INFORMATION INEQUALITIES FOR THE BAYES RISK FOR A FAMILY OF NON-REGULAR DISTRIBUTIONS

MASAFUMI AKAHIRA AND NAO OHYAUCHI

Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan

(Received April 20, 2001; revised September 20, 2001)

Abstract. For a family of non-regular distributions with a location parameter including the uniform and truncated distributions, the stochastic expansion of the Bayes estimator is given and the asymptotic lower bound for the Bayes risk is obtained and shown to be sharp. Some examples are also given.

Key words and phrases: Loss function, Bayes risk, Bayes estimator, lower bound, uniform distribution, truncated distributions.

1. Introduction

In the non-regular case, the Cramér-Rao type inequality was discussed by Vincze (1979), Khatri (1980), Móri (1983) and others (see also Akahira and Takeuchi (1995)). For a family of uniform distributions on $[\theta - (1/2), \theta + (1/2)]$, the information inequality for the Bayes risk of any estimator of $\theta$ under the quadratic loss and the uniform prior distribution on an interval $[-\tau, \tau]$ is exactly given and shown to have the sharp bound by Ohyauuchi and Akahira (2000, 2001), where $\tau > 1/2$. Akahira and Takeuchi (2001) also show that for any $\tau > 0$ the Bayes risk of any estimator in the interval of $\theta$ values of length $2\tau$ and centered at $\theta_0$ can not be smaller than that of the mid-range. And the lower bound for the limit inferior of the Bayes risk of any estimator of $\theta$ as $\tau \to \infty$ is attained by the mid-range, which involves the result for unbiased estimators of $\theta$ by Móri (1983). For a family of symmetrically truncated normal distributions with a location parameter, the Bayes estimator with respect to the quadratic loss and the uniform prior distribution on the interval $[-\tau, \tau]$ is obtained and its stochastic expansion is given up to the order $o_p(n^{-1})$ as the size $n$ of sample tends to infinity, and also the attainment of the lower bound is asymptotically discussed by Ohyauuchi (2002) and Ohyauuchi and Akahira (2001).

In this paper, for a family of non-regular distributions with a location parameter $\theta$ including the uniform and truncated ones, in a similar way to Akahira (1988) the stochastic expansion of the Bayes estimator is given up to the order $o_p(n^{-1})$, and the information inequality for the Bayes risk of any estimator of $\theta$ is asymptotically obtained.

2. Information inequalities for the Bayes risk

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed random variables with a density $p(x - \theta)$ w.r.t. a $\sigma$-finite measure $\mu$, where $\theta$ is a real-valued parameter. We also assume the following conditions (A1) to (A3).
(A1) \( p(x) > 0 \) for \( a < x < b \); \( p(x) = 0 \) otherwise, where \( a \) and \( b \) are constants with \( a < b \).

(A2) \( p \) is twice continuously differentiable in the open interval \((a, b)\), and
\[
\lim_{x \to a+0} p(x) = \lim_{x \to b-0} p(x) =: c > 0,
\]
\[
\lim_{x \to a+0} p'(x) = \lim_{x \to b-0} p'(x) =: h \leq 0.
\]

(A3) \( 0 < I := - \int_a^b \frac{d^2 \log p(x)}{dx^2} p(x) d\mu(x) < \infty \).

Let \( \theta := \max_{1 \leq i \leq n} X_i - b \) and \( \bar{\theta} := \min_{1 \leq i \leq n} X_i - a \), and \( L(u) \) be a loss function on \( \mathbb{R}^1 \) which is nonnegative-valued, three times continuously differentiable and monotone increasing in \( |u| \). Now, we consider a uniform distribution \( \pi \) on an interval \([-\tau, \tau]\) as a prior distribution of \( \theta \). Then the Bayes estimator w.r.t. \( L \) and \( \pi \) is the estimator \( \hat{\theta} \) minimizing
\[
\int_A^B L(\hat{\theta} - \eta) \prod_{i=1}^n p(x_i - \eta) d\eta
\]
for a.a. \( (x_1, \ldots, x_n) \), where \( A := \max\{-\tau, \theta\}, B := \min\{\tau, \bar{\theta}\} \). And the Bayes risk of an estimator \( \hat{\theta} \) of \( \theta \) is defined by
\[
r_r(\hat{\theta}) := \frac{1}{2\tau} \int_{-\tau}^\tau E_\theta[L(\hat{\theta} - \theta)] d\theta.
\]

Let \( l(x) := \log p(x) \) for \( a < x < b \), \( l^{(i)}(x) := (d^i/dx^i)l(x) \) \((i = 1, 2)\), and
\[
Z_1(\theta) := -\frac{1}{\sqrt{n}} \sum_{i=1}^n l^{(1)}(X_i - \theta), \quad Z_2(\theta) := \frac{1}{\sqrt{n}} \sum_{i=1}^n l^{(2)}(X_i - \theta) + \sqrt{n} l
\]
for \( A \leq \theta \leq B \). Then the Bayes estimator of \( \theta \) is obtained as a solution of the equation
\[
\int_A^B L^{(1)}(\hat{\theta} - \eta) \prod_{i=1}^n p(x_i - \eta) d\eta = 0,
\]
where \( L^{(1)}(u) := (\partial/\partial u)L(u) \). Let \( t := \sqrt{n}(\hat{\theta} - \theta) \) and \( u = \sqrt{n}(\eta - \theta) \). Since \( \sqrt{n}(A - \theta) = \max\{\sqrt{n}(\tau - \theta), \sqrt{n}(\bar{\theta} - \theta)\} \) and \( \sqrt{n}(B - \theta) = \min\{\sqrt{n}(\tau - \theta), \sqrt{n}(\bar{\theta} - \theta)\} \), it follows that, for \( -\tau \leq \theta \leq \tau \), \( \sqrt{n}(A - \theta) = \sqrt{n}(\theta - \theta) \) and \( \sqrt{n}(B - \theta) = \sqrt{n}(\theta - \theta) \). From (2.1) we have
\[
\int_{-\sqrt{n}(\theta - \theta)}^{\sqrt{n}(\bar{\theta} - \theta)} L^{(1)}(t - u) \left[ \exp \left( \sum_{i=1}^n \log p \left( \frac{x_i - \theta - u}{\sqrt{n}} \right) \right) \right] \frac{1}{\sqrt{n}} du = 0
\]
hence, in order to obtain the Bayes estimator, it is enough to get a solution of \( t \) of the equation (2.2). We also assume that \( L^{(1)}(0) = 0 \) and put \( U := n(\bar{\theta} - \theta) \) and \( V := n(\theta - \theta) \). Then we have the following (see also Akahira (1988)).

**Theorem 1.** Under the conditions (A1) to (A3), the Bayes estimator \( \hat{\theta}_B \) of \( \theta \) w.r.t. the loss function \( L \) and the prior uniform distribution \( \pi \) on \([-\tau, \tau]\] has the stochastic
\[ n(\hat{\theta}_B - \theta) = \frac{1}{2}(U + V) + \frac{1}{12\sqrt{n}}Z_1(\theta)(U - V)^2 - \frac{I}{24n}(U + V)(U - V)^2 - \frac{b_3}{24b_2n}(U - V)^2 + o_p\left(\frac{1}{n}\right), \]

where \(-\tau < \theta < \tau\), \(b_i := (d^i/du^i)L(0)\) \((i = 2, 3)\) and \(b_2 \neq 0\).

**Proof.** Since \(L^{(1)}(0) = 0\), it follows that

\[ L^{(1)}\left(\frac{1}{\sqrt{n}}(t - u)\right) = \frac{1}{\sqrt{n}}(t - u)L^{(2)}(0) + \frac{1}{2n}(t - u)^2L^{(3)}(0) + o_p\left(\frac{1}{n^2}\right) \]

\[ = \frac{b_2}{\sqrt{n}}(t - u) + \frac{b_3}{2n}(t - u)^2 + o_p\left(\frac{1}{n^2}\right). \]

We also have

\[ \exp\left\{\sum_{i=1}^{\infty} \log p(x_i - \theta - \frac{u}{\sqrt{n}})\right\} \]

\[ = \exp\left\{\sum_{i=1}^{\infty} t(x_i - \theta) - \frac{u}{\sqrt{n}} \sum_{i=1}^{\infty} t^{(1)}(x_i - \theta) + \frac{u^2}{2n} \sum_{i=1}^{\infty} t^{(2)}(x_i - \theta) \right\} \]

\[ + O_p\left(\frac{1}{n}\right) \]

\[ = \prod_{i=1}^{\infty} p(x_i - \theta) \left[1 - \frac{u}{\sqrt{n}} \sum_{i=1}^{\infty} t^{(1)}(x_i - \theta) + \frac{u^2}{2n} \sum_{i=1}^{\infty} t^{(2)}(x_i - \theta) \right] \]

\[ + \frac{u^2}{2n} \left\{ \sum_{i=1}^{\infty} t^{(1)}(x_i - \theta) \right\}^2 + O_p\left(\frac{1}{n}\right) \]

\[ = \prod_{i=1}^{\infty} p(x_i - \theta) \left[1 + uZ_1(\theta) + \frac{u^2}{2\sqrt{n}} \{Z_2(\theta) - \sqrt{n}I(\theta)\} \right] \]

\[ + \frac{u^2}{2} Z_1(\theta) + O_p\left(\frac{1}{n}\right). \]

From (2.2) to (2.4) we obtain

\[ 0 = \frac{1}{\sqrt{n}} \int_{\sqrt{n}(\theta - \theta)}^{\sqrt{n}(\theta - \theta)} \left\{ \frac{b_2}{\sqrt{n}}(t - u) + \frac{b_3}{2n}(t - u)^2 \right\} \]

\[ + \left\{1 + uZ_1 + \frac{u^2}{2}(Z^2_1 - I) \right\} du + O_p\left(\frac{1}{n^{5/2}}\right) \]

\[ = \frac{1}{\sqrt{n}} \int_{\sqrt{n}(\theta - \theta)}^{\sqrt{n}(\theta - \theta)} \left\{b_2(t - u) + b_2Z_1u(t - u) + \frac{b_3}{2}(Z^2_1 - I)u^2(t - u) \right\} \]

\[ + \frac{b_3}{2\sqrt{n}}(t - u)^2 + \frac{b_3}{2\sqrt{n}}Z_1u(t - u)^2 \]

\[ + \frac{b_3}{4\sqrt{n}}(Z^2_1 - I)u^2(t - u)^2 \right\} du + O_p\left(\frac{1}{n^{5/2}}\right), \]
where \( Z_1 = Z_1(\theta) \) and \( I = I(\theta) \). Let \( S := (U + V)/2 \) and \( T := (U - V)/2 \) for \(-\tau < \theta < \tau\). From (2.5) we have

\[
0 = 2b_2 T \left( t - \frac{S}{\sqrt{n}} \right) + 2b_2 Z_1 T \left\{ \frac{t}{\sqrt{n}} S - \frac{1}{3n} (3S^2 + T^2) \right\} \\
+ \frac{b_2}{2} (Z_1^2 - I) \left\{ \frac{2t}{3n} T(3S^2 + T^2) - \frac{2}{n\sqrt{n}} TS(S^2 + T^2) \right\} \\
+ \frac{b_2}{\sqrt{n}} T \left\{ t^2 - \frac{2}{\sqrt{n}} tS + \frac{1}{3n} (3S^2 + T^2) \right\} + \frac{b_3}{n} t^2 Z_1 TS \\
+ \frac{b_3 t^2}{6n\sqrt{n}} (Z_1^2 - I) T(3S^2 + T^2) + O_p \left( \frac{1}{n^2} \right),
\]

which implies

\[
2 \sqrt{n} t \left\{ 1 + \frac{1}{\sqrt{n}} Z_1 S + \frac{b_3}{2b_2 \sqrt{n}} t - \frac{b_3}{b_2 n} \left( 1 - \frac{t}{2} Z_1 \right) S + \frac{1}{6n} (Z_1^2 - I)(3S^2 + T^2) \right\} \\
= b_2 S + \frac{b_3}{3\sqrt{n}} Z_1 (3S^2 + T^2) + \frac{b_3}{2n} (Z_1^2 - I) S(S^2 + T^2) \\
- \frac{b_3}{6n} (3S^2 + T^2) + O_p \left( \frac{1}{n^{3/2}} \right).
\]

Hence we obtain

\[
n(\hat{\theta} - \theta) = \sqrt{n} t \\
= \left\{ S + \frac{1}{3\sqrt{n}} Z_1 (3S^2 + T^2) + \frac{1}{2n} (Z_1^2 - I) S(S^2 + T^2) \\
- \frac{b_3}{6b_2 n} (3S^2 + T^2) \right\} \\
\cdot \left\{ 1 - \frac{1}{\sqrt{n}} Z_1 S - \frac{b_3}{2b_2 \sqrt{n}} t + \frac{b_3}{b_2 n} \left( 1 + \frac{t}{2} Z_1 \right) S \\
- \frac{1}{6n} (Z_1^2 - I)(3S^2 + T^2) + \frac{1}{n} Z_1^2 S^2 + \frac{b_3}{4b_2^2 n} t^2 \right\} + o_p \left( \frac{1}{n} \right) \\
= S + \frac{1}{3\sqrt{n}} Z_1 T^2 - \frac{I}{3n} ST^2 - \frac{b_3}{6b_2 n} T^2 + o_p \left( \frac{1}{n} \right) \\
= \frac{1}{2} (U + V) + \frac{1}{12\sqrt{n}} Z_1 (U - V)^2 - \frac{I}{24n} (U + V)(U - V)^2 \\
- \frac{b_3}{24b_2 n} (U - V)^2 + o_p \left( \frac{1}{n} \right)
\]

for \(-\tau < \theta < \tau\). Thus we complete the proof.

Henceforth, we assume that \( \mu \) is the Lebesgue measure. Letting \( I_0 := \int_a^b \frac{(p'(x))^2}{p(x)} dx \), we have \( I = -2h + I_0 \). Let \( I_0 > 0 \). Then we also obtain the following from Theorem 1.

**Theorem 2.** Under the conditions (A1) to (A3), for any fixed \( \tau > 0 \) the information inequality for the Bayes risk w.r.t. the quadratic loss \( L \) and the uniform prior \( \pi \) on
\[ [-\tau, \tau] \] of any estimator \( \hat{\theta} = \hat{\theta}(X_1, \ldots, X_n) \) of \( \theta \) is given by

\[
\lim_{n \to \infty} n \left\{ n^2 r_\tau(\hat{\theta}) - \frac{1}{2c^2} \right\} \geq -\frac{1}{2c^4} (3c^2 - 4h) - \frac{5I_0}{6c^4} =: B_2(c).
\]

**Proof.** Letting \( L(u) = u^2 \), we have from Theorem 1

\[
n(\hat{\theta}_B - \theta) = \frac{1}{2} (U + V) + \frac{1}{12\sqrt{n}} Z_1(U - V)^2
\]

\[
-\frac{I}{24n}(U + V)(U - V)^2 + o_p \left( \frac{1}{n} \right)
\]

for \( -\tau < \theta < \tau \), which implies

\[
E[n^2(\hat{\theta}_B - \theta)^2] = \frac{1}{4} E[(U + V)^2] + \frac{I_0}{144n} [(U - V)^4 Z^2]
\]

\[
+ \frac{\sqrt{I_0}}{12\sqrt{n}} E[(U + V)(U - V)^2 Z]
\]

\[
-\frac{I}{24n} E[(U + V)^2(U - V)^2] + o \left( \frac{1}{n} \right).
\]

Now, the second order asymptotic joint density of \((U, V)\) is given by

\[
f_{U,V}(u, v) := \begin{cases} 
  c^2 e^{-c(u-v)} \left[ 1 + \frac{1}{n} \left\{ -1 + 2c(u-v) \right\} 
  + \frac{h}{4} ((u+v)^2 + (u-v)^2) - \frac{c^2}{2} (u-v)^2 
  -\frac{h}{c} (u-v) \right\} + o \left( \frac{1}{n} \right) 
  & \text{for } v < 0 < u, \\
  0 & \text{otherwise}
\end{cases}
\]

(see Akahira (1991), p. 191). Letting \( Z(\theta) := Z_1(\theta)/\sqrt{I_0} \), we have the second order asymptotic conditional density given \( U = u \) and \( V = v \) as

\[
f_Z(z \mid u, v) = \phi(z) - \frac{h(u + v)}{\sqrt{nI_0}} z \phi(z) - \frac{K}{6\sqrt{nI_0}} (z^3 - 3z) \phi(z) + O \left( \frac{1}{n} \right)
\]

for \( -\infty < z < \infty \), where \( \phi(z) = (1/\sqrt{2\pi}) e^{-z^2/2} \) and

\[
K = -\int_a^b \{f_{(1)}(x)\}^3 p(x) dx
\]

(see Akahira (1996), p. 358). Since, for nonnegative integer \( p \) and \( q \)

\[
\int_0^\infty cu^p e^{-cu} du = \frac{p!}{c^p}, \quad \int_0^\infty cu^q e^{cu} du = (-1)^q \frac{q!}{c^q},
\]
it follows from (2.9) that

\[
E(U^2) = \frac{2}{c^3} - \frac{6}{c^2n} + \frac{6h}{c^4n} + o \left( \frac{1}{n} \right),
\]

\[
E(V^2) = \frac{2}{c^2} - \frac{6}{c^2n} + \frac{6h}{c^4n} + o \left( \frac{1}{n} \right),
\]

\[
E(UV) = -\frac{1}{c^2} + \frac{3}{c^2n} - \frac{2h}{c^4n} + o \left( \frac{1}{n} \right),
\]

hence

\[
E \left[ (U + V)^2 \right] = \frac{2}{c^2} - \frac{6}{c^2n} + \frac{8h}{c^4n} + o \left( \frac{1}{n} \right).
\]

From (2.9) and (2.10) we have

\[
(2.12) \quad E \left[ (U - V)^4 Z^2 \right] = E \left[ (U - V)^4 E(Z^2 \mid U, V) \right]
\]

\[
= E \left[ (U - V)^4 \right] + O \left( \frac{1}{n} \right)
\]

\[
= \frac{120}{c^4} + O \left( \frac{1}{n} \right),
\]

(2.13)

\[
E \left[ (U + V)(U - V)^2 \right]
\]

\[
= E \left[ (U + V)(U - V)^2 \right] E(Z \mid U, V)
\]

\[
= E \left[ (U + V)(U - V)^2 \left\{ -\frac{h(U + V)}{\sqrt{nI_0}} + O \left( \frac{1}{n} \right) \right\} \right]
\]

\[
= -\frac{40h}{c^4\sqrt{nI_0}} + O \left( \frac{1}{n} \right),
\]

(2.14)

\[
\]

From (2.8) and (2.11) to (2.14), we obtain

\[
E[n^2(\hat{\theta}_B - \theta)]^2 = \frac{1}{2c^2} - \frac{1}{2c^4n}(3c^2 - 4h) - \frac{5I_0}{6c^4n} + o \left( \frac{1}{n} \right)
\]

for \(-\tau < \theta < \tau\), since \(I = -2h + I_0\). Hence for any estimator \(\hat{\theta}\) of \(\theta\) and any fixed \(\tau > 0\)

\[
\lim_{n \to \infty} n \left\{ n^2r_\tau(\hat{\theta}) - \frac{1}{2c^2} \right\}
\]

\[
\geq \lim_{n \to \infty} n \left\{ n^2r_\tau(\hat{\theta}_B) - \frac{1}{2c^2} \right\}
\]

\[
\geq \lim_{n \to \infty} n \left\{ \frac{1}{2\tau} \int_{-\tau}^{\tau} E[n^2(\hat{\theta}_B - \theta)^2] d\theta - \frac{1}{2c^2} \right\}
\]

\[
= -\frac{1}{2c^2}(3c^2 - 4h) - \frac{5I_0}{6c^4}.
\]
Thus we complete the proof.

**Remark.** In the second order asymptotic lower bound $B_2(c)(<0)$ in (2.6), the first term represents the information on the endpoints $a$ and $b$ of the density $p$ through $(U,V)$ and the second one means the information on the central part of the distribution with $p$ through $Z_1(\theta)$. The Bayes estimator $\hat{\theta}_B$ is seen to attain the lower bound $B(c)$.

**Corollary.** Under the conditions (A1) and (A2), for any fixed $\tau > 0$ the information inequality for the Bayes risk w.r.t. the quadratic loss $L$ and the uniform prior $\pi$ on $[-\tau, \tau]$ of any estimator $\hat{\theta}$ of $\theta$ is given by

$$\lim_{n \to \infty} n^2 \tau^2 (\hat{\theta}) \geq \frac{1}{2c^2} =: B_1(c).$$

The first order asymptotic lower bound $B_1(c)$ is attained by the mid-range $\hat{\theta}_0 := (\hat{\theta} + \bar{\theta})/2$.

The proof of the first half is straightforward from Theorem 2, and that of the latter half is given since from (2.7)

$$n(\hat{\theta}_B - \theta) = \frac{1}{2}(U + V) + o_p(1) = \frac{1}{2}(\hat{\theta}_0 - \theta) + o_p(1)$$

and from (2.9)

$$E[n^2(\hat{\theta}_B - \theta)^2] = \frac{1}{4} E[(U + V)^2] + o(1) = \frac{1}{2c^2} + o(1).$$

3. Examples

As the application of Theorem 2, we consider the uniform and truncated distribution cases.

**Example 1.** If $p$ is a density of the uniform distribution on the interval $[-1/2, 1/2]$, then $c = 1$, hence, from (2.15) we have $B_1(c) = 1/2$ as the first order asymptotic lower bound which coincides with that of Ohyaucli and Akahira (2000). In this case, the mid-range $\hat{\theta}_0 = (\min_{1 \leq i \leq n} X_i + \max_{1 \leq i \leq n} X_i)/2$ attains the lower bound $B_1(c)$.

**Example 2.** If $p$ is a symmetrically truncated normal density, i.e.

$$p(x) = \begin{cases} ke^{-x^2/2} & \text{for } |x| < 1/2, \\ 0 & \text{otherwise}, \end{cases}$$

where $k$ is some constant, then $c = ke^{-1/8}$, $h = -c/2$ and $I_0 = 1 - c$, hence, from (2.6) we have

$$B_2(c) = -\frac{3c + 2}{2c^3} - \frac{5(1 - c)}{6c^4}$$

as the second order asymptotic lower bound which coincides with that of Ohyaucli (2002) (see also Ohyaucli and Akahira (2001)). In this case, it is shown from (2.1) that the
Bayes estimator $\hat{\theta}_B$ w.r.t. the quadratic loss and the uniform prior $\pi$ on the interval $[-\tau, \tau]$ is given by

$$\hat{\theta}_B(X) = \int_A^B \theta \exp \left\{ -\frac{n}{2} (\bar{X} - \theta)^2 \right\} d\theta / \int_A^B \exp \left\{ -\frac{n}{2} (\bar{X} - \theta)^2 \right\} d\theta,$$

where $A := \max\{-\tau, X_{(n)} - (1/2)\}$ and $B := \min\{\tau, X_{(1)} + (1/2)\}$ with $X_{(1)} := \min_{1 \leq i \leq n} X_i$ and $X_{(n)} := \max_{1 \leq i \leq n} X_i$ and $\bar{X} = (1/n) \sum_{i=1}^n X_i$. Letting $\phi$ be the density of the standard normal distribution, we have

$$\hat{\theta}_B(X) = \int_A^B \theta \phi(\sqrt{n}(\theta - \bar{X})) d\theta / \int_A^B \phi(\sqrt{n}(\theta - \bar{X})) d\theta,$$

which attains the above lower bound $B_2(c)$ (see Ohyauchi (2002) and Ohyauchi and Akahira (2001)).

In this case the maximum likelihood estimator (MLE) $\hat{\theta}_{ML}$ of $\theta$ is given by

$$\hat{\theta}_{ML} := \begin{cases} \frac{\theta}{\bar{X}} & \text{for } \bar{X} \leq \bar{\theta}, \\ \frac{\bar{\theta}}{\bar{X}} & \text{for } \bar{X} \geq \bar{\theta}, \\ \bar{X} & \text{for } \bar{\theta} < \bar{X} < \bar{\theta}, \end{cases}$$

where $\bar{\theta} = X_{(n)} - (1/2)$ and $\bar{\theta} = X_{(1)} + (1/2)$, which is asymptotically equivalent to

$$\hat{\theta}_n := \begin{cases} \frac{\theta}{\bar{X}} & \text{with probability } 1/2 + o(1), \\ \frac{\bar{\theta}}{\bar{X}} & \text{with probability } 1/2 + o(1) \end{cases}$$

as $n \to \infty$ (see Akahira and Takeuchi (1981), Ohyauchi and Akahira (2001) and Ohyauchi (2001)). Since the asymptotic density of $n(\hat{\theta}_{ML} - \theta)$ is

$$f_{\hat{\theta}_{ML}}(t) = \frac{c}{2e^{-c|t|}} \quad (\infty < t < \infty),$$

for any fixed $\tau > 0$ the Bayes risk of the MLE is given by

$$(3.1) \quad R_{ML}(c) := \lim_{n \to \infty} n^2 \tau^r (\hat{\theta}_{ML} - \theta) = \frac{2}{c^2} > B_1(c) = \frac{1}{2c^2}.$$ 

From (3.1) it is seen that the MLE does not attain the first order asymptotic lower bound $B_1(c)$ and the Bayes risk of the MLE has four times as large magnitude as the bound $B_1(c)$.

Next we consider the maximum probability estimator (MPE) $\hat{\theta}_{MP}$ which is defined as that value of $d$ maximizing

$$\int_{d-(r/n)}^{d+(r/n)} \prod_{i=1}^n p(x_i - \theta) d\theta$$

for a fixed number $r > 0$ (Weiss and Wolowitz (1974)). Then we have

$$\hat{\theta}_{MP} = \begin{cases} \frac{1}{2} (\bar{\theta} + \bar{X}) & \text{for } \bar{\theta} - \bar{X} < 2r/n, \\ \bar{\theta} - (r/n) & \text{for } \bar{X} > \bar{\theta} - (r/n), \quad \bar{\theta} - \bar{X} > 2r/n, \\ \bar{\theta} + (r/n) & \text{for } \bar{X} < \bar{\theta} + (r/n), \quad \bar{\theta} - \bar{X} > 2r/n, \\ \bar{X} & \text{for } \bar{\theta} + (r/n) \leq \bar{X} \leq \bar{\theta} - (r/n). \end{cases}$$
(see Akahira and Takeuchi (1981)). Since the asymptotic density of $n(\hat{\theta}_{MP} - \theta)$ is given by

$$f_{\hat{\theta}_{MP}}(t) = \begin{cases} ce^{-2c|t|} & \text{for } |t| < r, \\ \frac{1}{2} ce^{-c(|t|+r)} & \text{for } |t| \geq r, \end{cases}$$

it follows that for any fixed $\tau > 0$

$$R_{MP}(r; c) := \lim_{n \to \infty} n^2 r_n(\hat{\theta}_{MP}^r) = \left( \frac{3}{2c^2} + \frac{r}{c} \right) e^{-2cr} + \frac{1}{2c^2} > B_1(c).$$

It is noted that the MPE $\hat{\theta}_{MP}$ tends to the mid-range $(\hat{\theta} + \bar{\theta})/2$ as $r \to \infty$ and the MLE $\hat{\theta}_{ML}$ as $r \to 0$ which corresponds to the fact

$$\lim_{r \to \infty} R_{MP}(r; c) = \frac{1}{2c^2} = B_1(c),$$

$$\lim_{r \to 0} R_{MP}(r; c) = \frac{2}{c^2} = R_{ML}(c).$$

**Example 3.** If $p$ is a symmetrically truncated density of the form

$$p(x) = \begin{cases} ce^{(1-x^2)^2} & \text{for } |x| < 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $c$ is some positive constant, then $h = 0$ and

$$0 < I_0 = \frac{72c}{5} \int_0^1 x^2 (1-x^2)^3 e^{(1-x^2)^2} dx < \infty,$$

hence, from (2.6) we have

$$B_2(c) = -\frac{3}{2c^2} - \frac{5I_0}{6c^4}$$

as the second order asymptotic lower bound which is attained by the Bayes estimator $\hat{\theta}_B$ w.r.t. the quadratic loss and the uniform prior $\pi$ on $[-\tau, \tau]$.

**References**


