Ann. Inst. Statist. Math.
Vol. 54, No. 4, 918–933 (2002)


# INDEPENDENCE OF LIKELIHOOD RATIO CRITERIA FOR HOMOGENEITY OF SEVERAL POPULATIONS

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(Received October 17, 2000; revised November 8, 2001)

**Abstract.** Let $\Pi_i$ be an $i$-th population with a probability density function $f(\cdot \mid \theta_i)$ with one dimensional unknown parameter $\theta_i$, $i = 1, 2, \dots, k$. Let $n_i$ sample be drawn from each $\Pi_i$. The likelihood ratio criteria $\lambda_{j|(j-1)}$ for testing hypothesis that the first $j$ parameters are equal against alternative hypothesis that the first $(j-1)$ parameters are equal and the $j$-th parameter is different with the previous ones are defined, $j = 2, 3, \dots, k$. The paper shows the asymptotic independence of $\lambda_{j|(j-1)}$'s up to the order $1/n$ under a hypothesis of equality of $k$ parameters, where $n$ is a number of total samples.

*Key words and phrases*: Likelihood ratio criterion, asymptotic expansion, homogeneity of parameters, asymptotic independence.

## 1. Introduction

Bartlett (1937) dealt with the case of homogeneity of variances of $k$ normal populations. As the exact distribution of a test statistic was unknown, he considered to give a good approximation based on a knowledge of the moments of it. The method consisted in multiplying $-2\log$ (likelihood ratio criterion) by a scalor factor which results in a statistic having the same moments as chi-square random variable ignoring quantities of order $1/n^2$, where $n$ is the size of the total sample. This correction factor was known as Bartlett correction factor in the sense of the moment.

For a case of a general population Lawley (1956) considered an asymptotic behavior of the likelihood ratio criterion for testing a composite hypothesis and obtained Bartlett correction factor in the sense of the moment. He decomposed a log-likelihood ratio criterion into a sum of log-likelihood ratio criteria corresponding to a sequence of a nested hypothesis and he showed that each log-likelihood ratio criteria has a Bartlett correction factor in the sense of the moment.

Hayakawa (1977, 1987) gave an asymptotic expansion of the distribution function of a likelihood ratio criterion for testing a simple hypothesis up to the order $1/n$ and showed that Bartlett correction factor in the sense of the moment yields a statistic having a chi-square distribution ignoring quantities of the order $1/n^2$. This implies that Bartlett correction factor in the sense of the moment is same as a Bartlett correction factor in the sense of the distribution for a likelihood ratio criterion. For other statistic it is usually hard to claim this fact, for example, this does not hold for Rao's score statistic. Thus the concept of Bartlett correctness in the sense of the distribution is stronger than that of Bartlett correctness in the sense of the moment. Thus if a statistic has Bartlett correction factor in the sense of the distribution, we call that it is Bartlett correctable.

Harris (1986) and Cordeiro (1987) pointed out an incompleteness of Hayakawa's 1977 result for the case of composite hypothesis testing.

Bickel and Ghosh (1990) considered independence of a sequence of signed likelihood ratio criterion which corresponds to Lawley's decomposition of log-likelihood ratio criterion from Bayesian point of view, and Takemura and Kuriki (1996) also handled a similar problem from a frequentist point of view. Takemura and Kuriki introduced a new parameter transformation which makes some higher order cross moments of derivatives of log-likelihood ratio criterion vanish.

Let $X_i = [x_{i1}, x_{i2}, \ldots, x_{in_i}]$ be a random sample from the $i$-th population $\Pi_i$ with probability density function (pdf) $f(x \mid \theta_i)$, $i = 1, 2, \ldots, k$ and $\theta_i$'s are one dimensional parameters. For testing a hypothesis of homogeneity of parameters

$$H : \theta_1 = \theta_2 = \cdots = \theta_k (= \theta, \text{say})$$

against the alternative

$$K : \text{violation of at least one equality,}$$

the likelihood ratio criterion $\lambda$ is defined as

$$\lambda = \frac{\prod_{i=1}^{k} \prod_{\alpha=1}^{n_i} f(x_{i\alpha} \mid \tilde{\theta})}{\prod_{i=1}^{k} \prod_{\alpha=1}^{n_i} f(x_{i\alpha} \mid \hat{\theta}_i)}, \tag{1.1}$$

where $\tilde{\theta}$ is the maximum likelihood estimator of $\theta$ based on $n = \sum_{i=1}^{k} n_i$ observations under $H$ and $\hat{\theta}_i$ is the maximum likelihood estimator of $\theta_i$ based on $n_i$ observations $X_i$.

This is a general set up of Bartlett's homogeneity of variances of $k$ normal populations. Hayakawa (1993) studied the asymptotic behavior of the distribution of $\lambda$, and Hayakawa (1994) dealt with the case of $p$ dimensional parameter and showed that Bartlett correction factor is closely related to the corresponding expression given by Hayakawa (1977) in the context of a one-sample problem. Hayakawa (2001) considered this problem by use of Rao's score statistic, and Hayakawa and Doi (1999) also considered this by use of Wald statistic.

Consider a sequence of hypotheses $H_{j|(j-1)}$ and $K_{j|(j-1)}$ defined as

$$H_{j|(j-1)} : \theta_1 = \cdots = \theta_{j-1} = \theta_j \quad \text{vs.} \quad K_{j|(j-1)} : \theta_1 = \cdots = \theta_{j-1} \neq \theta_j,$$
$$j = 2, 3, \ldots, k.$$

The likelihood rato criterion for testing $H_{j|(j-1)}$ is given by

$$\lambda_{j|(j-1)} = \frac{\prod_{i=1}^{j} \prod_{\alpha=1}^{n_i} f(x_{i\alpha} \mid \tilde{\theta}_i)}{\prod_{i=1}^{j-1} \prod_{\alpha=1}^{n_i} f(x_{i\alpha} \mid \tilde{\theta}_{j-1}) \prod_{\alpha=1}^{n_j} f(x_{j\alpha} \mid \hat{\theta}_j)}, \qquad j = 2, 3, \ldots, k, \tag{1.2}$$

where $\tilde{\theta}_i$ is the maximum likelihood estimator of $\theta_1 = \cdots = \theta_i$ based on $\tilde{n}_i = \sum_{j=1}^{i} n_j$ observations $[X_1, \ldots, X_i]$. The likelihood ratio criterion $\lambda$ for testing $H$ against $K$ is decomposed as

$$\lambda = \lambda_{2|1}\lambda_{3|2}\cdots\lambda_{k|(k-1)}. \tag{1.3}$$

The purpose of this paper is to show the independence of $\lambda_{j|(j-1)}$, $j = 2, 3, \ldots, k$ up to the order $1/n$.

It should be noted that by use of an appropriate parameter transformation this testing problem is reduced to the case of one sample problem dealt with by Bickel and Ghosh (1990) and Takemura and Kuriki (1996). However, it is not self-evident to have a concrete expression because of a complexity of a parameter structure even though our problem is included in a general set up, and it would be worth to express a final result by original parameters and to be able to handle this problem without any choice of an appropriate prior probability density function.

## 2. Asymptotic independence of LRC

Let $\theta_i$, $i = 1, 2, \ldots, k$ be a univariate parameter. Defining the log-likelihood function based on independent random sample $x_{i1}, \ldots, x_{in_i}$ by

$$L_i(\theta_i) = \sum_{\alpha=1}^{n_i} \log f(x_{i\alpha} \mid \theta_i), \qquad i = 1, 2, \ldots, k,$$

the following notations and convensions will be used. We assume that each $L_i(\theta_i)$ is regular with respect to $\theta_i$ derivatives.

(i) $y_i^{(l)} = n_i^{-l/2} \sum_{\alpha=1}^{n_i} \dfrac{\partial^l L_i(\theta_i)}{\partial \theta_i^l}$, $l = 1, 2, 3, 4, \quad y_i \equiv y_i^{(1)}$, $i = 1, 2, \ldots, k$,

(ii) $m_{(r_1^{\alpha_1}, r_2^{\alpha_2}, \ldots, r_l^{\alpha_l})}(\theta_i)$

$$= \int \left\{ \frac{\partial^{r_1} \log f(x \mid \theta_i)}{\partial \theta_i^{r_1}} \right\}^{\alpha_1} \cdots \left\{ \frac{\partial^{r_l} \log f(x \mid \theta_i)}{\partial \theta_i^{r_l}} \right\}^{\alpha_l} f(x \mid \theta_i)\, dx.$$

Bartlett identities (Barndorff-Nielsen and Cox (1994)) hold

$$\begin{aligned}
&m_{(2)}(\theta_i) + m_{(1^2)}(\theta_i) = 0,\\
&m_{(3)}(\theta_i) + 3m_{(21)}(\theta_i) + m_{(1^3)}(\theta_i) = 0,\\
&m_{(4)}(\theta_i) + 4m_{(31)}(\theta_i) + 3m_{(2^2)}(\theta_i) + 6m_{(21^2)}(\theta_i) + m_{(1^4)}(\theta_i) = 0,
\end{aligned}$$

(iii) $\rho_i = n_i/n (> 0)$, $\quad \sum_{i=1}^k \rho_i = 1$.

Under the hypothesis $H : \theta_1 = \cdots = \theta_k = \theta$ (say) all moments are expressed as $m_{(3)}(\theta) = m_{(3)}$, $m_{(21)}(\theta) = m_{(21)}$, $m_{(31)}(\theta) = m_{(31)}$, etc.

By noting

$$\lambda_{j|(j-1)} = \frac{\prod_{i=1}^{j}\prod_{\alpha=1}^{n_i} f(x_{i\alpha} \mid \tilde{\theta}_j)}{\prod_{i=1}^{j}\prod_{\alpha=1}^{n_i} f(x_{i\alpha} \mid \hat{\theta}_i)} \Bigg/ \frac{\prod_{i=1}^{j-1}\prod_{\alpha=1}^{n_i} f(x_{i\alpha} \mid \tilde{\theta}_{j-1})}{\prod_{i=1}^{j-1}\prod_{\alpha=1}^{n_i} f(x_{i\alpha} \mid \hat{\theta}_i)} = \lambda_{12\cdots j}/\lambda_{12\cdots(j-1)},$$

we have

$$-2\log\lambda_{j|(j-1)} = 2\log\lambda_{12\cdots(j-1)} - 2\log\lambda_{12\cdots j}.$$

By use of the asymptotic expansion (2) in Hayakawa (1994) we have

$$(2.1)\qquad -2\log\lambda_{j|(j-1)} = w_0^{(j)} + w_1^{(j)} + w_2^{(j)} + o_p\left(\frac{1}{n}\right), \qquad j = 2,3,\dots,k,$$

where

$$w_0^{(j)} = -\frac{y_j^2}{y_j^{(2)}} + \frac{\left(\sum_{i=1}^{j}\sqrt{\rho_i}y_i\right)^2}{\sum_{i=1}^{j}\rho_i y_i^{(2)}} - \frac{\left(\sum_{i=1}^{j-1}\sqrt{\rho_i}y_i\right)^2}{\sum_{i=1}^{j-1}\rho_i y_i^{(2)}},$$

$$w_1^{(j)} = -\frac{1}{3}\frac{y_j^{(3)}y_j^3}{(y_j^{(2)})^3} - \frac{1}{3}\sum_{i=1}^{j-1}\rho_i\sqrt{\rho_i}y_i^{(3)}\frac{\left(\sum_{i=1}^{j-1}\sqrt{\rho_i}y_i\right)^3}{\left(\sum_{i=1}^{j-1}\rho_i y_i^{(2)}\right)^3}$$

$$+\frac{1}{3}\sum_{i=1}^{j}\rho_i\sqrt{\rho_i}y_i^{(3)}\frac{\left(\sum_{i=1}^{j}\sqrt{\rho_i}y_i\right)^3}{\left(\sum_{i=1}^{j}\rho_i y_i^{(2)}\right)^3},$$

$$w_2^{(j)} = -\frac{1}{4}\frac{(y_j^{(3)})^2 y_j^4}{(y_j^{(2)})^5} + \frac{1}{12}\frac{y_j^{(4)}y_j^4}{(y_j^{(2)})^4}$$

$$-\frac{1}{4}\frac{\left(\sum_{i=1}^{j-1}\rho_i\sqrt{\rho_i}y_i^{(3)}\right)^2\left(\sum_{i=1}^{j-1}\sqrt{\rho_i}y_i\right)^4}{\left(\sum_{i=1}^{j-1}\rho_i y_i^{(2)}\right)^5} + \frac{1}{4}\frac{\left(\sum_{i=1}^{j}\rho_i\sqrt{\rho_i}y_i^{(3)}\right)^2\left(\sum_{i=1}^{j}\sqrt{\rho_i}y_i\right)^4}{\left(\sum_{i=1}^{j}\rho_i y_i^{(2)}\right)^5}$$

$$+\frac{1}{12}\sum_{i=1}^{j-1}\rho_i^2 y_i^{(4)}\frac{\left(\sum_{i=1}^{j-1}\sqrt{\rho_i}y_i\right)^4}{\left(\sum_{i=1}^{j-1}\rho_i y_i^{(2)}\right)^4} - \frac{1}{12}\sum_{i=1}^{j}\rho_i^2 y_i^{(4)}\frac{\left(\sum_{i=1}^{j}\sqrt{\rho_i}y_i\right)^4}{\left(\sum_{i=1}^{j}\rho_i y_i^{(2)}\right)^4}.$$

To find the moment generating function of these statistics we need to use the Edgeworth type expansion of the joint density function of $y_i$, $y_i^{(2)}$, $y_i^{(3)}$, $y_i^{(4)}$, $i = 1,2,\dots,k$, which is stated as follows.

$$(2.2)\qquad f = f_0\left[1 + \frac{1}{\sqrt{n}}F_1 + \frac{1}{n}F_2\right] + o\left(\frac{1}{n}\right),$$

where

$$f_0 = \prod_{i=1}^{k}(2\pi m_{(1^2)}(\theta_i))^{-1/2}\exp\{-y_i^2/2m_{(1^2)}(\theta_i)\}\prod_{l=2}^{4}\delta_{li},$$

$$F_1 = \frac{1}{6}\sum_{i=1}^{k}\frac{1}{\sqrt{\rho_i}}m_{(1^3)}(\theta_i)\,H_3(y_i) - \sum_{i=1}^{k}\frac{1}{\sqrt{\rho_i}}m_{(21)}(\theta_i)\,H_1(y_i)\,d_{2i}^{(1)},$$

$$\begin{aligned}
F_2 = &\frac{1}{2}\sum_{i=1}^{k}\frac{1}{\rho_i}\{m_{(2^2)}(\theta_i) - m_{(2)}^2(\theta_i)\}d_{2i}^{(2)}\\
&-\frac{1}{2}\sum_{i=1}^{k}\frac{1}{\rho_i}\{m_{(21^2)}(\theta_i) - m_{(2)}(\theta_i)\,m_{(1^2)}(\theta_i)\}H_2(y_i)\,d_{2i}^{(1)}\\
&-\sum_{i=1}^{k}\frac{1}{\rho_i}m_{(31)}(\theta_i)\,H_1(y_i)d_{3i}^{(1)} + \frac{1}{2}\sum_{i=1}^{k}\frac{1}{\rho_i}m_{(21)}^2(\theta_i)\,H_2(y_i)\,d_{2i}^{(2)}\\
&+\frac{1}{24}\sum_{i=1}^{k}\frac{1}{\rho_i}\{m_{(1^4)}(\theta_i) - 3m_{(1^2)}^2(\theta_i)\}H_4(y_i)\\
&-\frac{1}{6}\sum_{i=1}^{k}\frac{1}{\rho_i}m_{(21)}(\theta_i)\,m_{(1^3)}(\theta_i)\,H_4(y_i)\,d_{2i}^{(1)}\\
&+\frac{1}{72}\sum_{i=1}^{k}\frac{1}{\rho_i}m_{(1^3)}^2(\theta_i)\,H_6(y_i)\\
&+\frac{1}{2}\sum_{i\neq j}\frac{1}{\sqrt{\rho_i\,\rho_j}}m_{(21)}(\theta_i)\,m_{(21)}(\theta_j)\,H_1(y_i)\,H_1(y_j)\,d_{2i}^{(1)}d_{2j}^{(1)}\\
&-\frac{1}{6}\sum_{i\neq j}\frac{1}{\sqrt{\rho_i\,\rho_j}}m_{(21)}(\theta_i)\,m_{(1^3)}(\theta_j)\,H_1(y_i)\,H_3(y_j)\,d_{2i}^{(1)}\\
&+\frac{1}{72}\sum_{i\neq j}\frac{1}{\sqrt{\rho_i\,\rho_j}}m_{(1^3)}(\theta_i)\,m_{(1^3)}(\theta_j)\,H_3(y_i)\,H_3(y_j),
\end{aligned}$$

$$\delta_{li} = \delta(y_i^{(l)} - m_{(l)}(\theta_i)\,/n_i^{(l-2)/2}),$$

$$d_{li}^{(r)} = \delta^{(r)}(y_i^{(l)} - m_{(l)}(\theta_i)\,/n_i^{(l-2)/2})/\delta(y_i^{(l)} - m_{(l)}(\theta_i)\,/n_i^{(l-2)/2})$$

and $\delta^{(r)}$ is the $r$-th derivative of Dirac delta function $\delta$. $H_r(y)$ is defined by

$$\frac{d^n}{dy^n}\exp\left(-\frac{y^2}{2m_{(1^2)}}\right) = (-1)^n H_n(y)\exp\left(-\frac{y^2}{2m_{(1^2)}}\right). \tag{2.3}$$

PROPOSITION. *Under the hypothesis $H$, $-2\log\lambda_{j|(j-1)}$, $j = 2,3,\ldots,k$ are mutually independent in the limit.*

PROOF. With help of law of large numbers $y_i^{(2)}$ converges to $m_{(2)}(\theta_i) = -m_{(1^2)}(\theta_i)$ in the limit and $y_i^{(l)}$, $l \geq 3$ converges to zero in the limit, respectively. Thus we have

$$\begin{aligned}
\tilde{w}_0^{(2)} &= \operatorname*{p\,lim}_{\substack{n_i\to\infty\\ i=1,\cdots,k}}\{-2\log\lambda_{2|1}\}\\
&= \frac{1}{m_{(1^2)}(\theta)}\left\{\sum_{i=1}^{2}y_i^2 - \frac{1}{\hat{\rho}_2}\left(\sum_{i=1}^{2}\sqrt{\rho_i}y_i\right)^2\right\},
\end{aligned} \tag{2.4}$$

$$(2.5)\quad \tilde{w}_0^{(j)} = \underset{\substack{n_i\to\infty \\ i=1,\cdots,k}}{p\lim} \{-2\log\lambda_{j|(j-1)}\}$$
$$= \frac{1}{m_{(1^2)}(\theta)}\left\{y_j^2 - \frac{1}{\hat{\rho}_j}\left(\sum_{i=1}^{j}\sqrt{\rho_i}y_i\right)^2 + \frac{1}{\hat{\rho}_{j-1}}\left(\sum_{i=1}^{j-1}\sqrt{\rho_i}y_i\right)^2\right\}, \qquad 3\le j\le k,$$

where $\hat{\rho}_j = \sum_{i=1}^{j}\rho_i$.

Define $P_j = \sqrt{\rho_j^*}\sqrt{\rho_j^*}'$, $I_{jj} = e_je_j'$ and $y = (y_1, y_2, \dots, y_k)'$, where $\sqrt{\rho_j^*} = (\sqrt{\rho_1}, \dots, \sqrt{\rho_j}, 0, \dots, 0)'$ and $e_j' = (0, \dots, 0, \overset{j}{\check{1}}, 0, \dots, 0)$. Then (2.4), (2.5) are expressed as

$$(2.6)\qquad \frac{1}{m_{(1^2)}(\theta)}y'Q_jy, \qquad Q_j = I_{jj} - P_j + P_{j-1}, \quad j = 2, 3, \dots, k$$

and $y$ is $k$ dimensional random vector with mean 0 and covariance matrix $m_{(1^2)}(\theta)I_k$.

By noting

$$(2.7)\qquad Q_jQ_l = \delta_{jl}Q_j, \qquad rank\, Q_j = 1$$

and by use of Craig theorem (e.g. Ogawa (1949), and others) we have that $-2\log\lambda_{j|(j-1)}$, $j = 2, 3, \dots, k$ are mutually independent and these have chi-square distribution with one degree of freedom in the limit, respectively.

THEOREM. *Under the hypothesis $H$, the joint moment generating function of $-2\log\lambda_{j|(j-1)}$, $j = 2, 3, \dots, k$ is expressed as*

$$(2.8)\quad M(t_2, \dots, t_k) = E\left[\exp\left\{\sum_{j=2}^{k} t_j(-2\log\lambda_{j|(j-1)})\right\}\right]$$
$$= \prod_{j=2}^{k}\frac{1}{(1-2t_j)^{1/2}}\cdot\left\{1 + \frac{1}{n}A_j\left(\frac{1}{1-2t_j} - 1\right) + o\left(\frac{1}{n}\right)\right\},$$

*where*

$$(2.9)\qquad A_j = a\frac{1}{\hat{\rho}_j}\left(\frac{\hat{\rho}_{j-1}}{\rho_j} + 1 + \frac{\rho_j}{\hat{\rho}_{j-1}}\right),$$
$$a = \frac{1}{8}\frac{1}{(m_{(1^2)})^2}\{m_{(2^2)} - m_{(1^4)} - 2m_{(21^2)}\}$$
$$+\frac{1}{24}\frac{1}{(m_{(1^2)})^3}\{m_{(3)}m_{(21)} - 5m_{(3)}m_{(1^3)} - 8m_{(21)}m_{(1^3)}\}.$$

*This implies that $-2\log\lambda_{j|(j-1)}$, $j = 2, 3, \dots, k$ are mutually independent up to the order $1/n$.*

PROOF. The proof is given in Section 3.

*Note.* If we set $t_2 = t_3 = \cdots = t_k = t$, then the moment generating function is reduced to the one of $-2\log\lambda$ for testing $H$ against $K$. By noting $\hat{\rho}_1 = \rho_1$ and $\hat{\rho}_k = 1$, and

$$\sum_{j=2}^{k} \frac{1}{\hat{\rho}_j}\left(\frac{\hat{\rho}_{j-1}}{\rho_j} + 1 + \frac{\rho_j}{\hat{\rho}_{j-1}}\right) = \sum_{j=1}^{k} \frac{1}{\rho_j} - 1,$$

the moment generating function is expressed as

$$(2.10)\qquad (1-2t)^{-1/2(k-1)}\left[1 + \frac{a}{n}\left(\sum_{i=1}^{k}\frac{1}{\rho_i} - 1\right)\left(\frac{1}{1-2t} - 1\right) + o\left(\frac{1}{n}\right)\right].$$

This is the one given by Hayakawa (1994).

*Note.* $-2\log\lambda_{j|(j-1)}$, $j = 2, 3, \ldots, k$ are Bartlett correctable.

## 3. Proof of theorem

The joint moment generating function of $-2\log\lambda_{j|(j-1)}$, $j = 2, 3, \ldots, k$ is expressed as

$$(3.1)\qquad M(t_2, \ldots, t_k) = \int \exp\left\{\sum_{j=2}^{k} t_j w_0^{(j)} + \sum_{j=2}^{k} t_j w_1^{(j)} + \sum_{j=2}^{k} t_j w_2^{(j)}\right\}$$
$$\times f_0\left[1 + \frac{1}{\sqrt{n}}F_1 + \frac{1}{n}F_2\right] dy dy^{(2)} dy^{(3)} dy^{(4)} + o\left(\frac{1}{n}\right),$$

where $y^{(l)} = (y_1^{(l)}, y_2^{(l)}, \ldots, y_k^{(l)})'$, $l = 2, 3, 4$.

The limit of the moment generation function is expressed as

$$(3.2)\qquad \int \exp\left\{\sum_{j=2}^{k} t_j \tilde{w}_0^{(j)}\right\}\prod_{i=1}^{k} n(0, m_{(1^2)}(\theta))dy$$
$$= |\Omega|^{1/2}\int \frac{1}{(2\pi m_{(1^2)}(\theta))^{k/2}|\Omega|^{1/2}} \exp\left\{-\frac{1}{2m_{(1^2)}(\theta)} y'\Omega^{-1}y\right\} dy,$$

where

$$(3.3)\qquad \Omega = \sum_{j=2}^{k} c_j Q_j + \sqrt{\rho}\sqrt{\rho}', \qquad \sqrt{\rho}' = (\sqrt{\rho_1}, \sqrt{\rho_2}, \ldots, \sqrt{\rho_k}),$$
$$c_j = (1-2t_j)^{-1}, \qquad j = 2, 3, \ldots, k, \qquad |\Omega| = \prod_{j=2}^{k} c_j.$$

This implies that $y$ is dealt with as a normal random vector with mean zero and covariance matrix $m_{(1^2)}(\theta)\Omega$.

We use following different expectation notations according to the order situations.

(3.4) $|\Omega|^{1/2}\hat{E}[g] = \int g \cdot \exp\left\{\sum_{j=2}^{k} t_j w_0^{(j)}\right\} f_0 dy dy^{(2)} dy^{(3)} dy^{(4)},$

(3.5) $|\Omega|^{1/2}\hat{\hat{E}}[g] = \int g \cdot \exp\left\{\sum_{j=2}^{k} t_j w_0^{(j)} + \sum_{j=2}^{k} t_j w_1^{(j)}\right\} f_0 dy dy^{(2)} dy^{(3)} dy^{(4)},$

(3.6) $|\Omega|^{1/2}\hat{\hat{\hat{E}}}[g] = \int g \cdot \exp\left\{\sum_{j=2}^{k} t_j w_0^{(j)} + \sum_{j=2}^{k} t_j w_1^{(j)} + \sum_{j=2}^{k} t_j w_2^{(j)}\right\} f_0 dy dy^{(2)} dy^{(3)} dy^{(4)}.$

Thus we have variance and covariance of $y$ with respect to an operator $\hat{E}$ as follows.

(3.7) $\hat{E}[1] = 1,$

(3.8) $\hat{E}[y_1^2/m_{(1^2)}] = w_{11} = \rho_1\left[c_2\frac{\rho_2}{\rho_1\hat{\rho}_2} + c_3\frac{\rho_3}{\hat{\rho}_2\hat{\rho}_3} + \cdots + c_k\frac{\rho_k}{\hat{\rho}_{k-1}\hat{\rho}_k} + 1\right],$

(3.9) $\hat{E}[y_j^2/m_{(1^2)}] = w_{jj} = \rho_j\left[c_j\frac{\hat{\rho}_{j-1}}{\rho_j\hat{\rho}_j} + c_{j+1}\frac{\rho_{j+1}}{\hat{\rho}_j\hat{\rho}_{j+1}} + \cdots + c_k\frac{\rho_k}{\hat{\rho}_{k-1}\hat{\rho}_k} + 1\right],$
$j = 2, 3, \ldots, k,$

(3.10) $\hat{E}[y_j y_l/m_{(1^2)}] = w_{jl} = \sqrt{\rho_j\rho_l}\left[c_l\left(-\frac{1}{\hat{\rho}_l}\right) + c_{l+1}\frac{\rho_{l+1}}{\hat{\rho}_l\hat{\rho}_{l+1}} + \cdots + c_k\frac{\rho_k}{\hat{\rho}_{k-1}\hat{\rho}_k} + 1\right],$
$1 \le j < l \le k.$

where $c_j = (1-2t_j)^{-1}$, $j = 2, 3, \ldots, k$.

Hereafter we give several moments.

(I) The integration of the first term in (2.2).

$$\int \exp\left\{\sum_{j=2}^{k} t_j w_0^{(j)} + \sum_{j=2}^{k} t_j w_1^{(j)} + \sum_{j=2}^{k} t_j w_2^{(j)}\right\} f_0 dy dy^{(2)} dy^{(3)} dy^{(4)}$$
$$= \int \exp\left\{\sum_{j=2}^{k} t_j \tilde{w}_0^{(j)}\right\} n(0, m_{(1^2)}(\theta) I)$$
$$\cdot\left[1 + \frac{1}{\sqrt{n}}\sum_{j=2}^{k} t_j \tilde{w}_1^{(j)} + \frac{1}{n}\left\{\sum_{j=2}^{k} t_j \tilde{w}_2^{(j)} + \frac{1}{2}\left(\sum_{j=2}^{k} t_j \tilde{w}_1^{(j)}\right)^2\right\}\right] dy + o\left(\frac{1}{n}\right),$$

where

$$\tilde{w}_1^{(2)} = \frac{1}{3}\frac{m_{(3)}}{(m_{(1^2)})^3}\left[\sum_{i=1}^{2}\frac{y_i^3}{\sqrt{\rho_i}} - \frac{1}{\hat{\rho}_2^2}\left(\sum_{i=1}^{2}\sqrt{\rho_i}y_i\right)^3\right],$$
$$\tilde{w}_1^{(j)} = \frac{1}{3}\frac{m_{(3)}}{(m_{(1^2)})^3}\left[\frac{y_j^3}{\sqrt{\rho_j}} - \frac{1}{\hat{\rho}_j^2}\left(\sum_{i=1}^{j}\sqrt{\rho_i}y_i\right)^3 + \frac{1}{\hat{\rho}_{j-1}^2}\left(\sum_{i=1}^{j-1}\sqrt{\rho_i}y_i\right)^3\right],$$
$3 \le j \le k,$

and

$$\tilde{w}_2^{(2)} = \left\{ \frac{1}{4}\frac{(m_{(3)})^2}{(m_{(1^2)})^5} + \frac{1}{12}\frac{m_{(4)}}{(m_{(1^2)})^4} \right\} \left[ \sum_{i=1}^{2} \frac{y_i^4}{\rho_i} - \frac{1}{\hat{\rho}_2^3}\left( \sum_{i=1}^{2} \sqrt{\rho_i} y_i \right)^4 \right],$$

$$\tilde{w}_2^{(j)} = \left\{ \frac{1}{4}\frac{(m_{(3)})^2}{(m_{(1^2)})^5} + \frac{1}{12}\frac{m_{(4)}}{(m_{(1^2)})^4} \right\}$$
$$\times \left[ \frac{y_j^4}{\rho_j} - \frac{1}{\hat{\rho}_j^3}\left( \sum_{i=1}^{j} \sqrt{\rho_i} y_i \right)^4 + \frac{1}{\hat{\rho}_{j-1}^3}\left( \sum_{i=1}^{j-1} \sqrt{\rho_i} y_i \right)^4 \right], \quad 3 \le j \le k.$$

Thus by setting $u_j = c_j - 1$, $j = 2, 3, \ldots, k$, we have

$$(3.11)\quad \hat{E}\left[ \sum_{j=2}^{k} t_j \tilde{w}_1^{(j)} \right] = 0,$$

$$(3.12)\quad \hat{E}\left[ \sum_{j=2}^{k} t_j \tilde{w}_2^{(j)} \right] = \left\{ \frac{3}{8}\frac{(m_{(3)})^2}{(m_{(1^2)})^3} + \frac{1}{8}\frac{m_{(4)}}{(m_{(1^2)})^2} \right\} \left[ \sum_{j=2}^{k} u_j^2 \frac{1}{\hat{\rho}_j}\left( \frac{\hat{\rho}_{j-1}}{\rho_j} - 1 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) \right.$$
$$\left. + \sum_{j=2}^{k} u_j \frac{1}{\hat{\rho}_j}\left( \frac{\hat{\rho}_{j-1}}{\rho_j} + 1 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) + 2 \sum_{2 \le p < q \le k} u_p u_q \frac{\rho_q}{\hat{\rho}_{q-1}\hat{\rho}_q} \right],$$

$$(3.13)\quad \hat{E}\left[ \frac{1}{2}\left\{ \sum_{j=2}^{k} t_j \tilde{w}_1^{(j)} \right\}^2 \right] = \frac{(m_{(3)})^2}{(m_{(1^2)})^3} \left[ \frac{5}{24} \sum_{j=2}^{k} u_j^3 \frac{1}{\hat{\rho}_j}\left( \frac{\hat{\rho}_{j-1}}{\rho_j} - 2 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) \right.$$
$$+ \frac{1}{24} \sum_{j=2}^{k} u_j^2 \frac{1}{\hat{\rho}_j}\left( 5\frac{\hat{\rho}_{j-1}}{\rho_j} - 1 + 5\frac{\rho_j}{\hat{\rho}_{j-1}} \right) + \frac{1}{4} \sum_{2 \le p < q < r \le k} u_p u_q u_r \frac{\rho_r}{\hat{\rho}_{r-1}\hat{\rho}_r}$$
$$+ \frac{3}{8} \sum_{2 \le p < q \le k} u_p^2 u_q \frac{\rho_q}{\hat{\rho}_{q-1}\hat{\rho}_q} + \frac{1}{4} \sum_{2 \le p < q \le k} u_p u_q^2 \frac{(\rho_q - \hat{\rho}_{q-1})}{\hat{\rho}_{q-1}\hat{\rho}_q}$$
$$\left. + \frac{1}{4} \sum_{2 \le p < q \le k} u_p u_q \frac{\rho_q}{\hat{\rho}_{q-1}\hat{\rho}_q} \right].$$

(II) The integration of the second term in (2.2).

To have terms of order $1/n$ it is enough only to use up to the second term in the exponent,

$$\int \exp\left\{ \sum_{j=2}^{k} t_j w_0^{(j)} + \sum_{j=2}^{k} t_j w_1^{(j)} \right\} f_0 F_1 dy dy^{(2)} dy^{(3)} dy^{(4)}.$$

$$(\text{II-1})\quad \int \exp\left\{ \sum_{j=2}^{k} t_j w_0^{(j)} + \sum_{j=2}^{k} t_j w_1^{(j)} \right\} f_0 \frac{m_{(1^3)}}{6} \sum_{i=1}^{k} \frac{1}{\sqrt{\rho_i}} H_3(y_i) dy dy^{(2)} dy^{(3)} dy^{(4)}$$
$$= \frac{1}{\sqrt{n}} \int \exp\left\{ \sum_{j=2}^{k} t_j \tilde{w}_0^{(j)} \right\} n(0, m_{(1^2)}(\theta) I_k) \left\{ \sum_{j=2}^{k} t_j \tilde{w}_1^{(j)} \right\}$$

$$\times\frac{m_{(1^3)}}{6}\sum_{i=1}^{k}\frac{1}{\sqrt{\rho_i}}H_3(y_i)dy+o\left(\frac{1}{\sqrt{n}}\right),$$

$$(3.14)\quad \hat{E}\left[\sum_{j=2}^{k}t_j\tilde{w}_1^{(j)}\frac{m_{(1^3)}}{6}\sum_{i=1}^{k}\frac{1}{\sqrt{\rho_i}}H_3(y_i)\right]$$

$$=\frac{m_{(3)}m_{(1^3)}}{(m_{(1^2)})^3}\left[\frac{5}{12}\sum_{j=2}^{k}u_j^3\frac{1}{\hat{\rho}_j}\left(\frac{\hat{\rho}_{j-1}}{\rho_j}-2+\frac{\rho_j}{\hat{\rho}_{j-1}}\right)\right.$$

$$+\frac{1}{12}\sum_{j=2}^{k}u_j^2\frac{1}{\hat{\rho}_j}\left(7\frac{\hat{\rho}_{j-1}}{\rho_j}-5+7\frac{\rho_j}{\hat{\rho}_{j-1}}\right)+\frac{1}{6}\sum_{j=2}^{k}u_j\frac{1}{\hat{\rho}_j}\left(\frac{\hat{\rho}_{j-1}}{\rho_j}+1+\frac{\rho_j}{\hat{\rho}_{j-1}}\right)$$

$$+\frac{1}{2}\sum_{2\le p<q<r\le k}u_pu_qu_r\frac{\rho_r}{\hat{\rho}_{r-1}\hat{\rho}_r}+\frac{3}{4}\sum_{2\le p<q\le k}u_p^2u_q\frac{\rho_q}{\hat{\rho}_{q-1}\hat{\rho}_q}$$

$$\left.+\frac{1}{2}\sum_{2\le p<q\le k}u_pu_q^2\frac{(\rho_q-\hat{\rho}_{q-1})}{\hat{\rho}_{q-1}\hat{\rho}_q}+\sum_{2\le p<q\le k}u_pu_q\frac{\rho_q}{\hat{\rho}_{q-1}\hat{\rho}_q}\right].$$

$$(\text{II-2})\qquad -\frac{m_{(21)}}{m_{(1^2)}}\int\exp\left\{\sum_{j=2}^{k}t_jw_0^{(j)}+\sum_{j=2}^{k}t_jw_1^{(j)}\right\}f_0\sum_{i=1}^{k}\frac{y_i}{\sqrt{\rho_i}}d_{2i}^{(1)}dydy^{(2)}dy^{(3)}dy^{(4)}.$$

Noting the integration with respect to $d_{2i}^{(l)}$,

$$(3.15)\qquad \int h(y_i^{(2)})f_0d_{2i}^{(l)}dy_i^{(2)}=(-1)^l\frac{\partial^l h}{\partial(y_i^{(2)})^l}\bigg|_{y_i^{(2)}=m_{(2)}=-m_{(1^2)}},$$

we have

$$(3.16)\quad -\frac{m_{(21)}}{m_{(1^2)}}\hat{E}\left[\sum_{i=1}^{k}\frac{y_i}{\sqrt{\rho_i}}d_{2i}^{(1)}\right]=\frac{1}{\sqrt{n}}\frac{m_{(3)}m_{(21)}}{(m_{(1^2)})^3}\left[\frac{5}{4}\sum_{j=2}^{k}u_j^2\frac{1}{\hat{\rho}_j}\left(\frac{\hat{\rho}_{j-1}}{\rho_j}-2+\frac{\rho_j}{\hat{\rho}_{j-1}}\right)\right.$$

$$+\frac{1}{4}\sum_{j=2}^{k}u_j^2\frac{1}{\hat{\rho}_j}\left(11\frac{\hat{\rho}_{j-1}}{\rho_j}-7+11\frac{\rho_j}{\hat{\rho}_{j-1}}\right)+\frac{3}{2}\sum_{j=2}^{k}u_j\frac{1}{\hat{\rho}_j}\left(\frac{\hat{\rho}_{j-1}}{\rho_j}+1+\frac{\rho_j}{\hat{\rho}_{j-1}}\right)$$

$$+\frac{3}{2}\sum_{2\le p<q<r\le k}u_pu_qu_r\frac{\rho_r}{\hat{\rho}_{r-1}\hat{\rho}_r}+\frac{9}{4}\sum_{2\le p<q\le k}u_p^2u_q\frac{\rho_q}{\hat{\rho}_{q-1}\hat{\rho}_q}$$

$$\left.+\frac{3}{2}\sum_{2\le p<q\le k}u_pu_q^2\frac{(\rho_q-\hat{\rho}_{q-1})}{\hat{\rho}_{q-1}\hat{\rho}_q}+\frac{9}{2}\sum_{2\le p<q\le k}u_pu_q\frac{\rho_q}{\hat{\rho}_{q-1}\hat{\rho}_q}\right]+O\left(\frac{1}{n}\right).$$

(III) The terms of order $1/n$ in (2.2) are obtained after some lengthy algebra as follows.

$$(3.17)\qquad \hat{E}\left[\frac{1}{2}\sum_{i=1}^{k}\frac{1}{\rho_i}\left\{m_{(2^2)}-(m_{(2)})^2\right\}d_{2i}^{(2)}\right]$$

$$= \frac{(m_{(2^2)} - (m_{(2)})^2)}{(m_{(1^2)})^2} \left[ \frac{3}{8} \sum_{j=2}^{k} u_j^2 \frac{1}{\hat{\rho}_j} \left( \frac{\hat{\rho}_{j-1}}{\rho_j} - 1 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) \right.$$
$$\left. + \frac{1}{2} \sum_{j=2}^{k} u_j \frac{1}{\hat{\rho}_j} \left( \frac{\hat{\rho}_{j-1}}{\rho_j} + 1 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) + \frac{3}{4} \sum_{2 \le p < q \le k} u_p u_q \frac{\rho_q}{\hat{\rho}_{q-1} \hat{\rho}_q} \right],$$

$$(3.18) \qquad \hat{E}\left[ -\frac{1}{2}(m_{(21^2)} - m_{(2)} m_{(1^2)}) \sum_{i=1}^{k} \frac{1}{\rho_i} H_2(y_i) d_{2i}^{(1)} \right]$$
$$= \frac{(m_{(21^2)} - m_{(2)} m_{(1^2)})}{(m_{(1^2)})^2} \left[ \frac{3}{4} \sum_{j=2}^{k} u_j^2 \frac{1}{\hat{\rho}_j} \left( \frac{\hat{\rho}_{j-1}}{\rho_j} - 1 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) \right.$$
$$\left. + \frac{1}{2} \sum_{j=2}^{k} u_j \frac{1}{\hat{\rho}_j} \left( \frac{\hat{\rho}_{j-1}}{\rho_j} + 1 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) + \frac{3}{2} \sum_{2 \le p < q \le k} u_p u_q \frac{\rho_q}{\hat{\rho}_{q-1} \hat{\rho}_q} \right],$$

$$(3.19) \quad -m_{(31)}\hat{\hat{E}} \left[ \sum_{i=1}^{k} \frac{1}{\rho_i} H_1(y_i) d_{3i}^{(1)} \right] = \frac{m_{(31)}}{(m_{(1^2)})^2} \left[ \frac{1}{2} \sum_{j=2}^{k} u_j^2 \frac{1}{\hat{\rho}_j} \left( \frac{\hat{\rho}_{j-1}}{\rho_j} - 1 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) \right.$$
$$\left. + \frac{1}{2} \sum_{j=2}^{k} u_j \frac{1}{\hat{\rho}_j} \left( \frac{\hat{\rho}_{j-1}}{\rho_j} + 1 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) + \sum_{2 \le p < q \le k} u_p u_q \frac{\rho_q}{\hat{\rho}_{q-1} \hat{\rho}_q} \right] + O\left( \frac{1}{\sqrt{n}} \right),$$

$$(3.20) \quad \frac{1}{24} \left\{ m_{(1^4)} - 3(m_{(1^2)})^2 \right\} \hat{E} \left[ \sum_{i=1}^{k} \frac{1}{\rho_i} H_4(y_i) \right] = \frac{m_{(1^4)} - 3(m_{(1^2)})^2}{(m_{(1^2)})^2}$$
$$\times \left[ \frac{1}{8} \sum_{j=2}^{k} u_j^2 \frac{1}{\hat{\rho}_j} \left( \frac{\hat{\rho}_{j-1}}{\rho_j} - 1 + \frac{\rho_j}{\hat{\rho}_{j-1}} \right) + \frac{1}{4} \sum_{2 \le p < q \le k} u_p u_q \frac{\rho_q}{\hat{\rho}_{q-1} \hat{\rho}_q} \right],$$

$$(3.21) \, \frac{1}{2}(m_{(21)})^2 \hat{E} \left[ \sum_{i=1}^{k} \frac{1}{\rho_i} H_2(y_i) d_{2i}^{(2)} \right]$$
$$= \frac{(m_{(21)})^2}{(m_{(1^2)})^3} \left[ \frac{15}{8} \sum_{j=2}^{k} u_j^3 \frac{1}{\hat{\rho}_j^3} \left\{ \left( \sum_{\alpha=1}^{j-1} \rho_\alpha^2 \right) \frac{\rho_j^3}{\hat{\rho}_{j-1}^3} + \frac{\hat{\rho}_{j-1}^3}{\rho_j} \right\} \right.$$
$$+ \sum_{j=2}^{k} u_j^2 \left[ \left( \sum_{\alpha=1}^{j-1} \rho_\alpha^2 \right) \left\{ \frac{3}{2} \frac{\rho_j^3}{\hat{\rho}_{j-1}^3 \hat{\rho}_j^3} + \frac{9}{2} \frac{\rho_j^2}{\hat{\rho}_{j-1}^2 \hat{\rho}_j^3} + \frac{3}{2} \frac{\rho_j^2}{\hat{\rho}_{j-1}^3 \hat{\rho}_j^2} \right\} \right.$$
$$\left. + \frac{15}{8} \frac{\hat{\rho}_{j-1}^3}{\rho_j \hat{\rho}_j^3} + \frac{39}{8} \frac{\hat{\rho}_{j-1}^2}{\hat{\rho}_j^3} + \frac{9}{8} \frac{\hat{\rho}_{j-1}^2}{\rho_j \hat{\rho}_j^2} \right]$$
$$+ \sum_{j=2}^{k} u_j \left[ \left( \sum_{\alpha=1}^{j-1} \rho_\alpha^2 \right) \left\{ \frac{\rho_j^2}{\hat{\rho}_{j-1}^3 \hat{\rho}_j^2} - \frac{1}{2} \frac{\rho_j^2}{\hat{\rho}_{j-1}^2 \hat{\rho}_j^3} + \frac{1}{2} \frac{\rho_j}{\hat{\rho}_{j-1} \hat{\rho}_j^3} + \frac{5}{2} \frac{\rho_j}{\hat{\rho}_{j-1}^2 \hat{\rho}_j^2} \right\} \right.$$

$$+\frac{3}{2}\frac{\hat\rho_{j-1}^2}{\rho_j\hat\rho_j^2}-\frac{1}{2}\frac{\hat\rho_{j-1}\rho_j}{\hat\rho_j^3}-\frac{3}{2}\frac{\rho_j^2}{\hat\rho_j^3}+\frac{\hat\rho_{j-1}}{\hat\rho_j^2}+\frac{3}{2}\frac{1}{\hat\rho_j}-\frac{1}{2}\frac{\hat\rho_{j-1}}{\rho_j\hat\rho_j}\Bigg]$$

$$+\frac{45}{4}\sum_{2\le p<q<r\le k}u_pu_qu_r\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\frac{\rho_r}{\hat\rho_{r-1}\hat\rho_r}$$

$$+\frac{45}{8}\sum_{2\le p<q\le k}u_p^2u_q\frac{1}{\hat\rho_{p-1}^2\hat\rho_p^2}\left\{\left(\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\rho_p^2+\hat\rho_{p-1}^4\right\}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}$$

$$+\frac{45}{8}\sum_{2\le p<q\le k}u_pu_q^2\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q^2}{\hat\rho_{q-1}^2\hat\rho_q^2}$$

$$+\sum_{2\le p<q\le k}u_pu_q\left\{\frac{17}{4}\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p^2}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}+\frac{3}{4}\frac{\rho_p}{\hat\rho_{p-1}^2\hat\rho_p}\right.$$

$$\times\left(\hat\rho_{p-1}^2-\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}+3\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\left(\frac{1}{\hat\rho_{q-1}^2}-\frac{1}{\hat\rho_q^2}\right)$$

$$+\frac{3}{4}\frac{1}{\hat\rho_{p-1}^2\hat\rho_p^2}\left\{\rho_p^2\left(\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)+\hat\rho_{p-1}^4\right\}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}-\frac{3}{4}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}$$

$$\left.+\frac{1}{2}\left\{\left(\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_p(6\hat\rho_p-\hat\rho_{p-1})}{\hat\rho_{p-1}^2\hat\rho_p^2}+\frac{\hat\rho_{p-1}(6\hat\rho_p-\rho_p)}{\hat\rho_p^2}\right\}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right\}\Bigg],$$

$$(3.22)\quad -\frac{1}{6}m_{(21)}m_{(1^3)}\hat E\left[\sum_{i=1}^{k}\frac{1}{\rho_i}H_4(y_i)d_{2i}^{(1)}\right]$$

$$=\frac{m_{(21)}m_{(1^3)}}{(m_{(1^2)})^3}\Bigg[\frac{5}{4}\sum_{j=2}^{k}u_j^3\frac{1}{\hat\rho_j^3}\left\{\left(\sum_{\alpha=1}^{j-1}\rho_\alpha^2\right)\frac{\rho_j^3}{\hat\rho_{j-1}^3}+\frac{\hat\rho_{j-1}^3}{\rho_j}\right\}$$

$$+\sum_{j=2}^{k}u_j^2\left\{\left(\sum_{\alpha=1}^{j-1}\rho_\alpha^2\right)\left\{\frac{\rho_j^3}{\hat\rho_{j-1}^3\hat\rho_j^3}+2\frac{\rho_j^2}{\hat\rho_{j-1}^2\hat\rho_j^3}\right\}+\frac{5}{2}\frac{\hat\rho_{j-1}^3}{\rho_j\hat\rho_j^3}-\frac{3}{2}\frac{\hat\rho_{j-1}^2}{\rho_j\hat\rho_j^2}+\frac{7}{2}\frac{\hat\rho_{j-1}^2}{\hat\rho_j^3}\right\}$$

$$+\sum_{j=2}^{k}u_j\left\{\frac{5}{4}\frac{\hat\rho_{j-1}^3}{\rho_j\hat\rho_j^3}-\frac{3}{2}\frac{\hat\rho_{j-1}^2}{\rho_j\hat\rho_j^2}+\frac{9}{4}\frac{\rho_j\hat\rho_{j-1}}{\hat\rho_j^3}+\frac{7}{2}\frac{\hat\rho_{j-1}^2}{\hat\rho_j^3}-\frac{5}{2}\frac{\hat\rho_{j-1}}{\hat\rho_j^2}+\frac{1}{4}\frac{\hat\rho_{j-1}}{\rho_j\hat\rho_j}\right\}$$

$$+\frac{15}{2}\sum_{2\le p<q<r\le k}u_pu_qu_r\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\frac{\rho_r}{\hat\rho_{r-1}\hat\rho_r}$$

$$+\frac{15}{4}\sum_{2\le p<q\le k}u_p^2u_q\frac{1}{\hat\rho_{p-1}^2\hat\rho_p^2}\left\{\rho_p^2\left(\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)+\hat\rho_{p-1}^4\right\}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}$$

$$+\frac{15}{4}\sum_{2\le p<q\le k}u_pu_q^2\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\left(\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right)^2$$

$$+\sum_{2\le p<q\le k} u_p u_q \left\{ 4\frac{1}{\hat\rho_{p-1}^2\hat\rho_p^2}\left\{\rho_p^2\left(\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)+\hat\rho_{p-1}^4\right\}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right.$$

$$+\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\left(\frac{1}{\hat\rho_{q-1}^2}-\frac{1}{\hat\rho_q^2}\right)$$

$$+5\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p^2}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}$$

$$\left.+3\frac{\rho_p}{\hat\rho_{p-1}^2\hat\rho_p}\left(\hat\rho_{p-1}^2-\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}-3\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right\}\Bigg],$$

$$(3.23)\qquad \frac{1}{72}(m_{(1^3)})^2\hat E\left[\sum_{i=1}^{k}\frac{1}{\rho_i}H_6(y_i)\right]$$

$$=\frac{(m_{(1^3)})^2}{(m_{(1^2)})^3}\left[\frac{5}{24}\sum_{j=2}^{k}u_j^3\frac{1}{\hat\rho_j^3}\left\{\left(\sum_{\alpha=1}^{j-1}\rho_\alpha^2\right)\frac{\rho_j^3}{\hat\rho_{j-1}^3}+\frac{\hat\rho_{j-1}^3}{\rho_j}\right\}\right.$$

$$+\frac{5}{4}\sum_{2\le p<q<r\le k}u_pu_qu_r\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\frac{\rho_r}{\hat\rho_{r-1}\hat\rho_r}$$

$$+\frac{5}{8}\sum_{2\le p<q\le k}u_p^2u_q\frac{1}{\hat\rho_{p-1}^2\hat\rho_p^2}\left\{\rho_p^2\left(\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)+\hat\rho_{p-1}^4\right\}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}$$

$$\left.+\frac{5}{8}\sum_{2\le p<q\le k}u_pu_q^2\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q^2}{\hat\rho_{q-1}^2\hat\rho_q^2}\right],$$

$$(3.24)\quad \frac{1}{72}(m_{(1^3)})^2\hat E\left[\sum_{i\ne j}\frac{1}{\sqrt{\rho_i\rho_j}}H_3(y_i)H_3(y_j)\right]$$

$$=\frac{(m_{(1^3)})^2}{(m_{(1^2)})^3}\left[-\frac{5}{24}\sum_{j=2}^{k}u_j^3\frac{1}{\hat\rho_j^3}\left\{\left(\sum_{\alpha=1}^{j-1}\rho_\alpha^2\right)\frac{\rho_j^3}{\hat\rho_{j-1}^3}+2\rho_j\hat\rho_{j-1}-\frac{\rho_j^3}{\hat\rho_{j-1}}\right\}\right.$$

$$+\sum_{2\le p<q<r\le k}u_pu_qu_r\left\{-\frac{5}{4}\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\frac{\rho_r}{\hat\rho_{r-1}\hat\rho_r}\right.$$

$$\left.+\frac{1}{4}\frac{\rho_r}{\hat\rho_{r-1}\hat\rho_r}\right\}$$

$$+\sum_{2\le p<q\le k}u_p^2u_q\left\{-\frac{1}{4}\frac{\rho_p\hat\rho_{p-1}}{\hat\rho_p^2}\left(\frac{\hat\rho_{p-1}}{\rho_p}-3+\frac{\rho_p}{\hat\rho_{p-1}}\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right.$$

$$\left.+\frac{5}{8}\frac{\rho_p^2}{\hat\rho_{p-1}^2\hat\rho_p^2}\left(\hat\rho_{p-1}^2-\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right\}$$

$$+\sum_{2\le p<q\le k} u_p u_q^2 \left\{ \frac{1}{4}\frac{(\rho_q-\hat\rho_{q-1})}{\hat\rho_{q-1}\hat\rho_q} - \frac{5}{8}\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q^2}{\hat\rho_{q-1}^2\hat\rho_q^2}\right\}\Bigg],$$

$$(3.25)\ \frac{1}{2}(m_{(21)})^2\hat E\left[\sum_{i\ne j}\frac{1}{\sqrt{\rho_i\,\rho_j}}H_1(y_i)H_1(y_j)d_{2i}^{(1)}d_{2j}^{(1)}\right]$$

$$= \frac{(m_{(21)})^2}{(m_{(1^2)})^3}\left[\sum_{j=2}^{k}u_j^3\left\{-\frac{15}{8}\left(\sum_{\alpha=1}^{j-1}\rho_\alpha^2\right)\frac{\rho_j^3}{\hat\rho_{j-1}^3\hat\rho_j^3}+\frac{15}{8}\frac{\rho_j^3}{\hat\rho_{j-1}\hat\rho_j^3}-\frac{15}{4}\frac{\rho_j\hat\rho_{j-1}}{\hat\rho_j^3}\right\}\right.$$

$$+\sum_{j=2}^{k}u_j^2\left\{-\frac{3}{2}\left(\sum_{\alpha=1}^{j-1}\rho_\alpha^2\right)\left\{\frac{\rho_j^3}{\hat\rho_{j-1}^3\hat\rho_j^3}+3\frac{\rho_j^2}{\hat\rho_{j-1}^2\hat\rho_j^3}+\frac{\rho_j^2}{\hat\rho_{j-1}^3\hat\rho_j^2}\right\}\right.$$

$$\left.+\frac{3}{2}\frac{\rho_j^3}{\hat\rho_{j-1}\hat\rho_j^3}+3\frac{\rho_j\hat\rho_{j-1}}{\hat\rho_j^3}+3\frac{\rho_j^2}{\hat\rho_j^3}-\frac{3}{2}\frac{\hat\rho_{j-1}^2}{\hat\rho_j^3}+\frac{3}{2}\frac{\rho_j^2}{\hat\rho_{j-1}\hat\rho_j^2}\right\}$$

$$+\sum_{j=2}^{k}u_j\left\{\left(\sum_{\alpha=1}^{j-1}\rho_\alpha^2\right)\left\{\frac{1}{2}\frac{\rho_j^3}{\hat\rho_{j-1}^3\hat\rho_j^3}-\frac{\rho_j^2}{\hat\rho_{j-1}^3\hat\rho_j^2}-\frac{1}{2}\frac{\rho_j}{\hat\rho_{j-1}\hat\rho_j^3}-\frac{5}{2}\frac{\rho_j}{\hat\rho_{j-1}^2\hat\rho_j^2}\right\}\right.$$

$$\left.-\frac{3}{2}\frac{\rho_j\hat\rho_{j-1}}{\hat\rho_j^3}-\frac{1}{2}\frac{\rho_j^2}{\hat\rho_j^3}+\frac{\rho_j^2}{\hat\rho_{j-1}\hat\rho_j^2}+\frac{5}{2}\frac{\rho_j}{\hat\rho_j^2}\right\}$$

$$+\sum_{2\le p<q<r\le k}u_pu_qu_r\left\{-\frac{45}{4}\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\frac{\rho_r}{\hat\rho_{r-1}\hat\rho_r}\right.$$

$$\left.+\frac{9}{4}\frac{\rho_r}{\hat\rho_{r-1}\hat\rho_r}\right\}$$

$$+\sum_{2\le p<q\le k}u_p^2u_q\left\{-\frac{9}{4}\frac{\rho_p\hat\rho_{p-1}}{\hat\rho_p}\left(\frac{\hat\rho_{p-1}}{\rho_p}-3+\frac{\rho_p}{\hat\rho_{p-1}}\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right.$$

$$\left.+\frac{45}{8}\left(\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\right)^2\left(\hat\rho_{p-1}^2-\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right\}$$

$$+\sum_{2\le p<q\le k}u_pu_q^2\left\{-\frac{45}{8}\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\left(\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right)^2+\frac{9}{4}\frac{(\rho_q-\hat\rho_{q-1})}{\hat\rho_{q-1}\hat\rho_q}\right\}$$

$$+\sum_{2\le p<q\le k}u_pu_q\left\{-\frac{3}{4}\frac{1}{\hat\rho_{p-1}^2\hat\rho_p^2}\left(\rho_p^2\left(\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)+\hat\rho_{p-1}^4\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right.$$

$$-\frac{1}{2}\left\{\frac{\rho_p}{\hat\rho_{p-1}^2\hat\rho_p^2}(6\hat\rho_p-\hat\rho_{p-1})\left(\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)+\frac{\hat\rho_{p-1}}{\hat\rho_p^2}(6\hat\rho_p-\rho_p)\right\}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}$$

$$-3\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\left(\frac{1}{\hat\rho_{q-1}^2}-\frac{1}{\hat\rho_q^2}\right)$$

$$+\left(-\frac{3}{4}\frac{\rho_p}{\hat\rho_{p-1}^2\hat\rho_p}+\frac{17}{4}\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p^2}\right)\left(\hat\rho_{p-1}^2-\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}$$
$$\left.\left.+\frac{21}{4}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}-\frac{17}{2}\frac{\rho_p\hat\rho_{p-1}}{\hat\rho_p^2}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right\}\right],$$

$$(3.26)\quad -\frac{1}{6}m_{(21)}m_{(1^3)}\hat E\left[\sum_{i\neq j}\frac{1}{\sqrt{\rho_i\rho_j}}H_1(y_i)H_3(y_j)d_{2i}^{(1)}\right]$$
$$=\frac{m_{(21)}m_{(1^3)}}{(m_{(1^2)})^3}\left[-\frac{5}{4}\sum_{j=2}^k u_j^3\left\{\left(\sum_{\alpha=1}^{j-1}\rho_\alpha^2\right)\frac{\rho_j^3}{\hat\rho_{j-1}^3\hat\rho_j^3}-\frac{\rho_j^3}{\hat\rho_{j-1}\hat\rho_j^3}+2\frac{\rho_j\hat\rho_{j-1}}{\hat\rho_j^3}\right\}\right.$$
$$+\sum_{j=2}^k u_j^2\left\{-\left(\sum_{\alpha=1}^{j-1}\rho_\alpha^2\right)\left(\frac{\rho_j^3}{\hat\rho_{j-1}^3\hat\rho_j^3}+2\frac{\rho_j^2}{\hat\rho_{j-1}^2\hat\rho_j^3}\right)+\frac{\rho_j^3}{\hat\rho_{j-1}\hat\rho_j^3}\right.$$
$$\left.+\frac{1}{4}\frac{\rho_j\hat\rho_{j-1}}{\hat\rho_j^3}-\frac{1}{2}\frac{\hat\rho_{j-1}^2}{\hat\rho_j^3}+\frac{3}{4}\frac{\rho_j^2}{\hat\rho_j^3}+\frac{3}{4}\frac{\rho_j}{\hat\rho_j^2}\right\}$$
$$+\sum_{j=2}^k u_j\left\{-\frac{1}{2}\frac{\rho_j\hat\rho_{j-1}}{\hat\rho_j^3}-\frac{1}{2}\frac{\hat\rho_{j-1}^2}{\hat\rho_j^3}+\frac{1}{2}\frac{\hat\rho_{j-1}}{\hat\rho_j^2}\right\}$$
$$+\sum_{2\le p<q<r\le k}u_pu_qu_r\left\{-\frac{15}{2}\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\frac{\rho_r}{\hat\rho_{r-1}\hat\rho_r}\right.$$
$$\left.+\frac{3}{2}\frac{\rho_r}{\hat\rho_{r-1}\hat\rho_r}\right\}$$
$$+\sum_{2\le p<q\le k}u_p^2u_q\left\{-\frac{3}{2}\frac{\rho_p\hat\rho_{p-1}}{\hat\rho_p^2}\left(\frac{\hat\rho_{p-1}}{\rho_p}-3+\frac{\rho_p}{\hat\rho_{p-1}}\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right.$$
$$\left.+\frac{15}{4}\left(\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\right)^2\left(\hat\rho_{p-1}^2-\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right\}$$
$$+\sum_{2\le p<q\le k}u_pu_q^2\left\{\frac{3}{2}\frac{(\rho_q-\hat\rho_{q-1})}{\hat\rho_{q-1}\hat\rho_q}-\frac{15}{4}\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\left(\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right)^2\right\}$$
$$+\sum_{2\le p<q\le k}u_pu_q\left\{-\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2+\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\left\{\frac{3}{2}\left(\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}\right)^2+\frac{5}{2}\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q^2}\right\}\right.$$
$$+\left(\hat\rho_{p-1}^2-\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\left(\frac{1}{\hat\rho_{p-1}^2}-\frac{1}{\hat\rho_p^2}\right)\frac{\rho_q}{\hat\rho_{q-1}\hat\rho_q}$$
$$-\frac{1}{2}\frac{\rho_p}{\hat\rho_{p-1}\hat\rho_p}\left(\hat\rho_{p-1}^2-\sum_{\alpha=1}^{p-1}\rho_\alpha^2\right)\frac{\rho_q}{\hat\rho_{q-1}^2\hat\rho_q}$$

$$+\frac{1}{2}\frac{\rho_q}{\hat{\rho}_{q-1}\hat{\rho}_q} - 2\frac{\rho_p\hat{\rho}_{p-1}}{\hat{\rho}_p^2}\frac{\rho_q}{\hat{\rho}_{q-1}\hat{\rho}_q} + \frac{\rho_p\hat{\rho}_{p-1}}{\hat{\rho}_p}\frac{\rho_q}{\hat{\rho}_{q-1}^2\hat{\rho}_q}\Bigg\}\Bigg]$$

$$+O\left(\frac{1}{\sqrt{n}}\right).$$

Combining (3.12), (3.17), (3.18), (3.19) and (3.20) for the fourth moments and using Bartlett identities, we have

$$\frac{1}{8}\sum_{j=2}^{k}(c_j-1)\frac{1}{\hat{\rho}_j}\left(\frac{\hat{\rho}_{j-1}}{\rho_j}+1+\frac{\rho_j}{\hat{\rho}_{j-1}}\right)\frac{(m_{(2^2)}-2m_{(21^2)}-m_{(1^4)})}{(m_{(1^2)})^2}. \tag{3.27}$$

Similary combining the third moments, we have after some lengthy algebra

$$\frac{1}{24}\sum_{j=2}^{k}(c_j-1)\frac{1}{\hat{\rho}_j}\left(\frac{\hat{\rho}_{j-1}}{\rho_j}+1+\frac{\rho_j}{\hat{\rho}_{j-1}}\right)\frac{(m_{(3)}m_{(21)}-5m_{(3)}m_{(1^3)}-8m_{(21)}m_{(1^3)})}{(m_{(1^2)})^3}, \tag{3.28}$$

which gives joint moment generating function (2.8).

## References


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