linear on every interval $[t_i,t_{i+1}]$, $i=0,\dots,k$ and with $u_0:=u(0)>0$, $u_{k+1}:=u(1)>0$ and put $\boldsymbol{u}:=(u(t_1),\dots,u(t_k))^\top$. Let Σ be the covariance matrix pertaining to the random vector $(B_0(t_1),\dots,B_0(t_k))$ of a Brownian bridge $B_0(t)$ with paths in $C([0,1])$. Then, we have

$$\begin{aligned}(2.2)\quad &\boldsymbol{P}\{\forall t\in[0,1]: B_0(t)<b_\gamma(t)\}\\ &=\boldsymbol{P}\{\forall t\in[0,1]: B_0(t)<u(t)-\gamma h(t)\}\\ &=c_{\boldsymbol{u},t_1,\dots,t_k}\exp\left(-\gamma^2\|\tilde h\|^2/2-\gamma\int u\,d\tilde h'\right)\gamma^{-|I|-2|II|-|I\hat J|-|\hat JI|}(1+o(1)),\\ &\qquad\gamma\to\infty,\end{aligned}$$

where

$$\begin{aligned}c_{\boldsymbol{u},t_1,\dots,t_k}&=(2\pi)^{-|I|/2}|\Sigma_I|^{-1/2}\\ &\quad\times\exp(-\boldsymbol{u}_I^\top(\Sigma_I)^{-1}\boldsymbol{u}_I/2)\left[\prod_{i\in II\cup I\hat J\cup\hat JI}2/(t_{i+1}-t_i)\right]u_0^{\mathbf{1}\{1\in I\}}u_{k+1}^{\mathbf{1}\{k\in I\}}D\\ &\quad\times\prod_{i-1\in KI}r_i^-\prod_{i\in IK}r_i^+\prod_{i\in I}\left[\frac{r_i^-+r_i^++2q_i}{q_i(r_i^-+q_i)(r_i^++q_i)(r_i^-+r_i^++q_i)}\right],\end{aligned}$$

$$q_i=\frac{h(t_i)-\tilde h(t_{i-1})}{t_i-t_{i-1}}-\frac{\tilde h(t_{i+1})-h(t_i)}{t_{i+1}-t_i}=\tilde h'(t_i-)-\tilde h'(t_i+),$$
$$r_i^-=2(\tilde h(t_{i-1})-h(t_{i-1}))/(t_i-t_{i-1}),$$
$$r_i^+=2(\tilde h(t_{i+1})-h(t_{i+1}))/(t_{i+1}-t_i),\qquad i\in I,$$
$$\begin{aligned}D&=(2\pi)^{-|J|/2}|((\Sigma_{\bar K})^{-1})_J|^{1/2}\\ &\quad\times\int_{\boldsymbol{x}_J\geq 0_J}\prod_{i\in JI}x_i\prod_{i-1\in IJ}x_i\prod_{i\in\hat J\hat J}[1-\exp(-2x_ix_{i+1}/(t_{i+1}-t_i))]\\ &\quad\times\exp(-(\boldsymbol{x}_J-\boldsymbol{u}_J+\Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{u}_I)^\top((\Sigma_{\bar K})^{-1})_J(\boldsymbol{x}_J-\boldsymbol{u}_J+\Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{u}_I)/2)d\boldsymbol{x}_J,\end{aligned}$$

with $\prod_{i\in\emptyset}\cdots := 1$ *and if* $J=\emptyset$ *then we put* $D := 1$.

Remark 2.1. a) Under the weaker assumption: the function $b_\gamma(t)$ is lower semi-continuous piecewise linear, we may derive along the same lines the exact asymptotic for the boundary crossing problem.

b) In the asymptotic result obtained by (2.2) the term in the exponential remains the same as long as we consider piecewise linear functions h that have the same $\tilde{h}$ and fixed u. The exponent of γ takes its minimum if $h=\tilde{h}$.

c) Note that the asymptotic result (2.2) depends only on the shape of h near the concave hull $\tilde{h}$.

d) Note further that $K=\emptyset$ implies $J=\emptyset$ and that $K=\emptyset$ is equivalent to $h=\tilde{h}$.

Putting specifically $h=\tilde{h}$ in the above theorem, we obtain:

THEOREM 2.2. *Under the assumptions of Theorem* 2.1 *we get for* $h=\tilde{h}$

$$\begin{aligned}(2.3)\qquad &\boldsymbol{P}\{\forall t\in[0,1]: B_0(t)<b_\gamma(t)\}\\ &= 2^{3k/2+1}\pi^{-k/2}\exp(-\boldsymbol{u}^\top\Sigma^{-1}\boldsymbol{u}/2)u_0u_{k+1}\prod_{i=0}^{k}(t_{i+1}-t_i)^{-3/2}\prod_{i=1}^{k}q_i^{-3}\\ &\quad\times\exp\left(-\gamma^2\|\tilde{h}\|^2/2-\gamma\int u\,d\tilde{h}'\right)\gamma^{-3k}(1+o(1)),\qquad \gamma\to\infty,\end{aligned}$$

with $q_i := \frac{h(t_i)-\tilde{h}(t_{i-1})}{t_i-t_{i-1}} - \frac{\tilde{h}(t_{i+1})-h(t_i)}{t_{i+1}-t_i} = \tilde{h}'(t_i-)-\tilde{h}'(t_i+)$.

PROOF OF THEOREM 2.1. We use the following abbreviations: $h_i := h(t_i)$, $\tilde{h}_i := \tilde{h}(t_i)$, $i=0,\dots,k+1$. In Bischoff *et al.* (2003*a*, 2003*b*) it was shown that for $\gamma>0$

$$\begin{aligned}(2.4)\qquad &\boldsymbol{P}\{\forall t\in[0,1]: B_0(t)<b_\gamma(t)\}\\ &= (2\pi)^{-k/2}\left[\prod_{i=0}^{k}(t_{i+1}-t_i)^{-1/2}\right]\int_{\boldsymbol{x}\le\boldsymbol{u}-\gamma\boldsymbol{h}}\exp(-\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}/2)q(\boldsymbol{x})d\boldsymbol{x}\end{aligned}$$

with

$$\begin{aligned}(2.5)\qquad q(\boldsymbol{x}) := &[1-\exp(-2u_0(u_1-\gamma h_1-x_1)/t_1)]\\ &\times\prod_{i=1}^{k-1}[1-\exp(-2(u_i-\gamma h_i-x_i)(u_{i+1}-\gamma h_{i+1}-x_{i+1})/(t_{i+1}-t_i))]\\ &\times[1-\exp(-2u_{k+1}(u_k-\gamma h_k-x_k)/(1-t_k))].\end{aligned}$$

By Lemma 4.1 of Bischoff *et al.* (2003*a*) the unique solution of the quadratic programming problem $\mathcal{P}_{\Sigma,\boldsymbol{h}}$: 'minimise $\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}$ with linear constraints $\boldsymbol{x}\ge\boldsymbol{h} := (h(t_1),\dots,h(t_k))^\top$' is given by $\boldsymbol{x}=\tilde{\boldsymbol{h}}=(\tilde{h}(t_1),\dots,\tilde{h}(t_k))^\top$ and

$$(2.6)\qquad \tilde{\boldsymbol{h}}_{I\cup J}=\boldsymbol{h}_{I\cup J},\qquad \text{and if }\ |K|>0,\quad \tilde{\boldsymbol{h}}_K>\boldsymbol{h}_K,$$

$$(2.7)\qquad (\Sigma_I)^{-1}\boldsymbol{h}_I>\boldsymbol{0}_I,$$

$$\begin{aligned}(2.8)\qquad \min_{\boldsymbol{x}\ge\boldsymbol{h}}\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x} &= \tilde{\boldsymbol{h}}^\top\Sigma^{-1}\tilde{\boldsymbol{h}} = \tilde{\boldsymbol{h}}_I^\top(\Sigma_I)^{-1}\tilde{\boldsymbol{h}}_I = \boldsymbol{h}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I\\ &= \int_0^1(\tilde{h}')^2d\lambda = \|\tilde{h}\|^2>0,\end{aligned}$$

with I, J, K as defined above. Clearly the index set I is unique and $|I| \geq 1$. Both index sets J and K can be empty. In order to avoid complicated notation, we assume in the following that $|J|, |K|$ are positive. The proof for J or K empty is easier, therefore omitted here.

We remark in passing that the inverse covariance matrix of Σ and its determinant are

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{t_1-t_0} + \frac{1}{t_2-t_1} & -\frac{1}{t_2-t_1} & 0 & \cdots & 0 \\ -\frac{1}{t_2-t_1} & \frac{1}{t_2-t_1} + \frac{1}{t_3-t_2} & -\frac{1}{t_3-t_2} & & \vdots \\ 0 & -\frac{1}{t_3-t_2} & \frac{1}{t_3-t_2} + \frac{1}{t_4-t_3} & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\frac{1}{t_k-t_{k-1}} \\ 0 & \cdots & 0 & -\frac{1}{t_k-t_{k-1}} & \frac{1}{t_k-t_{k-1}} + \frac{1}{t_{k+1}-t_k} \end{pmatrix},$$

$$|\Sigma^{-1}| = \prod_{i=0}^{k} (t_{i+1} - t_i)^{-1} > 0$$

respectively, and both $h, \tilde{h}$ are in the reproducing kernel Hilbert space corresponding to B_0.

Without loss of generality we can assume $\boldsymbol{x}^\top = (\boldsymbol{x}_{\bar{I}}^\top, \boldsymbol{x}_I^\top)$, $\boldsymbol{u}^\top = (\boldsymbol{u}_{\bar{I}}^\top, \boldsymbol{u}_I^\top)$, $\tilde{\boldsymbol{h}}^\top = (\tilde{\boldsymbol{h}}_{\bar{I}}^\top, \tilde{\boldsymbol{h}}_I^\top)$ with index set $\bar{I} := \{1, \ldots, k\} \setminus I = J \cup K$. To continue put $\boldsymbol{x}_{\gamma,\boldsymbol{u}} := (\boldsymbol{x}_{\bar{I}}^\top, \boldsymbol{x}_I^\top/\gamma)^\top - (\boldsymbol{u}_{\bar{I}}^\top, \boldsymbol{u}_I^\top)^\top + \gamma(\tilde{\boldsymbol{h}}_{\bar{I}}^\top, \tilde{\boldsymbol{h}}_I^\top)^\top$.

Next, transforming (2.4) by $\boldsymbol{x} \to -\boldsymbol{x}_{\gamma,\boldsymbol{u}}$, we obtain

$$\begin{aligned} \text{(2.9)} \qquad & \boldsymbol{P}\{\forall t \in [0,1] : B_0(t) < b_\gamma(t)\} \\ & = (2\pi)^{-k/2} \left[\prod_{i=0}^{k} (t_{i+1} - t_i)^{-1/2}\right] \gamma^{-|I|} \\ & \quad \times \int_{\boldsymbol{x} \geq \gamma(\boldsymbol{h} - \tilde{\boldsymbol{h}})} \exp(-\boldsymbol{x}_{\gamma,\boldsymbol{u}}^\top \Sigma^{-1} \boldsymbol{x}_{\gamma,\boldsymbol{u}}/2) q(-\boldsymbol{x}_{\gamma,\boldsymbol{u}}) d\boldsymbol{x}, \end{aligned}$$

with

$$\begin{aligned} q(-\boldsymbol{x}_{\gamma,\boldsymbol{u}}) = \prod_{i=0}^{k} [1 - \exp(-2(x_i/\gamma_{(i)} + \gamma(\tilde{h}_i - h_i)) \\ \times (x_{i+1}/\gamma_{(i+1)} + \gamma(\tilde{h}_{i+1} - h_{i+1}))/(t_{i+1} - t_i))], \end{aligned}$$

where $\gamma_{(i)} := \mathbf{1}(i \in \bar{I} \cup \{0, k+1\}) + \gamma \mathbf{1}(i \in I)$ for $0 \leq i \leq k+1$, $x_0 := u_0 > 0$, $x_{k+1} := u_{k+1} > 0$ are constant, and $h_0 = h_{k+1} = \tilde{h}_0 = \tilde{h}_{k+1} = 0$ has been used.

In the light of Lemma 4.2 of Bischoff *et al.* (2003*a*) we have for $\boldsymbol{y}, \boldsymbol{y}^* \in I\!R^k$ with $(\Sigma^{-1}\boldsymbol{y}^*)_{\bar{I}} = \boldsymbol{0}_{\bar{I}}$

$$\text{(2.10)} \qquad (\boldsymbol{y} + \boldsymbol{y}^*)^\top \Sigma^{-1} (\boldsymbol{y} + \boldsymbol{y}^*) = \boldsymbol{y}^\top \Sigma^{-1} \boldsymbol{y} + 2\boldsymbol{y}_I^\top (\Sigma_I)^{-1} \boldsymbol{y}_I^* + \boldsymbol{y}_I^{*\top} (\Sigma_I)^{-1} \boldsymbol{y}_I^*.$$

Since

$$(\tilde{h}_i - \tilde{h}_{i-1})/(t_i - t_{i-1}) = (\tilde{h}_{i+1} - \tilde{h}_i)/(t_{i+1} - t_i), \qquad i \in \bar{I}$$

we obtain

$$(\Sigma^{-1}\tilde{\boldsymbol{h}})_{\bar{I}} = \mathbf{0}_{\bar{I}}, \qquad \text{and} \qquad \forall i \in I : (\Sigma^{-1}\tilde{\boldsymbol{h}})_i \neq 0, \tag{2.11}$$

hence putting $\boldsymbol{y} = (\boldsymbol{x}_{\bar{I}}^\top, \boldsymbol{x}_I^\top/\gamma)^\top - \boldsymbol{u}$ and $\boldsymbol{y}^* = \gamma\tilde{\boldsymbol{h}}$, $\gamma > 0$ we get by (2.10)

$$\begin{aligned} \boldsymbol{x}_{\gamma,\boldsymbol{u}}^\top \Sigma^{-1} \boldsymbol{x}_{\gamma,\boldsymbol{u}} = {} & ((\boldsymbol{x}_{\bar{I}}^\top, \boldsymbol{x}_I^\top/\gamma) - \boldsymbol{u}^\top)\Sigma^{-1}((\boldsymbol{x}_{\bar{I}}^\top, \boldsymbol{x}_I^\top/\gamma)^\top - \boldsymbol{u}) \\ & + 2\boldsymbol{x}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I - 2\gamma \boldsymbol{u}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I + \gamma^2 \boldsymbol{h}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I. \end{aligned} \tag{2.12}$$

Using once again Lemma 4.1 of Bischoff *et al.* (2003*a*) we have

$$-\int u d\tilde{h}' = \boldsymbol{u}^\top \Sigma^{-1}\tilde{\boldsymbol{h}} = \boldsymbol{u}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I < \infty. \tag{2.13}$$

Thus we may write as $\gamma \to \infty$

$$\begin{aligned} \boldsymbol{x}_{\gamma,\boldsymbol{u}}^\top \Sigma^{-1} \boldsymbol{x}_{\gamma,\boldsymbol{u}} = {} & O(1/\gamma) + \gamma^2\|\tilde{h}\|^2 + 2\gamma \int u d\tilde{h}' \\ & + ((\boldsymbol{x}_{\bar{I}}^\top, \mathbf{0}_{\mathbf{I}}^\top) - \boldsymbol{u}^\top)\Sigma^{-1}((\boldsymbol{x}_{\bar{\mathbf{I}}}^\top, \mathbf{0}_{\mathbf{I}}^\top)^\top - \boldsymbol{u}) + 2\boldsymbol{x}_{\mathbf{I}}^\top(\Sigma_{\mathbf{I}})^{-1}\boldsymbol{h}_{\mathbf{I}} \end{aligned}$$

and

$$\begin{aligned} & 1 - \exp(-2(x_i/\gamma_{(i)} + \gamma(\tilde{h}_i - h_i))(x_{i+1}/\gamma_{(i+1)} + \gamma(\tilde{h}_{i+1} - h_{i+1}))/(t_{i+1} - t_i)) \\ & = \begin{cases} [1 - \exp(-2x_i(\tilde{h}_{i+1} - h_{i+1})/(t_{i+1} - t_i))](1 + o(1)), & i \in IK, \\ [1 - \exp(-2x_{i+1}(\tilde{h}_i - h_i)/(t_{i+1} - t_i))](1 + o(1)), & i \in KI, \\ [1 - \exp(-2x_i x_{i+1}/(t_{i+1} - t_i))](1 + o(1)), & i \in \hat{J}\hat{J}, \\ 2x_i x_{i+1}/((t_{i+1} - t_i)\gamma^2)(1 + o(1)), & i \in II, \\ 2x_i x_{i+1}/((t_{i+1} - t_i)\gamma)(1 + o(1)), & i \in I\hat{J} \text{ or } i \in \hat{J}I, \\ 1 + o(1), & \text{otherwise.} \end{cases} \end{aligned}$$

Next, substituting in (2.9) we obtain further

$$\begin{aligned} & \boldsymbol{P}\{\forall t \in [0,1] : B_0(t) < b_\gamma(t)\} \\ & = (1 + o(1))(2\pi)^{-k/2} \left[\prod_{i=0}^{k} (t_{i+1} - t_i)^{-1/2}\right] \left[\prod_{i \in II \cup I\hat{J} \cup \hat{J}I} 2/(t_{i+1} - t_i)\right] \\ & \quad \times \gamma^{-|I| - 2|II| - |I\hat{J}| - |\hat{J}I|} \exp\left(-\gamma^2\|\tilde{h}\|^2/2 - \gamma \int u d\tilde{h}'\right) \\ & \quad \times \int_{\boldsymbol{x} \geq \gamma(\boldsymbol{h} - \tilde{\boldsymbol{h}})} \exp(O(1/\gamma) - ((\boldsymbol{x}_{\bar{I}}^\top, \mathbf{0}_I^\top) - \boldsymbol{u}^\top)\Sigma^{-1} \\ & \qquad \times ((\boldsymbol{x}_{\bar{I}}^\top, \mathbf{0}_I^\top)^\top - \boldsymbol{u})/2 - \boldsymbol{x}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I) \prod_{i \in II \cup I\hat{J} \cup \hat{J}I} [x_i x_{i+1}] \\ & \quad \times \prod_{i \in IK} [1 - \exp(-2x_i(\tilde{h}_{i+1} - h_{i+1})/(t_{i+1} - t_i))] \\ & \quad \times \prod_{i \in KI} [1 - \exp(-2x_{i+1}(\tilde{h}_i - h_i)/(t_{i+1} - t_i))] \\ & \quad \times \prod_{i \in \hat{J}\hat{J}} [1 - \exp(-2x_i x_{i+1}/(t_{i+1} - t_i))] d\boldsymbol{x} \end{aligned}$$

$$=: c_{\boldsymbol{u},t_1,\dots,t_k}(\gamma)\exp\left(-\gamma^2\|\tilde{h}\|^2/2-\gamma\int u d\tilde{h}'\right)\gamma^{-|I|-2|II|-|I\hat{J}|-|\hat{J}I|}(1+o(1)) \quad \text{for} \quad \gamma\to\infty.$$

By the definition of the index set I and from (2.6) we get

$$\lim_{\gamma\to\infty}\gamma(\boldsymbol{h}-\tilde{\boldsymbol{h}})=\tilde{\boldsymbol{x}}_\infty, \quad \text{with} \quad (\tilde{\boldsymbol{x}}_\infty)_{I\cup J}=\boldsymbol{0}_{I\cup J}, \quad (\tilde{\boldsymbol{x}}_\infty)_K=-\boldsymbol{\infty}_K.$$

Further by Lemma A.1 and (2.13)

$$\begin{aligned}\boldsymbol{x}_{\gamma,\boldsymbol{u}}^\top\Sigma^{-1}\boldsymbol{x}_{\gamma,\boldsymbol{u}} &\geq \gamma^2\boldsymbol{h}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I-2\gamma\boldsymbol{u}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I\\ &\quad+(\boldsymbol{x}_{\bar{I}}-\boldsymbol{u}_{\bar{I}})^\top(\Sigma_{\bar{I}})^{-1}(\boldsymbol{x}_{\bar{I}}-\boldsymbol{u}_{\bar{I}})+2\boldsymbol{x}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I\\ &=\gamma^2\|\tilde{h}\|^2+2\gamma\int u d\tilde{h}'+(\boldsymbol{x}_{\bar{I}}-\boldsymbol{u}_{\bar{I}})^\top(\Sigma_{\bar{I}})^{-1}(\boldsymbol{x}_{\bar{I}}-\boldsymbol{u}_{\bar{I}})+2\boldsymbol{x}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I,\end{aligned}$$

hence Lebesgue's Bounded Convergence Theorem (see e.g. Theorem 1.21 of Kallenberg (1997)) implies

$$\begin{aligned}\lim_{\gamma\to\infty}c_{\boldsymbol{u},t_1,\dots,t_k}(\gamma) &= (2\pi)^{-k/2}\left[\prod_{i=0}^{k}(t_{i+1}-t_i)^{-1/2}\right]\left[\prod_{i\in II\cup I\hat{J}\cup\hat{J}I}2/(t_{i+1}-t_i)\right]\\ &\quad\times\int_{\boldsymbol{x}\geq\tilde{\boldsymbol{x}}_\infty}\exp(-((\boldsymbol{x}_{\bar{I}}^\top,\boldsymbol{0}_I^\top)-\boldsymbol{u}^\top)\Sigma^{-1}((\boldsymbol{x}_{\bar{I}}^\top,\boldsymbol{0}_I^\top)^\top-\boldsymbol{u})/2\\ &\qquad\qquad-\boldsymbol{x}_I^\top(\Sigma_I)^{-1}\boldsymbol{h}_I)\\ &\quad\times\prod_{i\in II\cup I\hat{J}\cup\hat{J}I}[x_ix_{i+1}]\prod_{i\in IK}[1-\exp(-2x_i(\tilde{h}_{i+1}-h_{i+1})/(t_{i+1}-t_i))]\\ &\quad\times\prod_{i\in KI}[1-\exp(-2x_{i+1}(\tilde{h}_i-h_i)/(t_{i+1}-t_i))]\\ &\quad\times\prod_{i\in\hat{J}\hat{J}}[1-\exp(-2x_ix_{i+1}/(t_{i+1}-t_i))]d\boldsymbol{x}\\ &=: c_{\boldsymbol{u},t_1,\dots,t_k}.\end{aligned}$$

It can be easily shown that $c_{\boldsymbol{u},t_1,\dots,t_k}$ is some positive constant. This constant can be calculated as follows. To this end we define

$$\begin{aligned}&\bar{J}:=\{1,\dots,k\}\setminus J=I\cup K,\\ &\bar{K}:=\{1,\dots,k\}\setminus K=I\cup J,\\ &\bar{K}I\bar{K}:=\{i\in I: i-1,i+1\in\bar{K}\cup\{0,k+1\}\},\\ &\bar{K}IK:=\{i\in I: i-1\in\bar{K}\cup\{0\}, i+1\in K\},\\ &KI\bar{K}:=\{i\in I: i-1\in K, i+1\in\bar{K}\cup\{k+1\}\},\\ &KIK:=\{i\in I: i-1,i+1\in K\}.\end{aligned}$$

Again we assume that the index sets defined above are non-empty. The other case can be dealt with similarly.

Using Schur complements, we have

$$
\begin{aligned}
&((\boldsymbol{x}_{\bar{I}}^\top, \boldsymbol{0}_I^\top) - \boldsymbol{u}^\top)\Sigma^{-1}((\boldsymbol{x}_{\bar{I}}^\top, \boldsymbol{0}_I^\top)^\top - \boldsymbol{u}) \\
&\quad = \boldsymbol{u}_I^\top (\Sigma_I)^{-1}\boldsymbol{u}_I + (\boldsymbol{x}_{\bar{I}} - \boldsymbol{u}_{\bar{I}} + \Sigma_{\bar{I}I}(\Sigma_I)^{-1}\boldsymbol{u}_I)^\top S^{-1}(\boldsymbol{x}_{\bar{I}} - \boldsymbol{u}_{\bar{I}} + \Sigma_{\bar{I}I}(\Sigma_I)^{-1}\boldsymbol{u}_I) \\
&\quad = \boldsymbol{u}_I^\top (\Sigma_I)^{-1}\boldsymbol{u}_I + (\boldsymbol{x}_J - \boldsymbol{u}_J + \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{u}_I)^\top (S_J)^{-1}(\boldsymbol{x}_J - \boldsymbol{u}_J + \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{u}_I) \\
&\qquad + \boldsymbol{y}_K^\top(\Sigma^{-1})_K\boldsymbol{y}_K,
\end{aligned}
$$

with

$$
\begin{aligned}
&S := ((\Sigma^{-1})_{\bar{I}})^{-1}, \\
&\boldsymbol{y}_K := \boldsymbol{x}_K - \boldsymbol{u}_K + \Sigma_{KI}(\Sigma_I)^{-1}\boldsymbol{u}_I - S_{KJ}(S_J)^{-1}(\boldsymbol{x}_J - \boldsymbol{u}_J + \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{u}_I).
\end{aligned}
$$

We get

$$
\begin{aligned}
c_{\boldsymbol{u},t_1,\dots,t_k} = {} & (2\pi)^{-k/2}\prod_{i=0}^{k}(t_{i+1}-t_i)^{-1/2}\left[\prod_{i\in II\cup I\hat{J}\cup\hat{J}I} 2/(t_{i+1}-t_i)\right] u_0^{\mathbf{1}\{1\in I\}}u_{k+1}^{\mathbf{1}\{k\in I\}} \\
& \times \exp(-\boldsymbol{u}_I^\top(\Sigma_I)^{-1}\boldsymbol{u}_I/2) \\
& \times \int_{\boldsymbol{x}_J\ge \boldsymbol{0}_J} \exp(-(\boldsymbol{x}_J - \boldsymbol{u}_J + \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{u}_I)^\top \\
& \qquad\qquad \times (S_J)^{-1}(\boldsymbol{x}_J - \boldsymbol{u}_J + \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{u}_I)/2)\prod_{i\in JI} x_i \prod_{i\in IJ} x_{i+1} \\
& \times \prod_{i\in\hat{J}\hat{J}}[1-\exp(-2x_ix_{i+1}/(t_{i+1}-t_i))]d\boldsymbol{x}_J \\
& \times \int_{\mathbb{R}^{|K|}} \exp(-\boldsymbol{y}_K^\top(\Sigma^{-1})_K\boldsymbol{y}_K/2)d\boldsymbol{y}_K \\
& \times \prod_{i\in\bar{K}I\bar{K}}\int_0^\infty x_i^2\exp(-x_iq_i)dx_i \\
& \times \prod_{i\in\bar{K}IK}\int_0^\infty x_i\exp(-x_iq_i)[1-\exp(-2x_i(\tilde{h}_{i+1}-h_{i+1})/(t_{i+1}-t_i))]dx_i \\
& \times \prod_{i\in KI\bar{K}}\int_0^\infty x_i\exp(-x_iq_i)[1-\exp(-2x_i(\tilde{h}_{i-1}-h_{i-1})/(t_i-t_{i-1}))]dx_i \\
& \times \prod_{i\in KIK}\int_0^\infty \exp(-x_iq_i)[1-\exp(-2x_i(\tilde{h}_{i-1}-h_{i-1})/(t_i-t_{i-1}))] \\
& \times [1-\exp(-2x_i(\tilde{h}_{i+1}-h_{i+1})/(t_{i+1}-t_i))]dx_i.
\end{aligned}
$$

Next, using

$$
\begin{aligned}
&\int_0^\infty s^2\exp(-qs)ds = 2/q^3, \\
&\int_0^\infty s\exp(-qs)(1-\exp(-cs))ds = 1/q^2 - 1/(c+q)^2,
\end{aligned}
$$

$$\int_0^\infty \exp(-qs)(1-\exp(-cs))(1-\exp(-as))ds$$
$$= \frac{ac(a+c+2q)}{q(c+q)(a+q)(a+c+q)}, \quad c,a,q>0,$$
$$((((\Sigma^{-1})_{\bar{I}})^{-1})_J)^{-1} = ((\Sigma_{\bar{I}} - \Sigma_{\bar{I}I}(\Sigma_I)^{-1}\Sigma_{I\bar{I}})_J)^{-1} = (\Sigma_J - \Sigma_{JI}(\Sigma_I)^{-1}\Sigma_{IJ})^{-1}$$
$$= ((\Sigma_{\bar{K}})^{-1})_J,$$
$$|\Sigma| = |\Sigma_{\bar{K}}||((\Sigma^{-1})_K)^{-1}|,$$
$$|\Sigma_{\bar{K}}| = |\Sigma_I||(((\Sigma_{\bar{K}})^{-1})_J)^{-1}|$$

and putting

$$D = (2\pi)^{-|J|/2}|((\Sigma_{\bar{K}})^{-1})_J|^{1/2}$$
$$\times \int_{\boldsymbol{x}_J \geq 0_J} \prod_{i\in JI} x_i \prod_{i-1\in IJ} x_i \prod_{i\in \hat{J}\hat{J}} [1-\exp(-2x_i x_{i+1}/(t_{i+1}-t_i))]$$
$$\times \exp(-(\boldsymbol{x}_J - \boldsymbol{u}_J + \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{u}_I)^\top((\Sigma_{\bar{K}})^{-1})_J(\boldsymbol{x}_J - \boldsymbol{u}_J + \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{u}_I)/2)d\boldsymbol{x}_J,$$

we get

$$c_{\boldsymbol{u},t_1,\dots,t_k} = (2\pi)^{-|I|/2}|\Sigma_I|^{-1/2}\exp(-\boldsymbol{u}_I^\top(\Sigma_I)^{-1}\boldsymbol{u}_I/2)$$
$$\times \left[\prod_{i\in II\cup I\hat{J}\cup \hat{J}I} 2/(t_{i+1}-t_i)\right] u_0^{\mathbf{1}\{1\in I\}} u_{k+1}^{\mathbf{1}\{k\in I\}} D$$
$$\times \prod_{i\in \bar{K}I\bar{K}} (2/q_i^3) \prod_{i\in \bar{K}IK} [1/q_i^2 - 1/(r_i^+ + q_i)^2]$$
$$\times \prod_{i\in KI\bar{K}} [1/q_i^2 - 1/(r_i^- + q_i)^2] \prod_{i\in KIK} \left[\frac{r_i^- r_i^+(r_i^- + r_i^+ + 2q_i)}{q_i(r_i^- + q_i)(r_i^+ + q_i)(r_i^- + r_i^+ + q_i)}\right]$$
$$= (2\pi)^{-|I|/2}|\Sigma_I|^{-1/2}\exp(-\boldsymbol{u}_I^\top(\Sigma_I)^{-1}\boldsymbol{u}_I/2)$$
$$\times \left[\prod_{i\in II\cup I\hat{J}\cup \hat{J}I} 2/(t_{i+1}-t_i)\right] u_0^{\mathbf{1}\{1\in I\}} u_{k+1}^{\mathbf{1}\{k\in I\}} D$$
$$\times \prod_{i-1\in KI} r_i^- \prod_{i\in IK} r_i^+ \prod_{i\in I} \left[\frac{r_i^- + r_i^+ + 2q_i}{q_i(r_i^- + q_i)(r_i^+ + q_i)(r_i^- + r_i^+ + q_i)}\right].$$

Thus the assertion of the theorem follows. □

3. Applications

In this section we explain to which extent our asymptotic results can be useful for a test problem. To this end we consider an ordinary regression model and we are interested in testing the hypothesis that the unknown regression function f is constant against the alternative that it is not constant. We want to test the above problem asymptotically with the help of the residual partial sums limit process, see MacNeill (1978*a*, 1978*b*), Bischoff (1998), Bischoff and Miller (2000), Bischoff *et al.* (2003*a*, 2003*b*). Let the regression function f have bounded variation. Moreover, we consider in the following f as a function in $L_2([0,1],\lambda)$ where λ is the Lebesgue measure. Putting

$$g_f(t) := \int_0^t f(s)ds - t\int_0^1 f(s)ds,$$

the residual partial sums limit process is given by the Gaussian process

$$B_0(t) + g_f(t), \qquad t \in [0,1].$$

Note that g_f is identically 0 if and only if f is identically constant (as a function in $L_2([0,1],\lambda)$). Thus our original test problem

$$H_0 : f \equiv \text{constant} \quad \text{against} \quad K : f \not\equiv \text{constant}$$

is equivalent to the test problem

$$H_0 : g_f \equiv 0 \quad \text{against} \quad K : g_f \not\equiv 0.$$

In the following we only consider the one sided test problem

$$H_0 : g_f \equiv 0 \quad \text{against} \quad K : g_f(t^*) > 0 \quad \text{for some } t^* \in [0,1].$$

By our result (Theorem 2.1) the following definitions are useful

$$\begin{aligned} &h(t) = h_f(t) = \|\tilde{g_f}\|^{-1} g_f, \quad \text{if} \quad f \not\equiv \text{constant},\\ &h(t) = h_f(t) \equiv 0, \quad \text{if} \quad f \equiv \text{constant},\\ &\gamma = \gamma_f = \|\tilde{g_f}\|. \end{aligned}$$

A suitable test for our test problem is the weighted Kolmogorov test. By the above defintions it can be expressed by

$$\text{Reject } H_0 \text{ if and only if } B_0(t^*) + \gamma h(t^*) \geq u(t^*) \text{ for some } t^* \in (0,1),$$

where $u : (0,1) \to (0,\infty]$ is chosen such that we obtain a size α test. For similar investigations for signal plus noise processes with the Brownian motion instead of the Brownian bridge see Bischoff *et al.* (2003*c*). It is worth mentioning that the residual partial sums limit process is obtained by a local limit theorem under the alternative, see Bischoff and Miller (2000). Thus our considerations on the power corresponds with the Pitman efficiency in case $\gamma = \gamma_f$ is large enough for the fixed alternative f.

To simplify the following considerations, we explain only the unweighted Kolmogorov test, that is $u(t) \equiv u$ (constant). For controlling the type I error by $\alpha \in (0,1)$, we have to fix u such that

$$\boldsymbol{P}\{B_0(t^*) \geq u \text{ for some } t^* \in (0,1)\} = \alpha.$$

This is fulfilled for $u = \sqrt{-\frac{1}{2}\log(\alpha)}$. In the succeeding examples, we fix $\alpha = 0.05$, hence $u = 1.22387$. Then the power of the Kolmogorov test in the direction of h is given by

$$p(\gamma, h) = \boldsymbol{P}\{B_0(t^*) + \gamma h(t^*) \geq u \text{ for some } t^* \in (0,1)\}. \tag{3.1}$$

Obviously, if $h : [0,1] \to I\!R$ fulfills the assumptions of Theorem 2.1 and if additionally $h(t_0) > 0$ for some $t_0 \in (0,1)$, then, for γ sufficiently large, the power of the Kolmogorov test in the direction of h can be approximated by

$$1 - p_{EA}(\gamma, h) = c_{\boldsymbol{u},t_1,\ldots,t_k} \exp\left(-\gamma^2/2 - \gamma \int u d\tilde{h}'\right) \gamma^{-|I|-2|II|-|I\hat{J}|-|\hat{J}I|}.$$

The following three examples show that we already have a good approximation for surprisingly small values of the parameter γ. We show this result by computing the power (3.1) numerically by formula (2.4) and by comparing these outcomes with the corresponding approximations received by the formula of Theorem 2.1. Note that the direct numerical calculation of (2.4) is only possible if k (the number of linear pieces) is small.

Example 3.1. For a fixed $\tau \in (0,1)$ we consider the trend

$$\tilde{h}(t) = h(t) = h_\tau(t) = \frac{\min\{t(1-\tau), (1-t)\tau\}}{\sqrt{\tau(1-\tau)}}, \qquad t \in [0,1].$$

The constants occurring in (2.3) of Theorem 2.2 are given by $k = 1$, $t_0 = 0$, $t_1 = \tau$, $t_2 = 1$ and

$$\|h_\tau\|^2 = 1, \qquad -\int u dh' = u/\sqrt{\tau(1-\tau)}, \qquad \boldsymbol{u}^\top \Sigma^{-1} \boldsymbol{u} = u^2(\tau(1-\tau))^{-1},$$
$$q_1 = 1/\sqrt{\tau(1-\tau)}.$$

Therefore, for γ sufficiently large the power is approximatively given by

$$\begin{aligned} \boldsymbol{P}\{\exists t \in [0,1] : B_0(t) \geq b_\gamma(t)\} &\approx p_{EA}(\gamma, h_\tau) \\ &= 1 - 4u^2(2/\pi)^{1/2}\gamma^{-3} \\ &\quad \times \exp\left(-\gamma^2 \frac{1}{2} + \gamma \frac{u}{\sqrt{\tau(1-\tau)}} - \frac{u^2}{2\tau(1-\tau)}\right). \end{aligned}$$

Next, we compare in Table 1 the (exact asymptotic) approximation $p_{EA}(\gamma, h_\tau)$ of the power given by the formula of Theorem 2.2 with the exact power $p(\gamma, h_\tau)$ numerically computed by formula (2.9) for several values for γ and τ. Figure 1 shows the exact power $p(\gamma, h_\tau)$ and the exact asymptotic approximation $p_{EA}(\gamma, h_\tau)$ for $\tau = 0.2$ and $\tau = 0.5$. Table 2 and Fig. 2 show the relative approximation error

$$\frac{1 - p_{EA}(\gamma, h_\tau)}{1 - p(\gamma, h_\tau)} - 1$$

by using p_{EA} instead of the exact p.

Example 3.2. Let us consider the following concave function as a trend

$$h(z) = \tilde{h}(z) = \begin{cases} \sqrt{2}z, & z \in [0,1/4), \\ \sqrt{2}/4, & z \in [1/4,3/4), \\ \sqrt{2}(1-z), & z \in [3/4,1], \end{cases} \qquad \tilde{h}'(z) = \begin{cases} \sqrt{2}, & z \in [0,1/4), \\ 0, & z \in [1/4,3/4), \\ -\sqrt{2}, & z \in [3/4,1]. \end{cases}$$

Table 1. Exact asymptotic approximation $p_{EA}(\gamma, h_\tau)$ and exact value $p(\gamma, h_\tau)$ of the power.

	$\gamma = 2$		$\gamma = 4$		$\gamma = 6$	
τ	p_{EA}	p	p_{EA}	p	p_{EA}	p
0.1	0.9312	0.2491	0.9255	0.7752	0.9965	0.9938
0.2	0.6592	0.4085	0.9520	0.9354	0.9997	0.9997
0.3	0.5228	0.4976	0.9691	0.9665	0.9999	0.9999
0.4	0.4722	0.5408	0.9758	0.9759	1.0000	1.0000
0.5	0.4594	0.5538	0.9776	0.9782	1.0000	1.0000

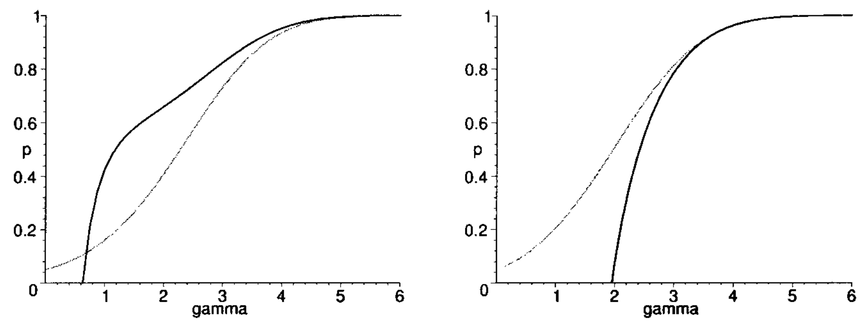

Fig. 1. Exact power (grey) and the exact asymptotic approximation (black) for $\tau = 0.2$ (left figure) and $\tau = 0.5$ (right figure) in Example 3.1.

Table 2. The relative approximation error $(1 - p_{EA}(\gamma, h_\tau))/(1 - p(\gamma, h_\tau)) - 1$ for Example 3.1.

τ	$\gamma = 2$	$\gamma = 4$	$\gamma = 6$	$\gamma = 8$	$\gamma = 10$	$\gamma = 20$
0.1	−91%	−67%	−43%	−27%	−19%	−5%
0.2	−42%	−26%	−15%	−9%	−6.2%	−1.6%
0.3	−5.0%	−7.7%	−4.8%	−3.0%	−2.0%	−0.6%
0.4	15%	0.3%	−0.6%	−0.5%	−0.4%	−0.1%
0.5	21%	2.6%	0.6%	0.2%	0.09%	0.008%

The constants occurring in (2.3) of Theorem 2.2 are given by $k = 2$, $t_0 = 0$, $t_1 = 1/4$, $t_2 = 3/4$, $t_3 = 1$ and

$$\|\tilde{h}\|^2 = 1, \qquad -\int u d\tilde{h}' = \boldsymbol{u}^\top \Sigma^{-1} \tilde{\boldsymbol{h}} = 2\sqrt{2}u, \qquad \boldsymbol{u}^\top \Sigma^{-1} \boldsymbol{u} = 8u^2,$$

$$\prod_{i=0}^{k} (t_{i+1} - t_i)^{-1} = 32, \qquad q_1 = q_2 = \sqrt{2}.$$

For γ sufficiently large the power is approximatively given by

$$\begin{aligned} \boldsymbol{P}\{\exists t \in [0,1] : B_0(t) \geq b_\gamma(t)\} &\approx p_{EA}(\gamma, h) \\ &= 1 - 256\frac{\sqrt{2}}{\pi} u^2 \exp(-4u^2)\gamma^{-6} \exp\left(-\gamma^2 \frac{1}{2} + \gamma 2\sqrt{2}u\right). \end{aligned}$$

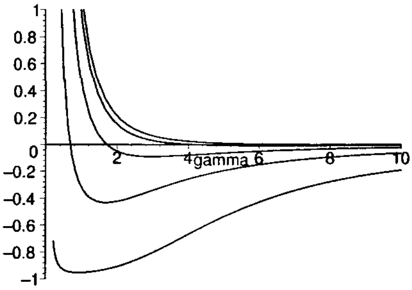

Fig. 2. Relative approximation error $\frac{1-p_{EA}(\gamma,h_\tau)}{1-p(\gamma,h_\tau)} - 1$ made using p_{EA} instead of the exact p for $\tau = 0.1, 0.2, 0.3, 0.4, 0.5$ (from bottom till top) for Example 3.1.

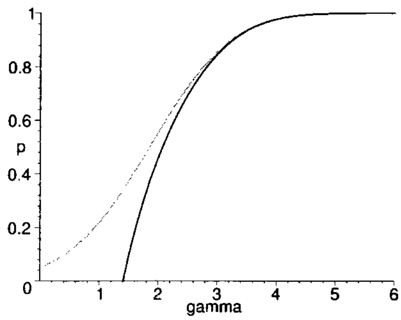

Fig. 3. Power $p(\gamma, h)$ (grey) and the exact asymptotic approximation $p_{EA}(\gamma, h)$ (black) for Example 3.2.

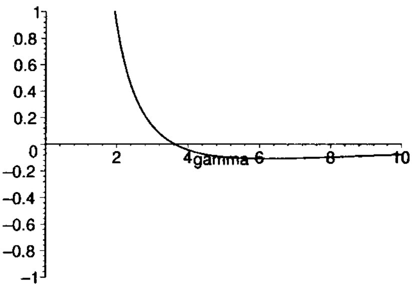

Fig. 4. Relative approximation error $\frac{1-p_{EA}(\gamma,h)}{1-p(\gamma,h)} - 1$ made using p_{EA} instead of the exact p for Example 3.2.

Next, we compare in Fig. 3 the (exact asymptotic) approximation $p_{EA}(\gamma, h)$ of the power given by the formula of Theorem 2.2 with the numerical results for the power $p(\gamma, h)$ as a function of γ. Figure 4 shows the relative approximation error.

Example 3.3. Let us consider the following non-concave function as a trend

$$h(z) = \begin{cases} \sqrt{2}z, & z \in [0, 1/4), \\ \sqrt{2}\left(\dfrac{1}{2} - z\right), & z \in [1/4, 1/2), \\ \sqrt{2}\left(z - \dfrac{1}{2}\right), & z \in [1/2, 3/4), \\ \sqrt{2}(1 - z), & z \in [3/4, 1]. \end{cases}$$

We have

$$\tilde{h}(z) = \begin{cases} \sqrt{2}z, & z \in [0, 1/4), \\ \sqrt{2}/4, & z \in [1/4, 3/4), \\ \sqrt{2}(1 - z), & z \in [3/4, 1], \end{cases} \qquad \tilde{h}'(z) = \begin{cases} \sqrt{2}, & z \in [0, 1/4), \\ 0, & z \in [1/4, 3/4), \\ -\sqrt{2}, & z \in [3/4, 1]. \end{cases}$$

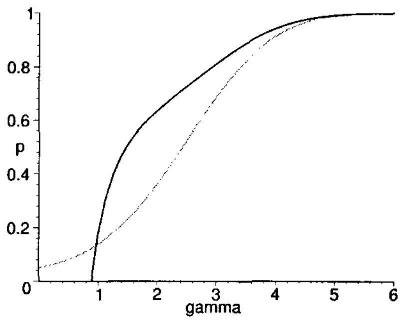

Fig. 5. Power $p(\gamma,h)$ (grey) and the exact asymptotic approximation $p_{EA}(\gamma,h)$ (black) for Example 3.3.

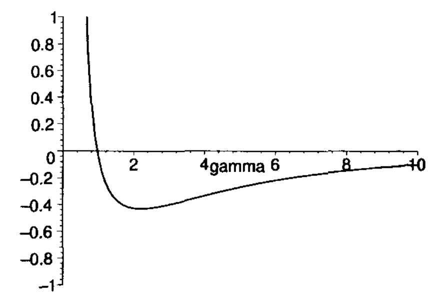

Fig. 6. Relative approximation error $\frac{1-p_{EA}(\gamma,h)}{1-p(\gamma,h)}-1$ made using p_{EA} instead of the exact p for Example 3.3.

The constants occurring in Theorem 2.1 are given by $k=3$, $t_0=0$, $t_1=1/4$, $t_2=1/2$, $t_3=3/4$, $t_4=1$, $I=\{1,3\}$, $J=\emptyset$, $K=\{2\}$, $\hat{J}=\{0,4\}$ and

$$\|\tilde{h}\|^2=1, \qquad -\int u d\tilde{h}'=\boldsymbol{u}^\top\Sigma^{-1}\tilde{\boldsymbol{h}}=2\sqrt{2}u, \qquad \boldsymbol{u}_I^\top(\Sigma_I)^{-1}\boldsymbol{u}_I=8u^2,$$

$$\prod_{i\in II\cup I\hat{J}\cup \hat{J}I}(2/(t_{i+1}-t_i))=2/t_1\cdot 2/(1-t_3)=64, \qquad |(\Sigma_I)^{-1}|^{1/2}=\sqrt{32},$$

$$q_1=q_3=\sqrt{2}, \qquad r_1^+=r_3^-=2\sqrt{2}.$$

Hence, the power is approximatively given by

$$\begin{aligned}\boldsymbol{P}\{\exists t\in[0,1]:B_0(t)\geq b_\gamma(t)\} &\approx p_{EA}(\gamma,h)\\ &=1-\frac{2048}{81}\frac{\sqrt{2}}{\pi}u^2\exp(-4u^2)\gamma^{-4}\exp\left(-\gamma^2\frac{1}{2}+\gamma 2\sqrt{2}u\right)\end{aligned}$$

for large γ. Figure 5 shows the values p_{EA} and the exact power p and Fig. 6 the relative approximation error as a function of γ.

Acknowledgements

We thank an anonymous referee for various helpful suggestions.

Appendix

LEMMA A.1. *Let $\boldsymbol{y}^*\in I\!R^d$, $d\geq 2$ be fixed and let Σ be a positively definite $d\times d$ matrix. Assume that $I\subseteq\{1,\dots,d\}$ is an non empty index set such that for all $i\in I$ we have $(\Sigma^{-1}\boldsymbol{y}^*)_i\neq 0$. Then for all $\boldsymbol{y}\in I\!R^d$*

$$(\boldsymbol{y}+\boldsymbol{y}^*)^\top\Sigma^{-1}(\boldsymbol{y}+\boldsymbol{y}^*)\geq \boldsymbol{y}_{\bar{I}}^\top(\Sigma_{\bar{I}})^{-1}\boldsymbol{y}_{\bar{I}}+2\boldsymbol{y}_I^\top(\Sigma_I)^{-1}\boldsymbol{y}_I^*+\boldsymbol{y}_I^{*\top}(\Sigma_I)^{-1}\boldsymbol{y}_I^*,$$

with $\bar{I}:=\{1,\dots,d\}\backslash I$ and $\boldsymbol{y}_{\bar{I}}^\top(\Sigma_{\bar{I}})^{-1}\boldsymbol{y}_{\bar{I}}=:0$ if $\bar{I}$ is empty.

PROOF. Clearly, if $I=\{1,\dots,d\}$, the claim follows immediately by the positive definiteness of the inverse matrix Σ^{-1}. Assume now that $\bar{I}$ is not empty and put in the sequel $B := \Sigma^{-1}$, $B_{\bar{I}} := (\Sigma^{-1})_{\bar{I}}$, $B_I := (\Sigma^{-1})_I$. Since Σ is positive definite, both matrices $B_I, B_{\bar{I}}$ exist and moreover

$$B_{\bar{I}} - B_{\bar{I}I}(B_I)^{-1}B_{I\bar{I}} = (\Sigma_{\bar{I}})^{-1} \quad \text{and} \quad B_I - B_{I\bar{I}}(B_{\bar{I}})^{-1}B_{\bar{I}I} = (\Sigma_I)^{-1}.$$

To this end, we obtain along the lines of Lemma 4.1 of Bischoff *et al.* (2003*a*)

$$\begin{aligned}
&(\boldsymbol{y}+\boldsymbol{y}^*)^\top B(\boldsymbol{y}+\boldsymbol{y}^*)\\
&= [\boldsymbol{y}_{\bar{I}} + (\boldsymbol{y}_{\bar{I}}^* + (B_{\bar{I}})^{-1}B_{\bar{I}I}\boldsymbol{y}_I^*) + (B_{\bar{I}})^{-1}B_{\bar{I}I}\boldsymbol{y}_I]^\top B_{\bar{I}}[\boldsymbol{y}_{\bar{I}} + (\boldsymbol{y}_{\bar{I}}^* + (B_{\bar{I}})^{-1}B_{\bar{I}I}\boldsymbol{y}_I^*)\\
&\quad + (B_{\bar{I}})^{-1}B_{\bar{I}I}\boldsymbol{y}_I] + \boldsymbol{y}_I^\top(\Sigma_I)^{-1}\boldsymbol{y}_I + 2\boldsymbol{y}_I^\top(\Sigma_I)^{-1}\boldsymbol{y}_I^* + \boldsymbol{y}_I^{*\top}(\Sigma_I)^{-1}\boldsymbol{y}_I^*\\
&= [\boldsymbol{y}_{\bar{I}} + (B_{\bar{I}})^{-1}B_{\bar{I}I}\boldsymbol{y}_I]^\top B_{\bar{I}}[\boldsymbol{y}_{\bar{I}} + (B_{\bar{I}})^{-1}B_{\bar{I}I}\boldsymbol{y}_I]\\
&\quad + \boldsymbol{y}_I^\top(\Sigma_I)^{-1}\boldsymbol{y}_I + 2\boldsymbol{y}_I^\top(\Sigma_I)^{-1}\boldsymbol{y}_I^* + \boldsymbol{y}_I^{*\top}(\Sigma_I)^{-1}\boldsymbol{y}_I^*\\
&= \boldsymbol{y}^\top B\boldsymbol{y} + 2\boldsymbol{y}_I^\top(\Sigma_I)^{-1}\boldsymbol{y}_I^* + \boldsymbol{y}_I^{*\top}(\Sigma_I)^{-1}\boldsymbol{y}_I^*\\
&= \boldsymbol{y}_{\bar{I}}^\top(B_{\bar{I}} - B_{\bar{I}I}(B_I)^{-1}B_{I,\bar{I}})\boldsymbol{y}_{\bar{I}} + [\boldsymbol{y}_I + (B_I)^{-1}B_{I\bar{I}}\boldsymbol{y}_{\bar{I}}]^\top B_I[\boldsymbol{y}_I + (B_I)^{-1}B_{I\bar{I}}\boldsymbol{y}_{\bar{I}}]\\
&\quad + 2\boldsymbol{y}_I^\top(\Sigma_I)^{-1}\boldsymbol{y}_I^* + \boldsymbol{y}_I^{*\top}(\Sigma_I)^{-1}\boldsymbol{y}_I^*\\
&\geq \boldsymbol{y}_{\bar{I}}^\top(\Sigma_{\bar{I}})^{-1}\boldsymbol{y}_{\bar{I}} + 2\boldsymbol{y}_I^\top(\Sigma_I)^{-1}\boldsymbol{y}_I^* + \boldsymbol{y}_I^{*\top}(\Sigma_I)^{-1}\boldsymbol{y}_I^*,
\end{aligned}$$

giving the proof. □

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