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A CENTRAL LIMIT THEOREM FOR THE L_2 ERROR OF POSITIVE WAVELET DENSITY ESTIMATOR

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Abstract. The asymptotic distribution of the integrated squared error of positive wavelet density estimator is derived. It is shown that three different cases arise depending on the smoothness of the unknown density. In each case the asymptotic distribution is shown to be normal. A Martingale central limit theorem is used to prove the results.

Key words and phrases: Positive wavelet density estimator, central limit theorem, U-statistic.

1. Introduction

Let $X_1, \ldots, X_n$ be an iid sequence of random variables with a common pdf f. Since the pioneering work of Rosenblatt (1956) numerous methods have been proposed for estimating f nonparametrically. Procedures based on the wavelets have gained considerable popularity in recent years. Estimators based on wavelets are classified either as linear or non-linear wavelet estimators. Various properties of linear wavelet estimators have been studied by Kerkacharian and Picard (1992), Masry (1994), Walter (1994). Nonlinear wavelet estimators were considered by Donoho *et al.* (1996), Hall and Patil (1995), Delyon and Judisky (1996). Donoho *et al.* (1996) have shown that nonlinear wavelet estimators have uniform optimal convergence rate over a large class of function spaces. Pointwise asymptotic normality of linear multiresolution wavelet estimator was established by Wu (1996). Zhang and Zheng (1999) derived the asymptotic distribution of L_2-error of the linear wavelet estimator under a very mild condition. Asymptotic distribution of L_2-error of orthogonal series type density estimator was derived by Ghorai (1980). Hall (1984) studied the asymptotic distribution of L_2-error of multivariate kernel density estimator. Asymptotic normality of L_p-norm of multivariate kernel density estimator was studied by Csörgő and Horváth (1988) and Horváth (1991).

One of the disadvantages of the nonparametric density estimators is that it can be negative. For kernel based method this can be avoided by using a nonnegative kernel. For wavelet based methods, Walter and Shen (1999) introduced a class of nonnegative wavelets and used them for estimation of density function. The estimates based on the positive wavelets are proper density functions. Walter and Shen (1999) studied the convergence of pointwise MSE of positive wavelet density estimator.

In this paper, the asymptotic normality of $I_n = \int(\hat{f}_n - f)^2 dx$, of the positive wavelet density estimator is derived. For general orthogonal wavelets, $I_n - EI_n$, can be expressed as a U-statistics. For positive wavelet density estimator $I_n - EI_n$ can be expressed as a sum of a U-statistic and a linear term. If the unknown density is smooth then the linear term can not be neglected. Three different cases arise depending on the smoothness of the true density. All three cases will be considered.

2. Notations and some preliminaries

Let ϕ and ψ denote the scalling function and the mother wavelet associated with the multiresolution analysis $\{V_j\}$ of $L_2(R)$ respectively. In the rest of the paper it will be assumed that ϕ and ψ satisfy the following conditions (Daubechies (1988)):

$\phi \in \mathcal{S}_r$, Schwartz space of order r, $r \geq 2$ and there exist constants C and D such that $\sup|\phi(x)| \leq C$, $\sup|\psi(x)| \leq C$, $\int \phi(x)dx = 1$, $\operatorname{supp}\phi \subset [-D, D]$, $\operatorname{supp}\psi \subset [-D, D]$.

Translations and dilations of ϕ and ψ are defined as

$$\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k), \qquad \psi_{jk}(x) = 2^{j/2}\psi(2^j x - k).$$

It is known that $\{\psi_{jk}, k \in \mathcal{Z}\}$ is an orthonormal basis of V_j and for fixed $j_0 \in \mathcal{Z}$, $\{\phi_{j_0k}, k \in \mathcal{Z}, \psi_{jk}, j \geq j_0, j \in \mathcal{Z}, k \in \mathcal{Z}\}$ is an orthonormal basis of $L_2(\mathcal{R})$. Wavelet kernels are defined as

$$q(x,y) = \sum_k \phi(x-k)\phi(y-k), \qquad \text{and} \qquad q_m(x,y) = \sum_k 2^m\phi(2^m x - k)\phi(2^m y - k).$$

Positive wavelet kernels are defined through $\rho_\alpha(x)$, where

$$\rho_\alpha(x) = \sum_j \alpha^{|j|}\phi(x-j).$$

Walter and Shen (1999) have shown that there exists $\alpha_0 > 0$, such that for $\alpha_0 \leq \alpha \leq 1$, $\rho_\alpha(x) > 0$ for all $x \in \mathcal{R}$. Positive wavelet kernels are defined as

$$K_\alpha(x,y) = \left(\frac{1-\alpha}{1+\alpha}\right)^2 \sum_k \rho_\alpha(x-k)\rho_\alpha(y-k) \tag{2.1}$$
$$K_{\alpha,m}(x,y) = 2^m K(2^m x, 2^m y).$$

Define

$$\tilde{K}_\alpha(u,v) = \int K_\alpha(u,w)K_\alpha(w,v)dw$$
$$f_m(x) = \int K_{\alpha,m}(x,y)f(y)dy$$
$$D_1(l,\alpha) = (|l|-1)\alpha^{|l|} + 2\alpha^{|l|}(1-\alpha^2)^{-1}$$
$$D_2(\alpha) = \left(\frac{1-\alpha}{1+\alpha}\right)^4 \left[\frac{(1+\alpha^2)^3 + 6\alpha^2(1+\alpha^2)}{(1-\alpha^2)^3}\right]$$
$$D_3(\alpha) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1 K_\alpha^2(u,v)K_\alpha^2(u,w)dudvdw$$
$$D_4(\alpha) = \int_{-\infty}^{\infty}\left[\int_0^1 (\tilde{K}_\alpha(u,v))^2 du\right] dv$$

$$D_5(\alpha) = \int_{-\infty}^{\infty}\int_0^1 (\tilde{K}_\alpha(u,v))^3 dudv$$

$$D_6(\alpha) = \int_{-\infty}^{\infty}\int_0^1 (\tilde{K}_\alpha(u,v))^4 dudv$$

$$D_7(\alpha) = \int_0^1 \iiint \tilde{K}_\alpha(u_1,u_3)\tilde{K}_\alpha(u_2,u_3)\tilde{K}_\alpha(u_1,u_4) \cdot \tilde{K}_\alpha(u_2,u_4)du_2du_3du_4du_1$$

$$D_8(\alpha) = \int_0^1 \left(\iint K_\alpha(x,z)K_\alpha(x,w)(x-z)^2dxdz\right) \cdot \left(\iint K_\alpha(y,z')K_\alpha(y,w)(y-z')^2dydz'\right)dw$$

$$D_9(\alpha) = \int_0^1 \iint K_\alpha(x,y)K_\alpha(x,z)|x-y|^2dxdydz$$

$$D_{10}(\alpha) = \iiint \left(\int_0^1 K_\alpha(x,z)K_\alpha^2(y,z)K_\alpha(x,w)dz\right)dxdydw$$

$$n^2H_n(X_i,X_j) = \int (K_{\alpha,m}(x,X_i) - f_m(x))(K_{\alpha,m}(x,X_j) - f_m(x))dx \tag{2.2}$$

$$G_n(x,y) = E(H_n(X_3,x)H_n(X_3,y))$$

$$W_i = \int (K_{\alpha,m}(x,X_i) - f_m(x))(f_m(x) - f(x))dx.$$

A positive wavelet kernel density estimate is defined as

$$\hat{f}_n(x) = \frac{1}{n}\sum_{i=1}^n K_{\alpha,m}(x,X_i).$$

The L_2-error associated with $\hat{f}_n$ is defined as

$$I_n = \int (\hat{f}_n(x) - f(x))^2dx.$$

The purpose of this paper is to derive the asymptotic distribution of $(I_n - EI_n)$. Define

$$I_{n1} = \int (\hat{f}_n(x) - f_m(x))^2dx$$

$$I_{n2} = 2\int (\hat{f}_n(x) - f_m(x))(f_m(x) - f(x))dx$$

$$I_{n3} = \int (f_m(x) - f(x))^2dx.$$

Since $EI_{n2} = 0$, we have $I_n - EI_n = (I_{n1} - EI_{n1}) + I_{n2}$. I_{n1} can be expressed as

$$\begin{aligned} I_{n1} &= \int (\hat{f}_n(x) - f_m(x))^2dx \\ &= \sum_i\sum_j H_n(X_i,X_j) \end{aligned}$$

$$= 2\sum_{i<j}\sum H_n(X_i, X_j) + \sum_{i=1}^{n} H_n(X_i, X_i)$$

$$= 2U_n + \sum_{i=1}^{n} H_n(X_i, X_i).$$

$$(2.3) \qquad (I_{n1} - EI_{n1}) + I_{n2} = 2U_n + \sum_{i=1}^{n}[H_n(X_i, X_i) - EH_n(X_i, X_i)] + \frac{2}{n}\sum_{i=1}^{n} W_i$$
$$= T_{n1} + T_{n2} + T_{n3}.$$

The first term is a multiple of a U-statistic. The second and third terms are sums of iid random variables. The asymptotic distribution of the first term can be derived using a CLT for the U-statistic. The asymptotic distribution of the second and the third term can be derived using the standard CLT for iid random variables. In the next section it will be shown that $V(T_{n1}) = O(2^{m+1}/n^2)$, $V(T_{n2}) = O(2^{2m}/n^3)$, and $V(T_{n3}) = O(2^{-4m}/n)$. For most choices of m, $V(T_{n2})/V(T_{n1}) \to 0$ and $V(T_{n2})/V(T_{n3}) \to 0$ as $n \to \infty$. Hence the distribution of $I_n - EI_n$ will be determined by the distributions of T_{n1} and T_{n3}. If m is such that $2^{5m}/n \to \infty$, then $I_n - EI_n \overset{\mathcal{D}}{\sim} T_{n1}$. If m is such that $2^{5m}/n \to 0$, then $I_n - EI_n \overset{\mathcal{D}}{\sim} T_{n3}$. If m is such that $2^{5m}/n \to \lambda$, then $I_n - EI_n \overset{\mathcal{D}}{\sim} T_{n1} + T_{n3}$.

3. Main results

In this section we will derive the asymptotic distribution of $I_n - EI_n$ via a sequence of lemmas. Several auxiliary results that are needed in the proof of the theorem are stated in Lemmas 3.1–3.14. The main theorem and its proof follows the statement of the lemmas. The proofs of the lemmas are in Section 4. The unknown density is assumed to satisfy the following condition.

(A): f is bounded, Riemann integrable, piecewise Hölder continuous on $[-L, L]$, for some $L > 0$, with index $\beta \in (0,1)$ and monotone on $(-\infty, -L]$ and $[L, \infty)$.

LEMMA 3.1. *Let ρ_α and $K_\alpha(x,y)$ be as defined in Section 2. Then*

(i) $\int_{-\infty}^{\infty}(\int_0^1 K_\alpha^2(u,v)du)dv = D_2(\alpha)$

(ii) $\int \rho_\alpha(x-k)\rho_\alpha(x-k')dx = D_1(k-k', \alpha)$

(iii) $K_\alpha(x+k, y+k) = K_\alpha(x,y)$ *for all* x *and* y.

LEMMA 3.2. *If f satisfies condition (A), then*

$$\int (EK_{\alpha,m}^2(x, X_1))dx = 2^m D_2(\alpha)(1 + o_m(1)).$$

LEMMA 3.3. *If f satisfies condition (A), then*

$$\int f_m^2(x)dx = \int f^2(x)dx + O(2^{-m\beta}).$$

LEMMA 3.4. *If f satisfies condition (A), then*

$$\iint E(K_{\alpha,m}^2(X_1, x)K_{\alpha,m}^2(X_1, y))dxdy = 2^{2m}D_3(\alpha) + O(2^{-m\beta}).$$

LEMMA 3.5. *Assume that f satisfies condition (A). Then*

$$\iint f_m(x) E K_{\alpha,m}(x, X_1) K^2_{\alpha,m}(y, X_1) dx dy$$
$$= 2^m D_{10}(\alpha) \left[\int f^2(x) dx + O(2^{-m\beta}) \right].$$

LEMMA 3.6. *If f satisfies condition (A), then*

$$\iint [E(K_{\alpha,m}(x, X_1) K_{\alpha,m}(y, X_1))]^2 dx dy$$
$$= 2^m D_4(\alpha)(1 + O(2^{-m\beta})).$$

LEMMA 3.7. *If f satisfies condition (A), then*
(a) $\int f_m^2(x) dx \leq \|f\|_\infty^2$
(b) $\iint f_m(x) f_m(y) E(K_{\alpha,m}(x, X_1) K_{\alpha,m}(y, X_1)) dx dy \leq \|f\|_\infty^2$.

LEMMA 3.8. *Let H_n be as defined in Section 2. If f satisfies condition (A), then* $En^4 H_n^2(X_1, X_2) = 2^m D_4(\alpha)(1 + O(2^{-m\beta}))$.

LEMMA 3.9. *If f satisfies condition (A), then*

$$\iiiint [E\Pi_{j=1}^4 K_{\alpha,m}(y_j, X_1)]^2 \Pi_{j=1}^4 dy_j = 2^{3m} D_6(\alpha) \left[\int f^2(x) dx + O(2^{-m\beta}) \right].$$

LEMMA 3.10. *If f satisfies condition (A), then*

$$\iiiint (E\Pi_{j=1}^3 K_{\alpha,m}(y_j, X_1))^2 f_m^2(y_4) \Pi_{j=1}^3 dy_j$$
$$\leq 2^{2m} D_5(\alpha) \|f\|_\infty^2 \left(\int f^2(x) dx + O(2^{-m\beta}) \right).$$

LEMMA 3.11. *If f satisfies condition (A), then*

$$n^8 E H_n^4(X_1, X_2) = 2^{3m} D_6(\alpha) \left[\int f^2(x) dx + O(2^{-m\beta}) \right].$$

LEMMA 3.12. *If f satisfies condition (A), then*

$$n^8 E G_n^2(X_1, X_2) = 2^m D_7(\alpha) \left[\int f^4(x) dx + O(2^{-m\beta}) \right].$$

LEMMA 3.13. *If $\phi \in \mathcal{S}_2$, then $\int u K_\alpha(x, x+u) du = 0$.*

LEMMA 3.14. *If f'' satisfies condition (A), then*

$$2^{4m}\operatorname{Var}(W_i) = \frac{1}{4}\left[D_8(\alpha)\int (f''(x))^2 f(x)dx - D_9^2(\alpha)\left[\int f''(x)fx(x)dx\right]^2\right] + O(2^{-m\beta}).$$

Remark 3.1. $(\int f''(x)f(x)dx)^2 \leq \int (f''(x))^2 f(x)dx$ and $D_9^2(\alpha) \leq D_8(\alpha)$ imply that $D_8(\alpha)\int (f'')^2 f(x)dx - D_9^2(\alpha)(\int f'' f(x)dx)^2 \geq 0$.

Define $\sigma_1^2 = 2D_4(\alpha)$ and $\sigma_3^2 = [D_8(\alpha)\int (f'')^2 f(x)dx - D_9^2(\alpha)(\int f'' f(x)dx)^2]$.

THEOREM 3.1. *Assume that f is two times piecewise differentiable and the piecewise second derivative, f'', satisfies condition (A). Define*

$$d(n) = \begin{cases} n2^{-m/2} & \text{if } 2^{5m}/n \to \infty \\ \sqrt{n}2^{2m} & \text{if } 2^{5m}/n \to 0 \\ n^{9/10} & \text{if } 2^{5m}/n \to \lambda,\ 0 < \lambda < \infty \end{cases}$$

and $\Delta_n = d(n)(I_n - EI_n)$. Assume that $m = m(n) \to \infty$ and $n/2^m \to \infty$ as $n \to \infty$. Then

$$\Delta_n \xrightarrow{\mathcal{D}} \begin{cases} N(0, \sigma_1^2) & \text{if } 2^{3m}/n \to \infty \\ N(0, \sigma_3^2) & \text{if } 2^{3m}/n \to 0 \\ N(0, \lambda^{1/5}\sigma_1^2 + \lambda^{-4/5}\sigma_3^2) & \text{if } 2^{3m}/n \to \lambda. \end{cases}$$

PROOF. From (2.3) we have $I_n - EI_n = T_{n1} + T_{n2} + T_{n3}$. Since $EH_n(X_i, X_j) = 0$, using Lemma 3.8 we get

$$\begin{aligned}\operatorname{Var}(T_{n1}) &= \operatorname{Var}(2U_n) \\ &= 4\sum\sum\nolimits_{i<j} EH_n^2(X_i, X_j) \\ &= 2n(n-1)n^{-4}2^m D_4(\alpha)(1 + O(2^{-m\beta})) \\ &= 2^m n^{-2} 2D_4(\alpha)\left(1 - \frac{1}{n}\right)(1 + O(2^{-m\beta})).\end{aligned}$$

Hence $\lim_{n\to\infty} 2^{-m}n^2 \operatorname{Var}(T_{n1}) = 2D_4(\alpha) = \sigma_1^2$. Using Lemmas 3.12, 3.11 and 3.8 it is easy to see that

$$\begin{aligned}[EG_n^2(X_1, X_2) + n^{-1}EH_n^4(X_1, X_2)]/[EH_n^2(X_1, X_2)]^2 &= O(2^{-m}) + O(2^m/n) \\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Since T_{n1} is a multiple of a U-statistic it follows from Therem 1 of Hall (1984) that $n2^{-m/2}T_{n1} \xrightarrow{\mathcal{D}} N(0, \sigma_1^2)$. Using Lemma 3.14 we get

$$\begin{aligned}\operatorname{Var}(T_{n3}) &= 4n^{-1}\operatorname{Var}(W_i) \\ &= 2^{-4m}n^{-1}\sigma_3^2(1 + O(2^{-m\beta})).\end{aligned}$$

Hence $\lim_{n\to\infty} 2^{4m} n \operatorname{Var}(T_{n3}) = \sigma_3^3$. Since T_{n3} is a sum of iid random variables, it follows that $2^{2m}\sqrt{n}T_{n3} \xrightarrow{\mathcal{D}} N(0, \sigma_3^2)$. Using Lemmas 3.2–3.5 we get

$$\begin{aligned}
n^4[EH_n(X_i, X_i)]^2 &= 2^{2m} D_2^2(\alpha)[1 + O(2^{-m\beta})] \\
n^4 EH_n^2(X_i, X_i) &= 2^{2m} D_3(\alpha)(1 + O(2^{-m\beta})) \\
\operatorname{Var}(T_{n2}) &= 2^{2m} n^{-3}(D_3(\alpha) - D_2^2(\alpha))(1 + O(2^{-m\beta})) \\
&= O(2^{2m} n^{-3}).
\end{aligned}$$

This shows that T_{n2} can be ignored. Since T_{n1} is a multiple of a U-statistic and $\operatorname{Var}(T_{n3})/\operatorname{Var}(T_{n1}) \to 0$ if $2^{3m}/n \to \infty$, the first part of the theorem follows from Theorem 1 of Hall (1984). Since $\operatorname{Var}(T_{n1})/\operatorname{Var}(T_{n3}) \to \infty$ if $2^{3m}/n \to 0$, the second part of the theorem follows from the standard CLT for iid random variables. If $2^{3m}/n \to \lambda$ then $\operatorname{Var}(T_{n1}) + \operatorname{Var}(T_{n3}) = n^{-9/5}(\lambda^{1/5}\sigma_1^2 + \lambda^{-4/5}\sigma_3^2)(1 + O(2^{-m\beta}))$. Since T_{n1} and T_{n3} are uncorrelated and are asymptotically normally distributed, third part of the theorm can be established using Cramér-Wold device. This completes the proof of the theorem.

4. Proofs of lemmas

PROOF OF LEMMA 3.1. Using the formula (2.1) we get

$$\begin{aligned}
&\left(\frac{1+\alpha}{1-\alpha}\right)^4 \int_{-\infty}^{\infty} \left(\int_0^1 K_\alpha^2(u, v)du\right) dv \\
&= \int_0^1 \left[\sum_k \sum_{k'} \rho_\alpha(u-k)\rho_\alpha(u-k') \int_{-\infty}^{\infty} \rho_\alpha(v-k)\rho_\alpha(v-k')dv\right] du \\
&= \int_0^1 \sum_k \sum_{k'} \rho_\alpha(u-k)\rho_\alpha(u-k') D_1(k-k', \alpha) \\
&= \sum_l (D_1(l, \alpha))^2 \\
&= (D_1(0, \alpha))^2 + 2\sum_{l=1}^{\infty} \left[(l-1)\alpha^l + \frac{2\alpha^l}{1-\alpha^2}\right]^2 \\
&= \frac{(1+\alpha^2)^3 + 6\alpha^2(1+\alpha^2)}{(1-\alpha^2)^3}.
\end{aligned}$$

The proof of other two parts are straight forward.

PROOF OF LEMMA 3.2. Using a scale and location transformation we get

$$\begin{aligned}
&2^{-m} \int (EK_{\alpha,m}^2(x, X_1))dx \qquad (4.1) \\
&= 2^{-m} \iint K_{\alpha,m}^2(x, y) f(y) dy dx \\
&= \iint_0^1 \frac{K_\alpha^2(u, v)}{2^m} \left[\sum_{s=-2^{-m}L}^{2^m L} + \sum_{s=2^{-m}L+1}^{\infty} + \sum_{s=-\infty}^{-2^m L-1}\right] f\left(\frac{v+s}{2^m}\right) du dv.
\end{aligned}$$

Let $v > 0$ be fixed. Choose $L > 0$ such that f is monotone on $(-\infty, L]$ and $[L, \infty)$. Then

$$2^{-m} \sum_{s=-2^m L}^{2^m L} f\left(\frac{s}{2^m}\right) + 2^{-m} \sum_{s=-2^m L}^{2^m L} \left[f\left(\frac{v+s}{2^m}\right) - f\left(\frac{s}{2^m}\right) \right]$$
$$= J_1(m, L) + R_1(m, L).$$

Since f is Riemann integrable,

$$\left| \int_{-L}^{L} f(x)dx - J_1(m, L) \right| = o_m(1). \tag{4.2}$$

To get a bound on $R_1(m, L)$ we proceed as follows.

$$R_1(m, L) = 2^{-m} \sum_{s=-2^m L}^{2^m L} \left[f\left(\frac{v+s}{2^m}\right) - f\left(\frac{[v]+s}{2^m}\right) \right]$$
$$+ 2^{-m} \sum_{s=-2^m L}^{2^m L} \sum_{j=1}^{[v]} \left[f\left(\frac{s+j}{2^m}\right) - f\left(\frac{s+j-1}{2^m}\right) \right]$$
$$|R_1(m, L)| \le 2^{-m} \sum_{s=-2^m L}^{2^m L} \left| f\left(\frac{v+s}{2^m}\right) - f\left(\frac{[v]+s}{2^m}\right) \right|$$
$$+ 2^{-m} \sum_{s=-2^m L}^{2^m L} \sum_{j=1}^{[v]} \left| f\left(\frac{s+j}{2^m}\right) - f\left(\frac{s+j-1}{2^m}\right) \right|$$
$$\le ([v]+1)[\bar{S}(2^{-m}, v, L) - \underline{S}(2^{-m}, v, L)]$$

where $\bar{S}(\Delta_m)$ and $\underline{S}(\Delta_m)$ denote the upper and lower Riemann sums with $\Delta_m = 2^{-m}$. Since f is Riemann integrable $|\bar{S}(2^{-m}) - \underline{S}(2^{-m})| \to 0$ as $m \to \infty$. Hence $|R_1(m, L)| \le ([v]+1)o_m(1)$. Similarly for $v < 0$, we get $|R_1(m, L)| \le ([|v|]+1)o_m(1)$. Hence for any fixed v, positive or negative,

$$|R_1(m, L)| \le ([|v|]+1)o_m(1). \tag{4.3}$$

Now using (4.2) and (4.3) in (4.1) we get

$$\int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u, v) \sum_{s=-2^m L}^{2^m L} f\left(\frac{v+s}{2^m}\right) dudv \tag{4.4}$$
$$= \int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u, v) J_1(m, L) dudv$$
$$+ \int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u, v) R_1(m, v, L) dudv$$
$$= \int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u, v) \left(\int_{-L}^{L} f(x)dx + o_m(1) \right) dudv$$
$$+ \int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u, v)(|v|+1)o_m(1) dudv$$

$$= D_2(\alpha)\left(\int_{-L}^{L} f(x)dx + o_m(1)\right) + o_m(1).$$

The second term above follows from the fact that K_α is rapidly decreasing and hence $\int_{-\infty}^{\infty}\int_0^1 K_\alpha^2(u,v)(|v|+1)dudv < \infty$. Next consider

$$\begin{aligned}
&\left|\int_L^\infty f(x)dx - \sum_{s=2^mL}^{\infty} 2^{-m} f\left(\frac{v+s}{2^m}\right)\right| \\
&\quad \le \left|\int_L^\infty f(x)dx - \sum_{s=2^mL}^{\infty} 2^{-m} f\left(\frac{s}{2^m}\right)\right| + \left|\sum_{s=2^mL}^{\infty} 2^{-m}\left(f\left(\frac{v+s}{2^m}\right) - f\left(\frac{s}{2^m}\right)\right)\right| \\
&\quad = J_2(m,L) + R_2(m,L).
\end{aligned}$$

An upper bound for $J_2(m,L)$ can be obtained as follows.

$$\begin{aligned}
J_2(m,L) &= \left|\int_L^\infty f(x) - \sum_{s=2^mL+1}^{\infty} 2^{-m} f\left(\frac{s}{2^m}\right)\right| \\
&\le \sum_{s=2^mL+1}^{\infty} 2^{-m}\left[f\left(\frac{s-1}{2^m}\right) - f\left(\frac{s}{2^m}\right)\right] \\
&= 2^{-m} f(L).
\end{aligned} \tag{4.5}$$

The last inequality above follows from the fact that f is monotone on $[L,\infty)$. Next consider the second term $R_2(m,L)$, with $v > 0$.

$$\begin{aligned}
|R_2(m,L)| &\le \sum_{s=2^mL+1}^{\infty} 2^{-m} \sum_{j=0}^{[v]+1}\left[f\left(\frac{s+j}{2^m}\right) - f\left(\frac{s+j+1}{2^m}\right)\right] \\
&= 2^{-m}\left[f(L) + f\left(L + \frac{1}{m}\right) + \cdots + f\left(L + \frac{[v]}{2^m}\right)\right] \\
&\le ([v]+1) f(L) 2^{-m}.
\end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6) we conclude that

$$\left|\int_L^\infty f(x)dx - \sum_{s=2^mL}^{\infty} 2^{-m} f\left(\frac{v+s}{2^m}\right)\right| \le ([v]+1) f(L) 2^{-m}. \tag{4.7}$$

Similarly for $v < 0$

$$\left|\int_L^\infty f(x)dx - \sum_{s=2^mL}^{\infty} 2^{-m} f\left(\frac{v+s}{2^m}\right)\right| \le (|v|+2) f(L) 2^{-m}.$$

Hence

$$\left|\int_{-\infty}^{\infty}\int_0^1 K_\alpha^2(u,v) \sum_{s=2^mL+1}^{\infty} 2^{-m} f\left(\frac{v+s}{2^m}\right) dudv\right. \tag{4.8}$$

$$- \int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u,v)dudv \int_L^\infty f(x)dx \Bigg|$$
$$\le 2^{-m} \left(\int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u,v)(|v|+2)dudv \right)$$
$$= O(2^{-m}).$$

Similarly

$$(4.9) \qquad \left| \int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u,v) \sum_{-\infty}^{-2^m L-1} 2^{-m} f\left(\frac{v+s}{2^m}\right) dudv \right.$$
$$\left. - \int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u,v)dudv \int_{-\infty}^{-L} f(x)dx \right|$$
$$= O(2^{-m}).$$

Now combining (4.4)–(4.9) we conclude that

$$\int_{-\infty}^{\infty} \int_0^1 K_\alpha^2(u,v) \sum_s 2^{-m} f\left(\frac{v+s}{2^m}\right) dudv$$
$$= D_2(\alpha)(1+o_m(1)).$$

PROOF OF LEMMA 3.3. As in the previous lemma, using a scale and a location tranformation we get

$$(4.10) \qquad \int f_m^2(x)dx$$
$$= \iint \left[2^{2m} \int K_\alpha(2^m x, 2^m y) K_\alpha(2^m x, 2^m z)dx \right] f(y)f(z)dydz$$
$$= \iint \left(\int_0^1 K_\alpha(u,v)K_\alpha(u,w)du \right)$$
$$\cdot \left[\sum_{s=-2^m L}^{2^m L} + \sum_{s=2^m L+1}^{\infty} + \sum_{-\infty}^{-2^m L-1} \right] 2^{-m} f\left(\frac{v+s}{2^m}\right) f\left(\frac{w+s}{2^m}\right) dvdw.$$

First consider

$$2^{-m} \sum_{s=-2^m L}^{2^m L} f\left(\frac{v+s}{2^m}\right) f\left(\frac{w+s}{2^m}\right)$$
$$= 2^{-m} \sum_{s=-2^m L}^{2^m L} f\left(\frac{v+s}{2^m}\right) \left[f\left(\frac{w+s}{2^m}\right) - f\left(\frac{s}{2^m}\right) \right]$$
$$+ 2^{-m} \sum_{s=-2^m L}^{2^m L} f\left(\frac{s}{2^m}\right) \left[f\left(\frac{v+s}{2^m}\right) - f\left(\frac{s}{2^m}\right) \right]$$

$$
\begin{aligned}
&+ 2^{-m} \sum_{s=-2^m L}^{2^m L} f^2\left(\frac{s}{2^m}\right) \\
&= R_3 + R_4 + J_3.
\end{aligned}
$$

A bound on J_3 can be obtained as follows:

$$
\begin{aligned}
&\left| \int_{-L}^{L} f^2(x)dx - J_3(m,L) \right| \\
&\quad = \left| 2^{-m} \int_{-2^m L}^{2^m L} f^2\left(\frac{x}{2^m}\right) - J_3(m,L) \right| \\
&\quad \le 2\|f\|_\infty 2^{-m} 2^{-m\beta} 2(2^m L) \\
&\quad = O(2^{-m\beta}).
\end{aligned}
$$

Next compute a bound on R_3. Assume $w > 0$.

$$
\begin{aligned}
&|R_3(m,L,v,w)| \\
&= \left| 2^{-m} \sum_{s=-2^m L}^{2^m L} f\left(\frac{v+s}{2^m}\right) \left[f\left(\frac{w+s}{2^m}\right) - f\left(\frac{s}{2^m}\right) \right] \right| \\
&\le \|f\|_\infty 2^{-m} w 2^{-m\beta} 2(2^m L) \\
&= O(2^{-m\beta} w).
\end{aligned}
$$

For $w < 0$, a similar argument can be used. Hence for any fixed w, positive or negative,

$$|R_3(m,L,v,w)| = O(2^{-m\beta}|w|).$$

Using a similar argument it can be shown that

$$|R_4(m,L,v,w)| = O(2^{-m\beta}|v|).$$

Now combining the above three bounds we conclude that

$$
\begin{aligned}
(4.11) \quad &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_0^1 K_\alpha(u,v) K_\alpha(u,w) du \right) \\
&\quad \cdot \left(\sum_{-2^m L}^{2^m L} 2^{-m} f\left(\frac{v+s}{2^m}\right) f\left(\frac{w+s}{2^m}\right) \right) dv dw \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_0^1 K_\alpha(u,v) K_\alpha(u,w) du \right) \left(\int_{-L}^{L} f^2(x) dx \right) + O(2^{-m\beta}).
\end{aligned}
$$

Next consider

$$
\begin{aligned}
&2^{-m} \sum_{2^m L+1}^{\infty} f\left(\frac{v+s}{2^m}\right) f\left(\frac{w+s}{2^m}\right) \\
&\quad = 2^{-m} \sum_{2^m L+1}^{\infty} f^2\left(\frac{s}{2^m}\right) + 2^{-m} \sum_{2^m L+1}^{\infty} f\left(\frac{v+s}{2^m}\right) \left[f\left(\frac{w+s}{2^m}\right) - f\left(\frac{s}{2^m}\right) \right]
\end{aligned}
$$

$$+2^{-m}\sum_{2^mL+1}^{\infty} f\left(\frac{s}{2^m}\right)\left[f\left(\frac{v+s}{2^m}\right)-f\left(\frac{s}{2^m}\right)\right]$$
$$= J_4 + R_5 + R_6.$$

Now compute a bound on J_4.

$$(4.12)\qquad \left|\int_L^\infty f^2(x)dx - J_4\right|$$
$$= \left|\sum_{s=2^mL}^{\infty} 2^{-m}\left(\int_s^{s+1} f^2\left(\frac{x}{2^m}\right) - f^2\left(\frac{s}{2^m}\right)\right)dx\right|$$
$$\leq f^2(L)2^{-m}.$$

Compute a bound on R_5. Assume $w > 0$.

$$|R_5(m,L,v,w)|$$
$$= 2^{-m}\left|\sum_{s=2^mL+1}^{\infty} f\left(\frac{v+s}{2^m}\right)\left[f\left(\frac{w+s}{2^m}\right)-f\left(\frac{s}{2^m}\right)\right]\right|$$
$$\leq 2^{-m}f(L)\left[f(L)+f\left(L+\frac{1}{2^m}\right)+\cdots+f\left(L+\frac{[w]}{2^m}\right)\right]$$
$$\leq 2^{-m}f(L)([w]f(L))$$
$$= O(w2^{-m}).$$

For $w < 0$, using the Hölder continuity condition on $[L+\frac{w}{2^m}, L]$ and the monotonocity of f on $[L,\infty)$, the following bound can be obtained.

$$(4.13)\qquad |R_5(m,L,v,w)| = O(|w|2^{-m\beta}).$$

Using a similar argument R_6 can be bounded as

$$(4.14)\qquad |R_6(m,L,v,w)| = O(|v|2^{-m\beta}).$$

Now combining (4.12)–(4.14) we conclude that

$$(4.15)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^1 K_\alpha(u,v)K_\alpha(u,w)du\right)$$
$$\cdot\sum_{2^mL+1}^{\infty}\left[2^{-m}f\left(\frac{v+s}{2^m}\right)f\left(\frac{w+s}{2^m}\right)\right]dvdw$$
$$= \left[\int_L^\infty f^2(x)dx + O(2^{-m\beta})\right].$$

Again using the arguments similar to the ones used to derive (4.15) we get

$$(4.16)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^1 K_\alpha(u,v)K_\alpha(u,w)du\right)$$
$$\cdot\sum_{-\infty}^{-2^mL-1}\left[2^{-m}f\left(\frac{v+s}{2^m}\right)f\left(\frac{w+s}{2^m}\right)\right]dvdw$$
$$= \left[\int_{-\infty}^{-L} f^2(x)dx + O(2^{-m\beta})\right].$$

Now combining (4.11), (4.15) and (4.16) we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^1 K_\alpha(u,v)K_\alpha(u,w)du\right)\sum_s\left[2^{-m}f\left(\frac{v+s}{2^m}\right)f\left(\frac{w+s}{2^m}\right)\right]dvdw$$
$$=\left[\int_{-\infty}^{\infty} f^2(x)dx+O(2^{-m\beta})\right].$$

PROOF OF LEMMA 3.4.

$$\begin{aligned}&\iint E\left(K_{\alpha,m}^2(X_1,x)K_{\alpha,m}^2(X_1,y)\right)dxdy\\&=2^m\iint\left(\int K_\alpha^2(z,x)K_\alpha^2(z,y)f\left(\frac{z}{2^m}\right)dz\right)dxdy\\&=2^m\iint\left(\sum_s\int_0^1 K_\alpha^2(w+s,u+s)K_\alpha^2(w+s,v+s)f\left(\frac{w+s}{2^m}\right)dw\right)dudv\\&=2^{2m}\int_0^1\iint K_\alpha^2(w,u)K_\alpha^2(w,v)dudv\left(2^{-m}\sum_s f\left(\frac{w+s}{2^m}\right)\right)dw.\end{aligned}$$

Now using the arguments similar to those used in Lemma 3.3, we conclude that

$$\begin{aligned}&\iint E\left(K_{\alpha,m}^2(X_1,x)K_{\alpha,m}^2(X_1,y)\right)dxdy\\&=2^{2m}D_3(\alpha)(1+O(2^{-m\beta})).\end{aligned}$$

PROOF OF LEMMA 3.5.

$$\begin{aligned}&\iint f_m(x)EK_{\alpha,m}(x,X_1)K_{\alpha,m}^2(y,X_1)dxdy\\&=\iiiint K_\alpha(x,w)K_\alpha(x,z)K_\alpha^2(y,z)f\left(\frac{w}{2^m}\right)f\left(\frac{z}{2^m}\right)dwdzdxdy\\&=\iiint\sum_s\int_{z=s}^{s+1}K_\alpha^2(y,z)f\left(\frac{z}{2^m}\right)K_\alpha(x,w)K_\alpha(x,z)f\left(\frac{w}{2^m}\right)dzdwdxdy\\&=2^m\iiiint_0^1 K_\alpha^2(y,z)K_\alpha(x,z)K_\alpha(x,w)\\&\qquad\cdot\left[2^{-m}\sum_s f\left(\frac{z+s}{2^m}\right)f\left(\frac{w+s}{2^m}\right)\right]dwdzdxdy\\&=2^mD_{10}(\alpha)\left[\int f^2(x)dx+O(2^{-m\beta})\right].\end{aligned}$$

PROOF OF LEMMA 3.6.

$$\iint\left[\int K_{\alpha,m}(x,z)K_{\alpha,m}(y,z)f(z)dz\right]^2dxdy$$

$$= \iint \tilde{K}_\alpha^2(z,z') f\left(\frac{z}{2^m}\right) f\left(\frac{z'}{2^m}\right) dz dz'$$
$$= 2^m \left[\int \left(\int_0^1 \tilde{K}_\alpha^2(u,v)du\right) dv\right] \left[\int f^2(x)dx + O(2^{-m\beta})\right]$$
$$= 2^m D_4(\alpha) \int f^2(x)dx + O(2^{-m(1-\beta)}).$$

PROOF OF LEMMA 3.8. The proof follows from Lemmas 3.6 and 3.7.

PROOF OF LEMMA 3.9.

$$\iiiint (E\Pi_{j=1}^4 K_{\alpha,m}(y_j, X_1))^2 \Pi_{j=1}^4 dy_j$$
$$= 2^{2m} \iint (\tilde{K}_\alpha(x,x'))^4 f\left(\frac{x}{2^m}\right) f\left(\frac{x'}{2^m}\right) dx dx'$$
$$= 2^{3m} \int_{-\infty}^{\infty} \int_0^1 (\tilde{K}_\alpha(u,v))^4 \left[2^{-m} \sum_s f\left(\frac{u+s}{2^m}\right) f\left(\frac{v+s}{2^m}\right)\right] du dv$$
$$= 2^{3m} D_5(\alpha) \left[\int f^2(x)dx + O(2^{-m\beta})\right].$$

PROOF OF LEMMA 3.10. Since $\int f_m^2(x)dx \le \|f\|_\infty^2$, it is enough to show the following.

$$\iiint [E\Pi_{j=1}^3 K_{\alpha,m}(y_j, X_1)]^2 \Pi_{j=1}^3 dy_j$$
$$= \iiint [K_{\alpha,m}(y_j, x) f(x) dx]^2 \Pi_{j=1}^3 dy_j$$
$$= 2^m \iint (\tilde{K}_{\alpha,m}(x,x'))^3 f\left(\frac{x}{2^m}\right) f\left(\frac{x'}{2^m}\right) dx dx'$$
$$= 2^{2m} \int_{-\infty}^{\infty} \int_0^1 (\tilde{K}_{\alpha,m}(u,v))^3 \left[2^{-m} \sum_s f\left(\frac{u+s}{2^m}\right) f\left(\frac{v+s}{2^m}\right)\right] du dv$$
$$= 2^{2m} D_6(\alpha) \left[\int f^2(x)dx + O(2^{-m\beta})\right].$$

PROOF OF LEMMA 3.11. The proof follows from Lemmas 3.10 and 3.11.

PROOF OF LEMMA 3.12.

$$EG_n^2(X_1, X_2) = E(H_n(X_3, X_1) H_n(X_3, X_2) H_n(X_4, X_1) H_n(X_4, X_2)).$$

Hence

$$n^8 EG_n^2(X_1, X_2)$$
$$= \iiiint E[(K_{\alpha,m}(x, X_1) - f_m(x))(K_{\alpha,m}(x, X_3) - f_m(x))$$

$$
\begin{aligned}
&\cdot (K_{\alpha,m}(y,X_2) - f_m(y))(K_{\alpha,m}(y,X_3) - f_m(y)) \\
&\cdot (K_{\alpha,m}(z,X_1) - f_m(z))(K_{\alpha,m}(z,X_4) - f_m(z)) \\
&\cdot (K_{\alpha,m}(w,X_2) - f_m(w))(K_{\alpha,m}(w,X_4) - f_m(w))]dxdydzdw \\
= \iiiint &[(EK_{\alpha,m}(x,X_1)K_{\alpha,m}(z,X_1) - f_m(x)f_m(z)) \\
&\cdot (EK_{\alpha,m}(y,X_2)K_{\alpha,m}(w,X_2) - f_m(y)f_m(w)) \\
&\cdot (EK_{\alpha,m}(x,X_3)K_{\alpha,m}(y,X_3) - f_m(x)f_m(y)) \\
&\cdot (EK_{\alpha,m}(z,X_4)K_{\alpha,m}(w,X_4) - f_m(z)f_m(w))]dxdydzdw.
\end{aligned}
$$

The integrand can be multiplied to get several terms. The dominating term is given below.

$$
\begin{aligned}
(4.17)\quad & \iiiint [E(K_{\alpha,m}(x,X_1)K_{\alpha,m}(z,X_1))E(K_{\alpha,m}(y,X_2)K_{\alpha,m}(w,X_2)) \\
&\cdot E(K_{\alpha,m}(x,X_3)K_{\alpha,m}(y,X_3))E(K_{\alpha,m}(z,X_4)K_{\alpha,m}(w,X_4))]dxdydzdw \\
&= \iiiint \tilde{K}_\alpha(u_1,u_3)\tilde{K}_\alpha(u_2,u_3)\tilde{K}_\alpha(u_1,u_4)\tilde{K}_\alpha(u_2,u_4)\Pi_{i=1}^4 f\left(\frac{u_i}{2^m}\right) du_i \\
&= \sum_s \int_{u_1=s}^{u_1=s+1} \iiint \tilde{K}_\alpha(u_1,u_3)\tilde{K}_\alpha(u_2,u_3)\tilde{K}_\alpha(u_1,u_4)\tilde{K}_\alpha(u_2,u_4) \\
&\qquad \cdot \Pi_{i=1}^4 f\left(\frac{u_i}{2^m}\right) du_i \\
&= \int_{u_1=0}^{u_1=1} \iiint \tilde{K}_\alpha(u_1,u_3)\tilde{K}_\alpha(u_2,u_3)\tilde{K}_\alpha(u_1,u_4)\tilde{K}_\alpha(u_2,u_4) \\
&\qquad \cdot \left(\sum_s \Pi_{i=1}^4 f\left(\frac{u_i+s}{2^m}\right) du_i\right).
\end{aligned}
$$

Now consider the sum in (4.17).

$$
\begin{aligned}
&\sum_s \Pi_{i=1}^4 f\left(\frac{u_i+s}{2^m}\right) \\
&= \sum_s \Pi_{i=1}^3 f\left(\frac{u_i+s}{2^m}\right)\left(f\left(\frac{u_4+s}{2^m}\right) - f\left(\frac{s}{2^m}\right)\right) \\
&+ \sum_s f\left(\frac{s}{2^m}\right)\Pi_{i=1}^2 f\left(\frac{u_i+s}{2^m}\right)\left(f\left(\frac{u_3+s}{2^m}\right) - f\left(\frac{s}{2^m}\right)\right) \\
&+ \sum_s f^2\left(\frac{s}{2^m}\right) f\left(\frac{u_1+s}{2^m}\right)\left(f\left(\frac{u_2+s}{2^m}\right) - f\left(\frac{s}{2^m}\right)\right) + \sum_s f^4\left(\frac{s}{2^m}\right) \\
&= T_{n4} + T_{n5} + T_{n6} + T_{n7}.
\end{aligned}
$$

If f satisfies condition (A), then it can be shown that

$$
\begin{aligned}
2^{-m}\sum_s f^4\left(\frac{s}{2^m}\right) &= \int f^4(x)dx + O(2^{-m\beta}) \\
|2^{-m}T_{n4}| &= O(|u_4|2^{-m\beta}) \\
|2^{-m}T_{n5}| &= O(|u_3|2^{-m\beta}) \\
|2^{-m}T_{n6}| &= O(|u_2|2^{-m\beta}).
\end{aligned}
$$

Since $\tilde{K}_\alpha(x,y)$ is rapidly decreasing, it can be shown that

$$\int_0^1 \iiint \tilde{K}_\alpha(u_1,u_3)\tilde{K}_\alpha(u_2,u_3)\tilde{K}_\alpha(u_1,u_4)\tilde{K}_\alpha(u_2,u_4)\Pi_{j=1}^4|u_i|du_i \leq \infty.$$

Hence the right hand side of (4.17) can be approximated by

$$|\text{RHS of (4.17)}| = 2^m D_7(\alpha)\left[\int f^4(x)dx + O(2^{-m\beta})\right].$$

The remaining terms in the expansion of the product in (4.17) are of lower order of magnitude. Hence

$$n^8 G_n^2(X_1,X_2) = 2^m D_7(\alpha)\left[\int f^4(x)dx + O(2^{-m\beta})\right].$$

This completes the proof.

PROOF OF LEMMA 3.13.

$$\begin{aligned}
&\int uK_\alpha(x,x+u)du \\
&= \int u\left(\frac{1-\alpha}{1+\alpha}\right)^2 \sum_j\sum_k \alpha^{|j|+|k|} q(x-k,x+u-j)du \\
&= \left(\frac{1-\alpha}{1+\alpha}\right)^2 \sum_j\sum_k \alpha^{|j|+|k|}\int uq(x-k,x+u-j)du \\
&= \left(\frac{1-\alpha}{1+\alpha}\right)^2 \sum_j\sum_k \alpha^{|j|+|k|}(j-k) \\
&= 0.
\end{aligned}$$

PROOF OF LEMMA 3.14. Since $EW_i = 0$

$$\begin{aligned}
\text{Var}(W_i) &= E\left[\int (K_{\alpha,m}(x,X_i) - f_m(x))(f_m(x) - f(x))dx\right]^2 \\
&= \iint (f_m(x) - f(x))(f_m(y) - f(y))(EK_{\alpha,m}(x,X_i)K_{\alpha,m}(y,X_i))dxdy \\
&\quad - \left[\int (f_m(x) - f(x))f_m(x)dx\right]^2 \\
&= T_{n8} - T_{n9}^2.
\end{aligned}$$

First compute an approximation to T_{n8}.

$$\text{(4.18)} \qquad \begin{aligned} T_{n8} = \iint &\left[\int K_{\alpha,m}(x,z)(f(z) - f(x))dz\right] \\ &\cdot\left[\int K_{\alpha,m}(y,z')(f(z') - f(y))dz'\right]\end{aligned}$$

$$
\begin{aligned}
&\cdot \left(\int K_{\alpha,m}(x,w)K_{\alpha,m}(y,w)f(w)dw\right) dxdy \\
&= 2^{-m} \iint \left[K_\alpha(x,z)\left(f\left(\frac{z}{2^m}\right) - f\left(\frac{x}{2^m}\right)\right) dz\right] \\
&\qquad \cdot \left[K_\alpha(y,z')\left(f\left(\frac{z'}{2^m}\right) - f\left(\frac{y}{2^m}\right)\right) dz'\right] \\
&\qquad \cdot \left[K_\alpha(x,w)K_\alpha(y,w)f\left(\frac{w}{2^m}\right) dw\right] dxdy \\
&= 2^{-m} \int \left[\iint K_\alpha(x,z)K_\alpha(x,w)\left(f\left(\frac{z}{2^m}\right) - f\left(\frac{x}{2^m}\right)\right) dxdz\right] \\
&\qquad \cdot \left[\iint K_\alpha(y,z')K_\alpha(y,w)\left(f\left(\frac{z'}{2^m}\right) - f\left(\frac{y}{2^m}\right)\right) dydz'\right] \\
&\qquad \cdot f\left(\frac{w}{2^m}\right) dw \\
&= \int_0^1 \iint K_\alpha(x,z)K_\alpha(x,w) \iint K_\alpha(y,z')K_\alpha(y,w) \\
&\qquad \cdot \left[2^{-m}\sum_s f\left(\frac{w+s}{2^m}\right)\right. \\
&\qquad\quad \cdot \left(f\left(\frac{z+s}{2^m}\right) - f\left(\frac{x+s}{2^m}\right)\right) \\
&\qquad\quad \left.\cdot \left(f\left(\frac{z'+s}{2^m}\right) - f\left(\frac{y+s}{2^m}\right)\right)\right] dxdzdydz'dw.
\end{aligned}
$$

Using Taylor's expansion, the quantity in the square bracket can be expressed as

$$
\begin{aligned}
&2^{-m}\sum_s f\left(\frac{w+s}{2^m}\right)\left[\left(\frac{z-x}{2^m}\right)f'\left(\frac{x+s}{2^m}\right) + \frac{1}{2}\left(\frac{z-x}{2^m}\right)^2 f''(\theta(z,x,s))\right] \\
&\qquad \cdot \left[\left(\frac{z'-y}{2^m}\right)f'\left(\frac{y+s}{2^m}\right) + \frac{1}{2}\left(\frac{z'-y}{2^m}\right)^2 f''(\theta'(z',y,s))\right] \\
&= \left(\frac{z-x}{2^m}\right)\left(\frac{z'-y}{2^m}\right)2^{-m}\sum_s f\left(\frac{w+s}{2^m}\right)f'\left(\frac{x+s}{2^m}\right)f'\left(\frac{y+s}{2^m}\right) \\
&\quad + \frac{1}{2}\left(\frac{z-x}{2^m}\right)\left(\frac{z'-y}{2^m}\right)^2 2^{-m}\sum_s f\left(\frac{w+s}{2^m}\right)f'\left(\frac{x+s}{2^m}\right)f''(\theta'(z',y,s)) \\
&\quad + \frac{1}{2}\left(\frac{z'-y}{2^m}\right)\left(\frac{z-x}{2^m}\right)^2 2^{-m}\sum_s f\left(\frac{w+s}{2^m}\right)f'\left(\frac{y+s}{2^m}\right)f''(\theta(z,x,s)) \\
&\quad + \frac{1}{4}\left(\frac{z-x}{2^m}\right)^2\left(\frac{z'-y}{2^m}\right)^2 2^{-m}\sum_s f\left(\frac{w+s}{2^m}\right)f''(\theta(z,x,s))f''(\theta'(z',y,s)).
\end{aligned}
$$

In view of Lemma 3.13, the integrals of first three terms in the above sum are zero. Hence if f'' satisfies condition (A), then for any fixed x, y, z, z' and w, the last term above can be approximated by

$$2^{-4m}|x-z|^2|y-z|^2\left[\frac{1}{4}\int (f'')^2 f(x)dx + O(2^{-m\beta})\right]. \tag{4.19}$$

Using (4.19) in (4.18) we conclude that

$$2^{4m}T_{n8}(m) = D_8(\alpha)\left[\frac{1}{4}\int (f''(x))^2 f(x)dx + O(2^{-m\beta})\right]. \tag{4.20}$$

Now compute an approximation to T_{n9}.

$$\begin{aligned} T_{n9}(m) &= \int (f_m(x)-f(x))f_m(x)dx \\ &= \int\left[\int K_{\alpha,m}(x,y)(f(y)-f(x))dy\right]\left[\int K_{\alpha,m}(x,z)f(z)dz\right]dx \\ &= \int_0^1 \iint K_\alpha(x,y)K_\alpha(x,z)2^{-m} \\ &\quad \cdot\sum_s f\left(\frac{z+s}{2^m}\right)\left[\left(\frac{y-x}{2^m}\right)f'\left(\frac{x+s}{2^m}\right)\right. \\ &\quad\quad \left.+\frac{1}{2}\left(\frac{y-x}{2^m}\right)^2 f''(\theta(x,y,z))\right]dxdydz. \end{aligned} \tag{4.21}$$

Again in view of Lemma 3.13, the integral of the first term wrt y vanishes. Hence if f'' satisfies condition (A), then the second term above can be approximated as

$$\begin{aligned} &\left[2^{-m}\sum_s f\left(\frac{z+s}{2^m}\right)\frac{1}{2}\left(\frac{y-x}{2^m}\right)^2 f''(\theta)\right] \\ &\quad = 2^{-2m}|x-y|^2\left[\frac{1}{2}\int f''(u)f(u)du + O(2^{m\beta})\right]. \end{aligned} \tag{4.22}$$

Using (4.22) in (4.21) we conclude that

$$2^{2m}T_{n9}(m) = D_9(\alpha)\left[\frac{1}{2}\int f''(x)f(x)dx + O(2^{-m\beta})\right]$$

and hence

$$2^{4m}T_{n9}^2(m) = D_9^2(\alpha)\left[\left(\frac{1}{2}\int f''(x)f(x)dx\right)^2 + O(2^{-m\beta})\right]. \tag{4.23}$$

Now using (4.20) and (4.23) we conclude that

$$\begin{aligned} &4[2^{4m}\operatorname{Var}(W_i)] \\ &\quad = 4[2^{4m}T_{n8} - (2^{2m}T_{n9})^2] \\ &\quad = \left[D_8(\alpha)\int (f''(x))^2 f(x)dx - D_9^2(\alpha)\left(\int f''(x)f(x)dx\right)^2\right] + O(2^{-m\beta}). \end{aligned}$$

This completes the proof.

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