FISHER INFORMATION IN AN ORDER STATISTIC AND
ITS CONCOMITANT

Z. A. ABO-ELENEEN* AND H. N. NAGARAJA

Department of Statistics, Ohio State University, Columbus OH 43210-1247, U.S.A.,
e-mail: zabeh@stat.washington.edu; hnn@stat ohio-state.edu

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Abstract. Let \((X, Y)\) have an absolutely continuous distribution with parameter \(\theta\). We suggest regularity conditions on the parent distribution that permit the definition of Fisher information (FI) about \(\theta\) in an \(X\)-order statistic and its \(Y\)-concomitant that are obtained from a random sample from \((X, Y)\). We describe some general properties of the FI in such individual pairs. For the Farlie-Gumbel-Morgenstern parent with dependence parameter \(\theta\), we investigate the properties of this FI, and obtain the asymptotic relative efficiency of the maximum likelihood estimator of \(\theta\) for Type II censored bivariate samples. Assuming \((X, Y)\) is Gumbel bivariate exponential of second type, and \(\theta\) is the mean of \(Y\), we evaluate the FI in the \(Y\)-concomitant of an \(X\)-order statistic and compare it with the FI in a single \(Y\)-order statistic.

Key words and phrases: Concomitants of order statistics, Fisher information, Farlie-Gumbel-Morgenstern family, Gumbel Type II bivariate exponential distribution, Type II censoring, maximum likelihood estimator.

1. Introduction

Suppose we have a random sample of size \(n\) from a continuous distribution with cumulative distribution function (cdf) \(F_1(x; \theta)\) and probability density function (pdf) \(f_1(x; \theta)\), where \(\theta\) is a real valued parameter and the sample is arranged in ascending order. The question about which part of the ordered sample has more information has been discussed by Tukey (1965) and Nagaraja (1994) in terms of linear sensitivity, a measure based on the first two moments of linear functions of order statistics. Mehrotra et al. (1979), Park (1996), and Zheng and Gastwirth (2000) have studied properties of the Fisher information (FI) measure in blocks and collections of order statistics. The FI plays a valuable role in statistical inference through the information (Cramer-Rao) inequality and its association with the asymptotic properties of the maximum likelihood estimators (MLE).

Now suppose \((X, Y)\) is absolutely continuous with joint cdf \(F(x, y; \theta)\) and pdf \(f(x, y; \theta)\). For \(1 \leq r \leq n\), let \(X_{r:n}\) be the \(r\)-th \(X\)-order statistic and \(Y_{r:n}\) be its concomitant obtained from a random sample of size \(n\) from \(f\). David and Nagaraja (1998) provide a review of the area of the concomitants of order statistics.

The joint pdf of \((X_{r:n}, Y_{r:n})\) is given by

\[
  f_{r:n}(x, y; \theta) = c f(x, y; \theta) [F_1(x; \theta)]^{r-1} [1 - F_1(x; \theta)]^{n-r},
\]

*Now at Faculty of Computer Science, Zagazig University, Zagazig, Egypt.
where
\begin{equation}
(1.2) \quad c = \frac{n!}{(r - 1)!(n - r)!}.
\end{equation}

The marginal pdf of $Y_{[r:n]}$ is given by
\begin{equation}
(1.3) \quad f_{[r:n]}(y; \theta) = c \int_{-\infty}^{\infty} f(x, y; \theta)[F_1(x; \theta)]^{r-1}[1 - F_1(x; \theta)]^{n-r}dx.
\end{equation}

For a fixed $n$, we use the notation $f_r$ instead of $f_{[r:n]}$ in (1.1) and we write $f_r$ for $f_{[r:n]}$ when $n$ is fixed in (1.3).

We investigate the properties of $I_\theta(X_{[r:n]}, Y_{[r:n]})$ and $I_\theta(Y_{[r:n]})$, the FI contained in $(X_{[r:n]}, Y_{[r:n]})$ and $Y_{[r:n]}$, respectively. These play an important role in the asymptotic properties of MLE's based on censored bivariate samples. In Section 2 we show that the “standard” regularity conditions used to define $I_\theta(X, Y)$, the FI contained in $(X, Y)$, are enough to define $I_\theta(X_{[r:n]}, Y_{[r:n]})$ and $I_\theta(Y_{[r:n]})$. Under the assumption that $F_1$ is free of $\theta$, we obtain a simple useful recurrence relation satisfied by $I_\theta(X_{[r:n]}, Y_{[r:n]})$ and show that it is additive in $r$. This permits easy computation of the FI in an arbitrary collection of order statistics and their concomitants. In Section 3 we give an explicit expression for $I_\theta(X_{[r:n]}, Y_{[r:n]})$, when $\theta$ is the dependence parameter of the Farlie-Gumbel-Morgenstern (FGM) cdf
\begin{equation}
(1.4) \quad F(x, y) = F_1(x)F_2(y)[1 + \theta(1 - F_1(x))(1 - F_2(y))],
\end{equation}
where $-1 < \theta < 1$, and $F_2$ is the marginal cdf of $Y$. We prove the monotonicity property of FI as a function of $r$ and evaluate $I_\theta(X_{[r:n]}, Y_{[r:n]})$ for selected $\theta$, $r$, and $n$. We find the asymptotic variance and the asymptotic relative efficiency (ARE) of the MLE of $\theta$ under Type II censoring. In Section 4 we consider the Gumbel Type II bivariate exponential distribution ($G_2_BVE$) where we take $\theta = E(Y)$ and give an explicit expression for $I_\theta(Y_{[r:n]})$. We compare it with $I_\theta(Y_{[r:n]})$ and discuss some implications.

2. Fisher information regularity conditions and properties

2.1 Regularity conditions for $I_\theta(X_{[r:n]}, Y_{[r:n]})$

The FI about the real parameter $\theta$ contained in $X$ is defined by $I_\theta(X) = E\left[\frac{\partial^2 \log f_1(X; \theta)}{\partial \theta^2}\right]^2$ under the following regularity conditions (see, for example, Rao (1973), p. 329):

1. The parameter space $\Omega$ is a real non-degenerate interval.
2. The pdf of $X$ w.r.t. the $\sigma$-finite measure $\mu$, $f_1(x; \theta)$, is differentiable w.r.t. $\theta$ for all $\theta \in \Omega$.
3. For every measurable set $C \subset S$, the sample space,
\begin{equation}
\frac{\partial}{\partial \theta} \int_C f_1(x; \theta)d\mu = \int_C \frac{\partial f_1(x; \theta)}{\partial \theta}d\mu,
\end{equation}
assuming the existence of the right side integral. One commonly used condition which validates (3) is:

(3') There exists an $H(x)$ integrable (w.r.t. $\mu$) such that $|\frac{\partial f_1(x; \theta)}{\partial \theta}| \leq H(x)$ for all $\theta \in \Omega$. 

Nagaraja (1983) showed that conditions (1), (2) and (3\textsuperscript{*}) on \(f_1(x, \theta)\) serve as regularity conditions for defining \(I_{\theta}(X_{r,n})\). Now assume:

(A1) \(\Omega\) is a real non-degenerate interval.
(A2) \(f(x, y; \theta)\) is differentiable w.r.t. \(\theta\) for all \(\theta \in \Omega\).
(A3\textsuperscript{*}) There exists an integrable \(H(x, y)\) such that \(|\frac{\partial f(x, y; \theta)}{\partial \theta}| \leq H(x, y)\) for all \(\theta\).

Note that (A3\textsuperscript{*}) validates the required assumption

(A3) For any measurable set \(C \subset S\), the sample space,

\[
\frac{\partial}{\partial \theta} \int_C f(x, y; \theta) d\mu = \int_C \frac{\partial f(x, y; \theta)}{\partial \theta} d\mu.
\]

Conditions (A2) and (A3\textsuperscript{*}) imply that \(f_r(x, y; \theta)\), given by (1.1), is differentiable w.r.t. \(\theta\), for all \(\theta \in \Omega\). Further, since \(0 \leq F_1(x; \theta) \leq 1\), (1.1) implies that

\[
(2.1) \quad \left| \frac{\partial f_r(x, y; \theta)}{\partial \theta} \right| \leq c \left| \frac{\partial f(x, y; \theta)}{\partial \theta} \right| + c(n-1)f(x, y; \theta) \frac{\partial F_1(x; \theta)}{\partial \theta}.
\]

Now since

\[
\frac{\partial F_1(x; \theta)}{\partial \theta} = \int_{-\infty}^{x} \int_{-\infty}^{\infty} \frac{\partial f(u, v; \theta)}{\partial \theta} dv du \text{ from (A3),}
\]

\[
\left| \frac{\partial F_1(x; \theta)}{\partial \theta} \right| \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(u, v) dv du \text{ from (A3\textsuperscript{*})}
\]

\[
= c_1 \text{ say.}
\]

Then

\[
(2.2) \quad \left| \frac{\partial f_r(x, y; \theta)}{\partial \theta} \right| \leq cH(x, y) + (n-1)c_1f(x, y; \theta)
\]

\[
\leq cH(x, y) + c_2f(x, y; \theta),
\]

where \(c_2 = (n-1)c_1\) and \(H(x, y)\) is integrable. We will now show that

\[
(2.3) \quad \left| \frac{\partial f_r(x, y; \theta)}{\partial \theta} \right| \leq cH(x, y) + (n-1)c_1f(x, y; \theta)
\]

for all interior points of \(\Omega\), and for all (two dimensional) measurable sets \(C\). Let \(\theta_0\) be an arbitrary but fixed point in \(\Omega\). Then from the mean value theorem

\[
\frac{f_r(x, y; \theta) - f_r(x, y; \theta_0)}{\theta - \theta_0} = \frac{\partial f_r(x, y; \theta)}{\partial \theta}
\]

for some \(\theta^{(1)}\) between \(\theta\) and \(\theta_0\). In view of (2.3), we have

\[
(2.4) \quad \left| \frac{f_r(x, y; \theta) - f_r(x, y; \theta_0)}{\theta - \theta_0} \right| \leq cH(x, y) + c_2f(x, y; \theta^{(1)}).
\]

But

\[
(2.5) \quad f(x, y; \theta^{(1)}) = f(x, y; \theta_0) + (\theta^{(1)} - \theta_0) \frac{\partial f(x, y; \theta_0)}{\partial \theta}
\]

\[
(2.6) \quad f(x, y; \theta^{(1)}) + (\theta^{(1)} - \theta_0) \frac{\partial f(x, y; \theta_0)}{\partial \theta}
\]
for some \( \theta^{(2)} \) between \( \theta_0 \) and \( \theta^{(1)} \). So from (2.5) and (2.6) we have, for \( |\theta - \theta_0| < 1 \),
\[
(2.7) \quad \left| \frac{f_r(x, y; \theta) - f_r(x, y; \theta_0)}{\theta - \theta_0} \right| \leq c_2 \left\{ f(x, y; \theta_0) + |\theta^{(1)} - \theta_0| \frac{\partial f(x, y; \theta^{(2)})}{\partial \theta} \right\} + cH(x, y)
\[
\leq c_2 \left\{ f(x, y; \theta_0) + |\theta^{(1)} - \theta_0| H(x, y) \right\} + cH(x, y)
\leq c_2 f(x, y; \theta_0) + (c + c_2) H(x, y).
\]

Let \( \{\theta_m, m \geq 1\} \) be a sequence converging to \( \theta_0 \). Without loss of generality one can take \( |\theta_m - \theta_0| < 1 \) for all \( m \). Then with \( g_m(x, y) = \frac{f_r(x, y; \theta_m) - f_r(x, y; \theta_0)}{\theta_m - \theta_0} \), we have \( |g_m(x, y)| \leq G(x, y) \), where \( G(x, y) \), the function on the right hand side of (2.7), is integrable. So at \( \theta = \theta_0 \),
\[
(2.8) \quad \int_C \frac{\partial f_r(x, y; \theta)}{\partial \theta} dxdy = \int_C \lim_{m \to \infty} g_m(x, y) dxdy
\[
= \lim_{m \to \infty} \int_C g_m(x, y) dxdy
\[
= \lim_{m \to \infty} \int_C f_r(x, y; \theta_m) dxdy - \int_C f_r(x, y; \theta_0) dxdy
\[
= \frac{\partial}{\partial \theta} \int_C f_r(x, y; \theta) dxdy.
\]

The second step above follows from the Lebesgue Dominated Convergence Theorem. Since \( \theta_0 \) is arbitrary, (2.4) holds for all \( \theta \). Hence we have the following result.

**Theorem 2.1.** Conditions (A1), (A2), and (A3*) on \( f(x, y; \theta) \) can serve as regularity conditions for defining \( I_{\theta}(X_{r,n}, Y_{[r,n]}) \), the FI in \( (X_{r,n}, Y_{[r,n]}) \).

2.2 Regularity conditions for \( I_{\theta}(Y_{[r,n]}) \)

Now we show that \( I_{\theta}(Y_{[r,n]}) \) exists if (A1), (A2) and (A3*) hold. Conditions (A2) and (A3) imply that \( f_{[r]}(y; \theta) \), given by (1.3), is differentiable w.r.t. \( \theta \). Since \( 0 \leq F_1(x; \theta) \leq 1 \),
\[
\left| \frac{\partial f_{[r]}(y; \theta)}{\partial \theta} \right| \leq c \int_{-\infty}^{\infty} \left| \frac{\partial f(x, y; \theta)}{\partial \theta} \right| dx
\]
\[
+ c(n - 1) \int_{-\infty}^{\infty} f(x, y; \theta) \left| \frac{\partial F_1(x; \theta)}{\partial \theta} \right| dx
\]
\[
\leq c \int_{-\infty}^{\infty} H(x, y) dx + (n - 1) c c_1 \int_{-\infty}^{\infty} f(x, y; \theta) dx
\]
from (A3*) and (2.2). Thus
\[
(2.9) \quad \left| \frac{\partial f_{[r]}(y; \theta)}{\partial \theta} \right| \leq c H_1(y) + (n - 1) c c_1 f_2(y; \theta),
\]
where \( H_1(y) = \int_{-\infty}^{\infty} H(x, y) dx \), is integrable. Note that (2.9) resembles (2.3) and we follow the approach used to prove (2.8) and conclude that
\[
(2.10) \quad \frac{\partial}{\partial \theta} \int_C f_{[r]}(y; \theta) dy = \int_C \frac{\partial f_{[r]}(y; \theta)}{\partial \theta} dy
\]
for all interior points \( \theta \in \Omega \), for all measurable sets \( C \). This leads to the following.

**Theorem 2.2.** Conditions (A1), (A2), and (A3*) on the bivariate pdf \( f(x, y; \theta) \)

\( \) can serve as regularity conditions for defining \( I_\theta(Y_{[r:n]}) \), for \( 1 \leq r \leq n \).

**Remark 1.** In addition to conditions (A1)–(A3), with a regularity condition that

permits the interchange of integration with respect to \( \mu \) and second derivative of \( f_1 \) w.r.t. \( \theta \), \( I_\theta(X) = -E^{2\log f_1(X, \theta)} \). Under a similar additional condition on \( f(x, y; \theta) \), parallel results hold for \( I_\theta(X_{[r:n]}, Y_{[r:n]}) \) and \( I_\theta(Y_{[r:n]}) \).

### 2.3 Properties of \( I_\theta(X_{[r:n]}, Y_{[r:n]}) \)

**Theorem 2.3.** If \( F_1 \) is free of \( \theta \), the basic triangle relationship satisfied by the moments of order statistics (see Arnold et al. (1992), p. 111) holds for \( I_\theta(X_{[r:n]}, Y_{[r:n]}) \); that is,

\[
(2.11) \quad nI_\theta(X_{[r:n]}, Y_{[r:n]}) = (n - r)I_\theta(X_{[r:n]}, Y_{[r:n]}) + rI_\theta(X_{[r+1:n]}, Y_{[r+1:n]})
\]

**Proof.** If \( F_1 \) is free of \( \theta \), from (1.1) it follows that

\[
(2.12) \quad \frac{\partial \log f_{r:n}(x, y; \theta)}{\partial \theta} = \frac{\partial \log f(x, y; \theta)}{\partial \theta} = g(x, y), \quad \text{say}.
\]

Further, (1.1) also implies that \( nf_{r:n-1}(x, y; \theta) = (n - r)f_{r:n}(x, y; \theta) + rf_{r+1:n}(x, y; \theta) \).

Consequently

\[
nEh(X_{[r:n-1]}, Y_{[r:n-1]}) = (n - r)Eh(X_{[r:n]}, Y_{[r:n-1]}) + rEh(X_{[r+1:n]}, Y_{[r+1:n]})
\]

for any \( h \). With \( h = \theta^2 \) in (2.12) we obtain (2.11). \( \square \)

**Theorem 2.4.** If \( F_1 \) is free of \( \theta \), \( I_\theta(X_{[r:n]}, Y_{[r:n]}) \) is additive; that is,

\[
I_\theta(X_{[r_1:n]}, \ldots, X_{r_k:n}, Y_{[r_1:n]}, \ldots, Y_{[r_k:n]}) = \sum_{i=1}^{k} I_\theta(X_{[r_i:n]}, Y_{[r_i:n]}).
\]

**Proof.** The joint pdf of the collection is a generalization of (1.1) and is given by

\[
f_r(x_1, \ldots, x_k, y_1, \ldots, y_k; \theta) = \frac{n!}{(r_1 - 1)!(n - r_k)!} \left\{ F_1(x_1) \right\}^{r_1 - 1} \left\{ 1 - F_1(x_k) \right\}^{n - r_k} \times \prod_{i=2}^{k} \frac{(F_1(x_i) - F_1(x_{i-1}))^{r_i - r_{i-1} - 1}}{(r_i - r_{i-1} - 1)!} \prod_{i=1}^{k} f(x_i, y_i; \theta).
\]

Hence,

\[
\left\{ \frac{\partial \log f_r}{\partial \theta} \right\}^2 = \left\{ \sum_{i=1}^{k} \frac{\partial \log f(x_i, y_i; \theta)}{\partial \theta} \right\}^2.
\]

We expand the right side before taking expectations, and note that the expected value of each of the cross product terms is zero. This yields the desired result. \( \square \)
Remark 2. When $F_1$ is free of $\theta$, Theorem 2.3 indicates that one needs to evaluate only $n$ expectations in order to find $I_\theta(X_{r:n}, Y_{r:n})$ for all $1 \leq r \leq m \leq n$. Further, all the recurrence relations that hold for the moments of $X_{r:n}$ do apply for the FI in $(X_{r:n}, Y_{r:n})$. The additive property of the FI, proven in Theorem 2.4, makes the computation of the FI in an arbitrary collection of order statistics and their concomitants to be trivial once the FI in individual pairs are known.

3. Farlie-Gumbel-Morgenstern Family

3.1 FI of the dependence parameter

Hutchinson and Lai ((1990); Sec. 5.2) provide an excellent introduction to the FGM family whose cdf is given by (1.4), and discuss its properties and applications to a variety of situations. One of its useful property is that the dependence parameter $\theta$ and the marginal distributions can be modeled separately. For the FGM family with normal, exponential, and logistic marginals, the correlation coefficient is a scalar multiple of $\theta$. With $F_1(x) = x$ and $F_2(y) = y$, one obtains the copula form of the FGM family. This copula is a special member of the family having copulas with quadratic sections (see Nelsen (1999), pp. 68–70) and for such distributions also, $F_1$ is free of the parameters determining the dependence structure. Another example is the bivariate normal distribution where $\theta$ is the correlation coefficient. Thus Theorems 2.3 and 2.4 are also applicable to these distributions. Here we focus on the FGM family.

Recently Smith and Moffatt (1999) have investigated FI about $\theta$ in FGM type bivariate logistic models with some special sampling schemes. Scarica and Nair (1999) have discussed some distributional properties of the concomitants of order statistics from the FGM family. The pdf for the cdf in (1.4) is

$$f(x, y; \theta) = f_1(x)f_2(y)[1 + \theta(1 - 2F_1(x))(1 - 2F_2(y))],$$

where $-1 < \theta < 1$, and $f_i$ is the pdf of $F_i$, $i = 1, 2$. The copula form of the above density (see Nelsen (1999), Chapter 2) is

$$f(x, y; \theta) = [1 + \theta C(x, y)], \quad 0 \leq x, y \leq 1,$$

where

$$C(x, y) = (1 - 2x)(1 - 2y).$$

Without loss of generality we use (3.2) to determine $I_\theta(X_{r:n}, Y_{r:n})$. Conditions (A1)-(A3) are satisfied; in fact, in (A3*) we can choose $H(x, y) = [1 - |C(x, y)|]^{-1}$. Now from (2.12) and (3.2)

$$\frac{\partial \log f_r(x, y; \theta)}{\partial \theta} = \frac{C(x, y)}{1 + \theta C(x, y)},$$

and

$$I_\theta(X_{r:n}, Y_{r:n}) = c \int_0^1 \int_0^1 \frac{|C(x, y)|^2}{[1 + \theta C(x, y)]} x^{r-1}(1-x)^{n-r} \, dx \, dy.$$

With $u = (1 - 2x)$ and $v = (1 - 2y)$, we obtain

$$I_\theta(X_{r:n}, Y_{r:n}) = \frac{c}{2^{n+1}} \int_{-1}^1 u^2(1+u)^{n-r}(1-u)^{r-1} \left[ \int_{-1}^1 \frac{v^2}{(1+\theta uv)} \, dv \right] \, du.$$
The factor \((1 + \theta w)^{-1}\) above can be expanded as \(\sum_{j=0}^{\infty} (-\theta)^j w^j v^j\). Since this representation is uniformly convergent for all \(x, y \in (0, 1)\), one can integrate term-by-term in (3.5). Upon doing so and noting that \(\int_{-1}^{1} v^{j+2} dv = 0\) whenever \(j\) is odd, we obtain

\[
I_\theta(X_{r:n}, Y_{r:n}) = \frac{c}{2n} \sum_{j=0}^{\infty} \frac{\theta^{2j}}{2j+3} \int_{-1}^{1} u^{2j+2}(1 + u)^{n-r}(1 - u)^{r-1} du.
\]

For \(r = 1\) we get

\[
I_\theta(X_{1:n}, Y_{1:n}) = \frac{n}{2n-1} \sum_{j=0}^{\infty} \frac{\theta^{2j}}{2j+3} \int_{-1}^{1} u^{2j+2}(1 + u)^{n-1} du,
\]

and on expanding \((1 + u)^{n-1}\) binomially, we obtain

\[
I_\theta(X_{1:n}, Y_{1:n}) = \frac{n}{2n-1} \sum_{j=0}^{\infty} \frac{\theta^{2j}}{2j+3} \sum_{i=0}^{\left\lfloor \frac{n-1}{2i} \right\rfloor} \left( \frac{n-1}{2i} \right) \frac{1}{2j+2i+3}.
\]

With \(n = 1\) in (3.7) we get the FI in a single pair to be

\[
I_\theta(X, Y) = \sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j+3)^2}.
\]

3.2 Properties of \(I_\theta(X_{r:n}, Y_{r:n})\)

From (3.6) it follows that \(I_\theta(X_{r:n}, Y_{r:n}) = I_\theta(X_{n-r+1:n}, Y_{[n-r+1:n]}\). Now we show that for \(1 \leq r < (\frac{n}{2} + 1)\), \(I_\theta(X_{r:n}, Y_{r:n})\) decreases as \(r\) increases. For this, consider

\[
a_r = c \int_{-1}^{1} w^k(1 + u)^{n-r}(1 - u)^{r-1} du,
\]

where \(n\) and \(k\) are fixed, \(k\) is an even number, and \(c\) is given by (1.2). With \(w = (1-u)/2\), \(a_r\) can be expressed as

\[
a_r = c 2^{n+k} \int_{0}^{1} (w - 0.5)^k w^{n-r}(1 - w)^{r-1} dw = 2^{n+k} E(W_r - 0.5)^k,
\]

where \(W_r\) is a Beta\((n-r+1, r)\) random variable. Note that since \(k\) is even, \(E(W_r - 0.5)^k = E(W_{n-r+1} - 0.5)^k\). Thus

\[
a_r = 2^{n+k} E(W_r^* - 0.5)^k,
\]

where the pdf of \(W_r^*\) is given by

\[
g_r(w) = \frac{c}{2} \{w^{-1} (1 - w)^{n-r} + w^{n-r}(1 - w)^{r-1}\}, \quad 0 < w < 1.
\]

Observe that \(g_r(w) = g_r(1 - w)\) and define \(h(w) = g_r(w)/g_{r+1}(w)\).
Lemma 3.1. When \(1 \leq r < n/2\), \(h(w)\) is increasing in \(w\) for \(0.5 < w < 1\).

Proof. We have, from (3.11),
\[
h(w) = \frac{r}{n-r} \left\{ \frac{w^{r-1}(1-w)^{n-r} + w^{n-r}(1-w)^{r-1}}{w^r(1-w)^{n-r-1} + w^{n-r}(1-w)^r} \right\} = \frac{r}{n-r} h_1(t),
\]
where \(t = \frac{w}{1-w}\) and \(h_1(t) = \frac{(1+t^{n-2r+1})}{1+t^{n-2r}}\). Since \(h_1'(t) > 0\) for \(t > 1\) whenever \(r < \frac{n}{2}\), it increases for \(t > 1\). Since \(t\) increases with \(w\) and \(t \geq 1\) whenever \(0.5 < w < 1\), we conclude that \(h(w)\) increases for \(w \in (0.5, 1)\). □

Lemma 3.2. Let \(g_r(w)\) be as defined in (3.11). Then, for \(x \in [0.5, 1]\),
\[(3.12) \int_0^x g_r(w) dw \leq \int_0^x g_{r+1}(w) dw\]
whenever \(r < \frac{n}{2}\) and equality holds in (3.12) only when \(x = 1\).

Proof. From Lemma 3.1 we know that \(h(w) = g_r(w)/g_{r+1}(w)\) increases in \((0.5, 1)\). Consider
\[(3.13) r(x) = \frac{\int_0^x g_r(w) dw}{\int_0^x g_{r+1}(w) dw}, \quad 0.5 < x < 1.
\]
Note that \(r'(x) > 0\) if, and only if,
\[h(x) > \frac{\int_0^x g_r(w) dw}{\int_0^x g_{r+1}(w) dw} = \frac{\int_0^x h(w) g_{r+1}(w) dw}{\int_0^x g_{r+1}(w) dw}.
\]
Since \(h(w)\) strictly increases, the last term above is less than \(h(x)\) for \(0.5 < x < 1\). Consequently, \(r(x)\) increases (strictly) in \((0.5, 1)\) and \(r(1) = 1\) since \(g_r(w)\) and \(g_{r+1}(w)\) are both densities in \((0,1)\) that are symmetric around 0.5. This establishes Lemma 3.2. □

To show that \(a_r > a_{r+1}\) for \(a_r\) in (3.9) whenever \(r < \frac{n}{2}\), we need to establish that \(E(W_r^* - 0.5)^k > E(W_{r+1}^* - 0.5)^k\) where \(W_r^*\) has pdf \(g_r(w)\) given by (3.11). Let \(V_r = (W_r^* - 0.5)^k\). Then, we have to show that \(E(V_r) > E(V_{r+1})\). This holds if \(P(V_r \leq v) < P(V_{r+1} \leq v), 0 < v < 0.5^k\), since both \(V_r\) and \(V_{r+1}\) are positive random variables. Consider
\[
P(V_r \leq v) = P((W_r^* - 0.5)^k \leq v)
\]= \(P(0.5 - v^* \leq W_r^* \leq 0.5 + v^*), \quad v^* = v^{1/k}\)
\]= \(2P(0.5 \leq W_r^* \leq x), \quad x = v^* + 0.5\)
\]= \(2 \int_0^x g_r(w) dw\)
\]< \(2 \int_0^x g_{r+1}(w) dw\) from Lemma 3.2
\]
\[= P(V_{r+1} \leq v).
\]
Thus, we have shown that when \(r < \frac{n}{2}\), \(a_r > a_{r+1}\). Also, when \(r = n/2\), \(a_r = a_{r+1}\) since (3.6) implies that
\[I_0(X_{r,n}, Y_{r,n}) = \frac{1}{2^n} \sum_{j=0}^{\infty} \frac{g^{2j}}{2j + 3} a_r(j)\]
where \( a_k(j) = a_r \) of (3.9) with \( k = 2j + 2 \), we have established that \( I_\theta(X_{r:n}, Y_{r:n}) \) decreases as \( r \) increases whenever \( r < (\frac{n}{2} + 1) \). Using (3.10), the above equation can be expressed as

\[
I_\theta(X_{r:n}, Y_{r:n}) = \sum_{i=1}^{\infty} \frac{\theta^{2(i-1)}}{2i + 1} E(2U_{r:n} - 1)^{2i}
\]

where \( U_{r:n} \) is a Beta\((r, n - r + 1)\) random variable (or the \( r \)-th order statistic from a random sample of size \( n \) from the standard uniform distribution). Note that (3.14) implies that \( I_\theta(X_{r:n}, Y_{r:n}) \) increases as \( \theta \) increases.

The above discussion leads to the following result.

**Theorem 3.1.** For the FGM family with dependence parameter \( \theta \), \( I_\theta(X_{r:n}, Y_{r:n}) \) is given by (3.14), and it has the following properties: (a) \( I_\theta(X_{r:n}, Y_{r:n}) = I_{-\theta}(X_{r:n}, Y_{r:n}) \), (b) \( I_\theta(X_{r:n}, Y_{r:n}) = I_\theta(X_{n-r+1:n}, Y_{[n-r+1:n]} \), (c) \( I_\theta(X_{r:n}, Y_{r:n}) \) decreases as \( r \) increases for \( 1 \leq r < (\frac{n}{2} + 1) \), and (d) \( I_\theta(X_{r:n}, Y_{r:n}) \) increases as \( \theta(>0) \) increases.

**Remark 3.** From (3.14) it follows that \( I_\theta(X_{r:n}, Y_{r:n}) < \sum_{i=1}^{\infty} \frac{\theta^{2(i-1)}}{2i + 1} (2i + 1)^{-1} = \theta^{-2} \{ 2\theta^{-1} \log(1 + \theta)/(1 - \theta) - 1 \} \) and hence the FI is bounded. For large \( n \) and \( r = \lfloor np \rfloor \), \( 0 < p < 1 \), \( U_{r:n} \rightarrow p \) in probability, and since \((2U_{r:n} - 1)^{2i}\) is bounded, \( E(2U_{r:n} - 1)^{2i} \rightarrow (2p - 1)^{2i} \) (Lehmann (1999), p. 71). Hence

\[
I_\theta(X_{r:n}, Y_{r:n}) \approx \sum_{i=1}^{\infty} \frac{\theta^{2(i-1)}}{2i + 1} (2p - 1)^{2i} = \frac{1}{\theta^2} \left\{ \frac{1}{2\theta_p} \log \frac{1 + \theta_p}{1 - \theta_p} - 1 \right\}
\]

where \( \theta_p = \theta(2p - 1) \).

### 3.3 Discussion

Table 1 provides the values of \( I_\theta(X_{r:n}, Y_{r:n}) \) as a function of \( n, r \) (\( \leq (n + 1)/2 \)) and \( \theta \), for \( n = 1(1)5(5)15 \), and \( \theta = 0.25, 0.5, 0.75, 0.99 \). The entries were computed using (3.6) and MATHEMATICA. For \( n = 1 \), (3.8) was used. The infinite series was truncated after 11 terms and this provided adequate accuracy. The first row represents \( I_\theta(X, Y) \), the FI in a single pair. Since the FI in a random sample of size \( n \) is \( nI_\theta(X, Y) \), the table can also be used to compute the proportion of the sample FI contained in a single pair \( I_\theta(X_{r:n}, Y_{r:n}) \). For example, when \( n = 10 \), the FI in the extreme pair ranges from 21% to 26% of the total FI as \( \theta \) ranges from 0.25 to 0.99. When \( n = 15 \) the FI in the extreme pair varies in the range of 16% to 21%. In contrast, the FI in the central pair is no more than 1% of what is available in the complete sample. One could use (3.7) and (2.11) to find \( I_\theta(X_{r:n}, Y_{r:n}) \) for all \( r \) and \( n \), \( 1 \leq r \leq n \), in a recursive manner. But this leads to significant rounding off errors for \( n \) as small as 15.

Since the FI is additive in this case, Table 1 can readily be used to obtain the information contained in singly or multiply censored bivariate samples from the FGM distribution. One just adds up the FI in individual pairs that constitute the censored sample. For example, when \( n = 10 \), the FI in the Type II censored sample consisting of the bottom (or the top) two pairs ranges from 35% to 39% of the total FI as \( \theta \) varies from 0.25 to 0.99. With the two extreme pairs together, it ranges from 42% to 52% of the total FI. We can also obtain an explicit expression for the FI in a Type II right censored bivariate sample using (3.14) and Theorem 2.4.
Table 1. Fisher information in \((X_{r:n}, Y_{r:n})\) about the dependence parameter \(\theta\) in the FGM family.

<table>
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<tr>
<th>(n)</th>
<th>(r)</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.99</th>
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<td>0.1226</td>
<td>0.1437</td>
<td>0.2073</td>
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<td>0.1226</td>
<td>0.1437</td>
<td>0.2073</td>
</tr>
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<td>0.1758</td>
<td>0.2619</td>
</tr>
<tr>
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<td>0.0714</td>
<td>0.0794</td>
<td>0.0979</td>
</tr>
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<td>0.0201</td>
<td>0.0208</td>
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</table>

**Theorem 3.2.** With \((X_{r,n}, Y_{r,n}) = ((X_{i:n}, Y_{i:n})\), \(1 \leq i \leq r < n)\),

\[
I_\theta(X_{r,n}, Y_{r,n}) = \frac{n}{2} \left\{ I_\theta(X, Y) + \sum_{i=1}^{\infty} \frac{\theta^{2(i-1)}}{(2i+1)^2} E(2U_{r:n} - 1)^{2i+1} \right\}.
\]

**Proof.** From (3.14) and Theorem 2.4 we have \(I_\theta(X_{r,n}, Y_{r,n}) = \sum_{i=1}^{\infty} \frac{\theta^{2(i-1)}}{2i+1} E(T_{i,r})\) where \(T_{i,r} = \sum_{j=1}^{r} (2U_{j:n} - 1)^{2i}\). Note that \(E(T_{i,r}) = E(E(T_{i,r} | U_{r+1:n}))\), and given \(U_{r+1:n} = u, U_{1:n} \cdots U_{r:n}\) behave like the order statistics of a random sample of size \(r\) from a uniform distribution over \((0, u)\). Hence

\[E(T_{i,r} | U_{r+1:n} = u) = rE((2U - 1)^{2i} | U \leq u) = \frac{r}{2u(2i+1)} \{1 + (2u - 1)^{2i+1}\}.
\]

Thus, one obtains

\[E(T_{i,r}) = \frac{n}{2(2i+1)} \{1 + E(2U_{r:n} - 1)^{2i+1}\},
\]

and

\[I_\theta(X_{r,n}, Y_{r,n}) = \frac{n}{2} \sum_{i=1}^{\infty} \frac{\theta^{2(i-1)}}{(2i+1)^2} \{1 + E(2U_{r:n} - 1)^{2i+1}\}.
\]
We now obtain (3.15) upon using (3.8). □

The above theorem can be used to obtain the asymptotic variance and ARE (Lehmann (1999), Chapter 7) of the MLE under Type II right censoring of $X$ values and associated concomitants. Let $\hat{\theta}_r$ and $\hat{\theta}_n$ be the MLE of $\theta$ based on the above censored sample, and the entire sample, respectively. When $r = [np]$, $0 < p < 1$, and $n$ is large, $\text{Var}(\hat{\theta}_r) \approx [I_0(X_{r,n}, Y_{r,n})]^{-1}$ and $\text{Var}(\hat{\theta}_n) \approx [nI_0(X,Y)]^{-1}$. With $r = [np]$, $U_{r,n-1} \to p$ in probability and, as argued in Remark 3, $E(2U_{r,n-1} - 1)^{2i+1} \rightarrow (2p - 1)^{2i+1}$. Hence from (3.15) and (3.8), it follows that

$$
\frac{1}{n} I_0(X_{r,n}, Y_{r,n}) \rightarrow \frac{1}{2} \left\{ I_0(X,Y) + (2p - 1)^3 \sum_{i=1}^{\infty} \frac{\theta(2p-1)^{2(i-1)}}{(2i+1)^2} \right\}
$$

$$
= \frac{1}{2} \left\{ I_0(X,Y) + (2p - 1)^3 I_{\theta_r}(X,Y) \right\},
$$

where $\theta_p = \theta(2p - 1)$. Thus

$$
(3.16) \quad \text{ARE}(\hat{\theta}_r, \hat{\theta}_n) = \frac{I_0(X,Y) + (2p - 1)^3 I_{\theta_r}(X,Y)}{2I_0(X,Y)},
$$

and the ARE can be computed for all $\theta$ and $p$, if $I_0(X,Y)$ is available for all $\theta > 0$. Hence Table 1 has another application. For example, when $\theta = 0.75$ and $p = 1/3$, $\theta_p = -0.25$, and upon using Table 1 entries, we obtain $\text{ARE}(\hat{\theta}_r, \hat{\theta}_n) = 0.49$. In other words, when $\theta = 0.75$, the MLE based on bottom 33% of the $X$'s and their concomitants is as efficient as using a random sample consisting of 49% of the pairs.

4. Gumbel's type II bivariate exponential distribution

4.1 $FI$ in $Y_{[r:n]}$ about $E(Y)$

A special distribution in the FGM family is the $G_2BV E$ (Gumbel (1960)), where the marginal distributions are exponential and the joint cdf takes the form

$$
(4.1) \quad F(x, y; \theta) = \left( 1 - \exp \left\{ -\frac{x}{\theta_1} \right\} \right) \left( 1 - \exp \left\{ -\frac{y}{\theta} \right\} \right) \left( 1 + \alpha \exp \left\{ -\frac{x}{\theta_1} - \frac{y}{\theta} \right\} \right),
$$

where $x > 0$, $y > 0$, $\theta_1, \theta > 0$, and $-1 < \alpha < 1$. Here, for convenience, we have labeled the mean of $F_2$, the parameter of interest, as $\theta$, and the dependence parameter as $\alpha$. The pdf of $Y_{[r:n]}$ (Balasubramanian and Dog (1997)) is

$$
(4.2) \quad f_{[r]}(y; \theta) = \frac{1}{\theta} \exp \left\{ -\frac{y}{\theta} \right\} \left( d_1 + d_2 \exp \left\{ -\frac{y}{\theta} \right\} \right)
$$

where

$$
(4.3) \quad d_1 = d_1(r, n; \alpha) = 1 - \alpha + \frac{2\alpha r}{n + 1} \quad \text{and} \quad d_2 = 2(1 - d_1).
$$

Since

$$
\frac{\partial \log f_{[r]}(y; \theta)}{\partial \theta} = \frac{1}{\theta} \left\{ 2 \frac{y}{\theta} - 1 - \frac{y}{\theta} \frac{d_1 + d_2 \exp \left\{ -\frac{y}{\theta} \right\}}{d_1} \right\},
$$
\[(4.4) \quad \left( \frac{\partial \log f_W(y; \theta)}{\partial \theta} \right)^2 = \frac{1}{\theta^2} \left\{ \frac{1}{d_1 + d_2 \exp\{-w\}} \left[ 1 - 4w + 4w^2 - \frac{4w^2d_1}{d_1 + d_2 \exp\{-w\}} + \frac{2wd_1}{d_1 + d_2 \exp\{-w\}} + \frac{w^2d_1^2}{(d_1 + d_2 \exp\{-w\})^2} \right] \right\}
\]

where \( W = Y/\theta \) has the pdf \( f_W(w) = e^{-w}(d_1 + d_2e^{-w}) \). Upon taking expectation in (4.4), we obtain
\[(4.5) \quad I_\theta(Y_{[r:n]}) = \frac{1}{\theta^2} \left\{ 1 - 2d_1 + d_1^2 \int_0^\infty \frac{w^2 \exp\{-w\}}{d_1 + d_2 \exp\{-w\}} dw \right\}
\]
upon simplification, with \( d_1 \) and \( d_2 \) given by (4.3).

4.2 Properties of \( I_\theta(Y_{[r:n]}) \)
Since \( d_1(r,n; \alpha) = d_1(n-r+1,n; -\alpha) \), \( I_\theta(Y_{[r:n]}; \alpha) = I_\theta(Y_{[n-r+1:n]}; -\alpha) \). We now fix \( \alpha > 0 \) and examine (4.5) as a function of \( r \) and \( n \). For this purpose we relabel \( d_1 \) as \( t = 1 - \alpha + 2\alpha r(n+1)^{-1} \) and note that \( I_\theta(Y_{[r:n]}) = \theta^{-2} g(t) \) with
\[(4.6) \quad g(t) = \frac{1}{\theta^2} \left\{ 1 - 2t + t^2 \int_0^\infty \frac{w^2 \exp\{-w\}}{t + 2(1-t) \exp\{-w\}} dw \right\},
\]
and \( 0 \leq 1 - \alpha \leq t \leq 1 + \alpha \leq 2 \). We used MAPLE to evaluate \( g(t) \) and as Fig. 1 shows, \( g(t) \) decreases in \((0, 0.383)\) and then increases. Further its maximum value is 1.80823, attained when \( t = 2 \), and the minimum value is 0.82617. Thus, \( I_\theta(Y_{[r:n]}) \) increases with \( r \) if \( 1 - \alpha > 0.383 \); otherwise it decreases for a while and then increases. The maximum FI is at \( Y_{[n:n]} \). Further \( I_\theta(Y_{[r:n]}) \) does not exceed 1.81 and thus no single concomitant is more efficient than the mean of a random sample of size 2 from the exponential parent. In contrast, for \( r = n_0_p \), \( 0 < p < 1 \), and \( n \) large, \( \theta^{-2} I_\theta(Y_{[r:n]} \approx n(1-p)p^{-1}[\log(1-p)]^2 \)

\( g(t) \) becomes unbounded and peaks around 65% of the total FI in the complete \( Y \)-sample, when \( p \approx 0.8 \) (Arnold et al. (1992), p. 166).

Table 2 provides \( I_\theta(Y_{[r:n]}) \) and \( I_\theta(Y_{r:n}) \) values for \( 1 \leq r \leq n \), \( n = 5, 15, \alpha = 0.25, 0.5, 0.75, 0.99, \) for \( \theta = 1 \). The \( I_\theta(Y_{r:n}) \) values are evaluated using (7.3.8) of Arnold et al. (1992). As Table 2 and the above discussion show, \( I_\theta(Y_{[r:n]; \alpha}) \) is much smaller than \( I_\theta(Y_{r:n}) \).

![Fig. 1](image_url)

**Fig. 1.** \( I_\theta(Y_{[r:n]}) \) values for the G2BVE parent with \( \theta = 1; t = 1 - \alpha + 2\alpha r(n+1)^{-1} \).
Table 2. Fisher information in $Y_{r,n}$, $Y_{r,n}$, and $(X,Y)$ for $\theta = 1$ where $\theta = E(Y)$ of the $G_2 BVE$ distribution with dependence parameter $\alpha$.

<table>
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<th>$n$</th>
<th>$r$</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.99</th>
<th>$I_\theta(Y_{r,n})$</th>
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</table>

$I_\theta(X,Y)$ = \[ \frac{1}{0.05612} \]

How does $I_\theta(Y_{r,n})$ compare with the total information in the full bivariate sample, $nI_\theta(X,Y)$? Using the $G_2 BVE$ cdf in (4.1), we obtain

$$
\frac{\partial \log f(x, y; \theta)}{\partial \theta} = \frac{1}{\theta} \left[ -1 + \frac{y}{\theta} + \frac{2\alpha y}{\theta} \left( \frac{\exp\left\{ -\frac{y}{\theta} \right\}}{1 + \alpha \left( 2 \exp\left\{ -\frac{x}{\theta} \right\} - 1 \right) \left( 2 \exp\left\{ -\frac{y}{\theta} \right\} - 1 \right)} \right] \right).
$$

On letting $u = x/\theta_1$ and $v = y/\theta$, we get

$$
(4.7) I_\theta(X,Y) = E \left( \frac{\partial \log f(X, Y; \theta)}{\partial \theta} \right)^2 = \frac{1}{\theta^2} \left\{ 1 + 4\alpha^2 \int_0^\infty \int_0^\infty v^2 \exp\{-3u\} \{2 \exp\{-u\} - 1\}^2 \exp\{u\} \frac{dv}{1 + \alpha \{2 \exp\{-v\} - 1\} \{2 \exp\{-u\} - 1\}} \right\}.
$$

upon simplification. By expanding the denominator of the integrand above as a power
series in $\alpha$ and carrying out term-by-term integration, we get

$$I_\theta(X, Y) = \frac{1}{\theta^2} \left( 1 + 4 \sum_{j=0}^{\infty} \frac{\alpha^{2j+2}}{2j+3} \int_0^{\infty} v^2 \exp\{-3v\} \{2 \exp\{-v\} - 1\} \text{d}v \right).$$

This indicates $I_\theta(X, Y)$ is increasing with $|\alpha|$ whereas $I_\theta(Y_{r:n})$ is nonmonotonic. The last row of Table 2 contains $I_\theta(X, Y)$ values computed using (4.7) and these do not exceed 1.115 suggesting that the additional gain in efficiency while using the bivariate sample is no more than 12% when compared to the complete $Y$-sample. see below*

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REFERENCES


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*It, along with the last column, also provides a comparison of $nI_\theta(X, Y)$ and $I_\theta(Y_{r:n})$ values.*