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STABILITY OF ORDER STATISTICS UNDER DEPENDENCE

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Abstract. We precisely evaluate the upper and lower deviations of the expectation of every order statistic from an i.i.d. sample under arbitrary violations of the independence assumption, measured in scale units generated by various central absolute moments of the parent distribution of a single observation. We also determine the distributions for which the bounds are attained. The proof is based on combining the Moriguti monotone approximation of functions with the Hölder inequality applied for proper integral representations of expected order statistics in the independent and dependent cases. The method allows us to derive analogous bounds for arbitrary linear combinations of order statistics.

Key words and phrases: Order statistics, identically distributed variables, independent sample, dependent sample, p-th central absolute moment, stability, sharp bound, Moriguti's monotone approximation, Hölder's inequality.

1. Introduction

Consider independent identically distributed random variables $X_1, \dots, X_n$ with common distribution function F, and finite mean $\mu = \int_0^1 F^{-1}(x)dx$. Let $Y_1, \dots, Y_n$ denote a sequence of identically distributed random variables with the same marginal distribution F and arbitrary joint n-dimensional one. Write $X_{j:n}, Y_{j:n}$, $1 \le j \le n$, for respective order statistics. In the paper, we present the extreme deviations of each order statistic under violations of independence assumption

$$(1.1) \quad D_p(j,n) = \sup_{F \in \mathcal{F}_p} (E_F Y_{j:n} - E_F X_{j:n})/\sigma_p, \qquad 1 \le j \le n < \infty, \quad 1 \le p \le \infty,$$

in terms of scale parameter units determined by central absolute moments

$$\sigma_p^p = E_F|X_1 - \mu|^p = \int_0^1 |F^{-1}(x) - \mu|^p dx, \qquad 1 \le p < \infty,$$
$$\sigma_\infty = \text{ess sup}|X_1 - \mu| = \sup_{x \in (0,1)} |F^{-1}(x) - \mu|.$$

The supremum in (1.1) is taken over the class $\mathcal{F}_p$ of all marginals with finite respective σ_p, and all possible dependences of Y_j, $1 \le j \le n$.

Once we determine optimal bounds (1.1) for the upper deviations, we immediately obtain those for the lower ones

$$D_p^-(j,n) = \sup_{F \in \mathcal{F}_p} (E_F X_{j:n} - E_F Y_{j:n})/\sigma_p.$$

Indeed, introducing $X_j^- = -X_j$, $1 \le j \le n$, with common distribution function $F^-(x) = 1 - F(-x-)$ and order statistics $X_{j:n}^- = -X_{n+1-j:n}$ (with the same notation for Y's), we have

$$\begin{aligned} D_p^-(j,n) &= -\inf_{F\in\mathcal{F}_p}(E_F Y_{j:n} - E_F X_{j:n})/\sigma_p \\ &= \sup_{F^-\in\mathcal{F}_p}(\mathrm{E}_{F^-} Y_{n+1-j:n}^- - \mathrm{E}_{F^-} X_{n+1-j:n}^-)/\sigma_p = D_p(n+1-j,n). \end{aligned}$$

For arbitrary fixed F with $\sigma_1 < \infty$, and all possible interdependences of Y_j, $1 \le j \le n$, we have

$$\begin{aligned} E_F X_{j:n} &= \int_0^1 F^{-1}(x) f_{j:n}(x)dx \\ &= \int_0^1 F^{-1}(x) n \binom{n-1}{j-1} x^{j-1}(1-x)^{n-j} dx \end{aligned}$$

(cf., e.g., Arnold *et al.* (1992), p. 109), and

$$\sup E_F Y_{j:n} = \int_0^1 F^{-1}(x) g_{j:n}(x)dx = \frac{n}{n+1-j}\int_{(j-1)/n}^1 F^{-1}(x)dx \tag{1.2}$$

(cf. Caraux and Gascuel (1992), Rychlik (1992)). Setting

$$\begin{aligned} h_{j:n}(x) &= g_{j:n}(x) - f_{j:n}(x) \\ &= \begin{cases} -f_{j:n}(x), & \text{if } \ 0 \le x < (j-1)/n, \\ \dfrac{n}{n+1-j} - f_{j:n}(x), & \text{if } \ (j-1)/n \le x \le 1\,, \end{cases} \end{aligned} \tag{1.3}$$

we therefore have

$$\begin{aligned} \sup(E_F Y_{j:n} - E_F X_{j:n}) &= \int_0^1 F^{-1}(x) h_{j:n}(x)dx \\ &= \int_0^1 [F^{-1}(x) - \mu][h_{j:n}(x) - c]dx \\ &\le \|F^{-1} - \mu\|_p \|h_{j:n} - c\|_q = \|h_{j:n} - c\|_q \sigma_p \end{aligned} \tag{1.4}$$

with conjugate exponent $q = p/(p-1)$ and arbitrary constant c. The equality in (1.4) holds iff $F^{-1}(x) - \mu = g_p(h_{j:n}(x) - c)$ for some monotone function g_p and properly chosen constant c (precise definitions will be presented further). The problem is that monotone increase of $F^{-1} - \mu$ is required, whereas $h_{j:n}(x) - c$ and accordingly $g_p(h_{j:n}(x) - c)$ are not so except of the case $j = 1$, and $D_p(j,n) < \|h_{j:n} - c\|_q$ for $j \ge 2$.

The idea of calculating general sharp bounds consists in replacing $h_{j:n}$ by its monotone approximation $\bar{h}_{j:n}$ due to Moriguti (1953) so that we have

$$\begin{aligned} \sup(E_F Y_{j:n} - E_F X_{j:n}) &\le \int_0^1 [F^{-1}(x) - \mu][\bar{h}_{j:n}(x) - c]dx \\ &\le \|\bar{h}_{j:n} - c\|_q \sigma_p. \end{aligned} \tag{1.5}$$

We shall see that some nondecreasing functions $F^{-1}(x)-\mu = g_p(\bar{h}_{j:n}(x)-c)$ provide the equality in the former inequality of (1.5) as well, which enable us to determine marginal distributions $F = F(p,j,n)$ which attain extreme deviations $D_p(j,n)$. Joint distributions providing equality in (1.2) are characterized by condition

$$Y_{j-1:n} \leq F^{-1}\left(\frac{j-1}{n}\right) \leq Y_{j:n} = Y_{n:n} \quad \text{almost surely.} \tag{1.6}$$

One can construct an example of identically F-distributed sample satisfying (1.6) by taking independent $V = Y_{1:n} = \cdots = Y_{j-1:n}$ and $Z = Y_{j:n} = \cdots = Y_{n:n}$ with distribution functions $\min\{nF/(j-1),1\}$ and $\max\{0,(nF+1-j)/(n+1-j)\}$, respectively, and random rearrangement of $Y_{1:n},\ldots,Y_{n:n}$. Constants c for $1<p<\infty$ are determined uniquely by moment conditions. Otherwise they are selected so to minimize $\|\bar{h}_{j:n}-c\|_q$.

In Section 2, we construct the Moriguti monotone approximations $\bar{h}_{j:n}$, based on greatest convex minorants of antiderivatives of $h_{j:n}$. In Section 3, we present main results, describing optimal bounds $D_p(j,n)$ for all $1 \leq p \leq \infty$ and $1 \leq j \leq n < \infty$ and distributions which attain the bounds. Most generally, the extreme marginal is a mixture of a three-point distribution with a smooth one which is the inverse of a polynomial of (noninteger in general) power of standarized argument. The respective joint distribution of the dependent sample is determined by (1.6). Simpler cases $j = 1, n$ of extreme order statistics will be studied separately. Numerical values of (1.1) for $p = 1, 2, \infty$ and $1 \leq j \leq n = 20$ are presented in Table 1.

It is worth pointing out that our method can be applied for calculating respective deviations for arbitrary linear combinations of order statistics. Then we take the counterpart of (1.2) for general L-statistics

$$\sup E_F \sum_{j=1}^{n} c_j Y_{j:n} = \int_0^1 F^{-1}(x) C'(x)\,dx, \tag{1.7}$$

where C' is the right continuous version of the piecewise linear function C which is the greatest convex one satisfying

$$C(j/n) \leq \sum_{i=1}^{j} c_i, \qquad j = 0,1,\ldots,n,$$

(cf. Rychlik (1993a)). Therefore C' has the form

$$C'(x) = \sum_{j=1}^{n} d_j \mathbf{1}_{[(j-1)/n,j/n)}(x)$$

for a well-defined nondecreasing sequence d_j, $1 \leq j \leq n$, and we can consider

$$\begin{aligned} \text{(1.8)} \qquad & \sup_{F\in\mathcal{F}_p}\left(E_F\sum_{j=1}^{n} c_j Y_{j:n} - E_F \sum_{j=1}^{n} c_j X_{j:n}\right) \\ & = \sup_{F\in\mathcal{F}_p}\int_0^1 F^{-1}(x)\sum_{j=1}^{n}\left[d_j\mathbf{1}_{[(j-1)/n,j/n)}(x) - c_j f_{j:n}(x)\right]dx, \end{aligned}$$

constructing the Moriguti approximations for the expression in braces and the modifications mentioned above. We can similarly determine sharper inequalities for symmetric parent marginals. Using the relation $F^{-1}(x) - \mu = \mu - F^{-1}(-x-)$ and changing variables in (1.8), we are left with the task of analyzing the sum in braces folded about 1/2 on the interval $[1/2, 1]$.

Problem (1.8), which describes bias-stability of L-statistics against departures from the independence assumptions, stems from robust statistics. Robustness of L-statistics, which are popular tools in robust and nonparametric inference, was studied by many authors (we refer merely to monographs by Huber (1981) and Hampel *et al.* (1986)). Mainly violations of marginals were considered. Dependence-robustness for location models was analyzed in Rychlik (1993*b*). Estimation of population quantiles is the simplest application of single order statistics, and the extreme ones are used for evaluating the population range. The best general reference to applications of order statistics in estimation is Balakrishnan and Cohen (1991).

Moriguti (1953) presented a simple numerical procedure for getting a bound on $E_F(X_{j:n} - \mu)/\sigma_2$ in the i.i.d. case. Analytic formulae for the sample maximum were given in Gumbel (1954) and Hartley and David (1954), and respective extensions for arbitrary p-th moments are in Arnold (1985), where the dependent case was also studied. Gascuel and Caraux (1992) evaluated $E_F(Y_{j:n} - \mu)/\sigma_2$, $1 \le j \le n$, using (1.2). A number of inequalities based on (1.7) for general L-statistics, and general and symmetric samples in various σ_p-units, $1 \le p \le \infty$, with some sample counterparts, can be found in Rychlik (1993*c*). Papadatos (1997) obtained bounds on expected order statistics and their differences for nonnegative i.i.d. samples in terms of μ. We also mention moment bounds of Gajek and Rychlik (1996, 1998) and Rychlik (1998, 2001*a*, 2001*b*), and quantile bounds of Blom (1958), van Zwet (1964), Ali and Chan (1965), Barlow and Proschan (1966), Lawrence (1975) and Rychlik (1998) for restricted families of marginals determined by some stochastic ordering relations. For a recent review of bounds on expectations of L-estimates we refer the reader to Rychlik (1998). The Moriguti monotone approximations, especially some discrete versions, have numerous applications in order restricted statistical inference (see Robertson *et al.* (1988)). Recently, Raqab (1997) used the Moriguti method for evaluating moment bounds on k-th records, and Gajek and Okolewski (2000) combined it with the Steffensen inequality and derived some integral bounds on expectations of order and record statistics.

2. The Moriguti monotone approximation

We first recall a result of Moriguti (1953) in a simpler form that will be used in the sequel. Suppose that a function h is defined and has a finite integral on some interval $[a, b]$. Let $H(x) = \int_a^x h(t)dt$, $a \le x \le b$, stand for its antiderivative, and $\bar{H}$ be the greatest convex minorant of H. Note that $\bar{H} < H$ in countably many open intervals at most, and $\bar{H}$ is linear in each of the intervals. Write $\bar{h}$ for the right-continuous version of the derivative of $\bar{H}$. Obviously, $\bar{h}$ is nondecreasing and constant in intervals where $\bar{h} \ne h$.

LEMMA 1. *For every nondecreasing function g on $[a, b]$ for which both the integrals in (2.1) are finite, we have*

$$\int_a^b g(x)h(x)dx \le \int_a^b g(x)\bar{h}(x)dx. \tag{2.1}$$

The equality in (2.1) *holds iff g is constant in every interval contained in the set where* $\bar{H} < H$.

Consider (1.3) with antiderivative

$$H_{j:n}(x) = \begin{cases} -F_{j:n}(x), & \text{if } 0 \le x \le (j-1)/n, \\ \dfrac{nx+1-j}{n+1-j} - F_{j:n}(x), & \text{if } (j-1)/n \le x \le 1, \end{cases} \tag{2.2}$$

where

$$F_{j:n}(x) = \int_0^x f_{j:n}(t)dt = \sum_{k=j}^{n} \binom{n}{k} x^k (1-x)^{n-k}.$$

If $j = 1$, then $h_{1:n}(x) = 1 - n(1-x)^{n-1}$ is increasing and so $\bar{h}_{1:n} = h_{1:n}$. For $j = n$ we have

$$H_{n:n}(x) = \begin{cases} -x^n, & \text{if } 0 \le x \le 1 - 1/n, \\ n(x-1) + 1 - x^n, & \text{if } 1 - 1/n \le x \le 1, \end{cases}$$

which is decreasing on $[0, 1-1/n]$ and increasing on $[1-1/n, 1]$, and concave on both the intervals. Therefore

$$\bar{H}_{n:n}(x) = \begin{cases} -(1-1/n)^{n-1}x, & \text{if } 0 \le x \le 1 - 1/n, \\ n(1-1/n)^n(x-1), & \text{if } 1 - 1/n \le x \le 1, \end{cases}$$

and

$$\bar{h}_{n:n}(x) = \begin{cases} -(1-1/n)^{n-1}, & \text{if } 0 \le x < 1 - 1/n, \\ n(1-1/n)^n, & \text{if } 1 - 1/n \le x \le 1. \end{cases} \tag{2.3}$$

A deeper analysis is needed for $2 \le j \le n-1$ for which each $f_{j:n}$ is first increasing and then decreasing, and vanishes at both ends of $[0,1]$. It is important to know the sign of $h_{j:n}$ at $\frac{j-1}{n-1}$ where $f_{j:n}$ is maximized. One can see that $h_{j:n}(\frac{j-1}{n-1}) < 0$ for small and moderate j with a given n. Also, the proportion of j's for which the relation holds is greater and increases to 1, as n becomes large. Indeed, sequence $f_{j:n}(\frac{j-1}{n-1})$ is decreasing-increasing in j, and symmetric about $(n+1)/2$, whereas $g_{j:n}(\frac{j-1}{n-1}) = \frac{n}{n+1-j}$, $2 \le j \le n-1$, increases from $\frac{n}{n-1}$ to $n/2$. For $h_{m+1:2m+1}(1/2)$, $m \ge 1$, corresponding to the sample medians, yields $0 = h_{2:3}(1/2) > h_{3:5}(1/2) > \cdots$. The equality is easily checked, and calculating

$$\frac{f_{m+1:2m+1}(1/2)}{f_{m-1:2m-1}(1/2)} = 1 + \frac{1}{2m},$$

$$\frac{g_{m+1:2m+1}(1/2)}{g_{m-1:2m-1}(1/2)} = 1 + \frac{1}{2m^2+m-1},$$

we see that $f_{m+1:2m+1}(1/2)$ increases faster than $g_{m+1:2m+1}(1/2)$, and verify the inequalities. On the other hand, we have $0 = h_{2:3}(1/2) < h_{3:4}(2/3) < \cdots$, because

$$\frac{f_{n-1:n}((n-2)/(n-1))}{g_{n-1:n}((n-2)/(n-1))} = \frac{\xi(1)}{\xi(n-2)} < 1,$$

where $\xi(n) = (1+1/n)^n$, $n \ge 1$, is a sequence increasing to e. For $n \to \infty$ and $j/n \to x \in (0,1)$, we can take an asymptotic representation

$$[2\pi x(1-x)/n]^{1/2} f_{j:n}\left(\frac{j-1}{n-1}\right) \to 1$$

(cf. Lorentz (1953), Theorem 1.5.2) for proving that $h_{j:n}(\frac{j-1}{n-1}) \to -\infty$.

We now present the Moriguti approximation of $h_{j:n}$ for $2 \le j \le n-1$. Let

$$S_{j:n}(x) = H_{j:n}(x)/x, \quad 0 < x \le 1, \tag{2.4}$$

$$T_{j:n}(x) = \left[H_{j:n}(x) - H_{j:n}\left(\frac{j-1}{n}\right)\right] \bigg/ \left[x - \frac{j-1}{n}\right], \quad \frac{j-1}{n} < x \le 1, \tag{2.5}$$

denote the slopes of straight lines secant to the graph of $H_{j:n}$ at points $(0, H_{j:n}(0) = 0)$, and $(\frac{j-1}{n}, H_{j:n}(\frac{j-1}{n}) = -F_{j:n}(\frac{j-1}{n}))$, respectively, and some $(x, H_{j:n}(x))$.

THEOREM 1. *If $h_{j:n}(\frac{j-1}{n-1}) < 0$, then there exist unique $r \in (\frac{j-1}{n-1}, 1)$ such that $h_{j:n}(r) = 0$ and $s \in [\frac{j-1}{n-1}, r)$ such that either $s = \frac{j-1}{n-1}$ if $S_{j:n}(\frac{j-1}{n-1}) \le h_{j:n}(\frac{j-1}{n-1})$ or s is the solution to $S_{j:n}(x) = h_{j:n}(x)$ otherwise.*

If

$$h_{j:n}\left(\frac{j-1}{n-1}\right) < 0 \quad \text{and} \quad S_{j:n}(s) \le S_{j:n}\left(\frac{j-1}{n}\right), \tag{2.6}$$

then

$$\bar{h}_{j:n}(x) = \begin{cases} S_{j:n}(s), & \text{if } 0 \le x < s, \\ h_{j:n}(x), & \text{if } s \le x \le 1. \end{cases} \tag{2.7}$$

If

$$\text{either } h_{j:n}\left(\frac{j-1}{n-1}\right) < 0 \quad \text{and} \quad S_{j:n}(s) > S_{j:n}\left(\frac{j-1}{n}\right) \quad \text{or } h_{j:n}\left(\frac{j-1}{n-1}\right) \ge 0 \tag{2.8}$$

then there exists a unique $t \in (\frac{j-1}{n-1}, 1)$ such that $T_{j:n}(t) = h_{j:n}(t)$, and

$$\bar{h}_{j:n}(x) = \begin{cases} S_{j:n}\left(\frac{j-1}{n}\right), & \text{if } 0 \le x < \frac{j-1}{n}, \\ h_{j:n}(t), & \text{if } \frac{j-1}{n} \le x \le t, \\ h_{j:n}(x), & \text{if } t \le x \le 1. \end{cases} \tag{2.9}$$

For the prevailing number of cases, we have (2.6) with $s > \frac{j-1}{n-1}$. Then (2.7) is continuous, and can be written as $\bar{h}_{j:n}(x) = h_{j:n}(\max\{s, x\})$. This form is easier to handle in numerical calculations. Function (2.9) has the only jump at $\frac{j-1}{n}$.

PROOF. Assume that $h_{j:n}(\frac{j-1}{n-1}) < 0$. Starting from the origin, $h_{j:n}$ continously decreases on $(0, \frac{j-1}{n})$, jumps up and again decreases to a negative value $h_{j:n}(\frac{j-1}{n-1})$, and eventually increases to $h_{j:n}(1) = \frac{n}{n+1-j} > 0$, passing through the horizontal axis at some $r \in (\frac{j-1}{n-1}, 1)$. It may happen that $h_{j:n}(\frac{j-1}{n}) > 0$, but by far more frequent is the other case. It follows that (2.2) is concave decreasing on $[0, \frac{j-1}{n}]$, and either concave decreasing on $[\frac{j-1}{n}, \frac{j-1}{n-1}]$ when $h_{j:n}(\frac{j-1}{n}) \le 0$ or concave increasing-decreasing on $[\frac{j-1}{n}, \frac{j-1}{n-1}]$ otherwise, and then convex decreasing on $[\frac{j-1}{n-1}, r]$, and eventually convex increasing on $[r, 1]$. Also, $H_{j:n}(x) < 0$ for $0 < x < 1$ with $H_{j:n}(0) = H_{j:n}(1) = 0$, and $H_{j:n}(x)$ is differentiable except of at $\frac{j-1}{n}$, where the left derivative is less than the right one.

We determine the left part of the greatest convex minorant by minimizing (2.4). Our purpose is to assert that

$$\inf_{x\in(0,1]} S_{j:n}(x) = \min\left\{S_{j:n}\left(\frac{j-1}{n}\right), S_{j:n}(s)\right\}. \tag{2.10}$$

The infimum cannot be attained in $(r,1]$, because

$$x^2 S'_{j:n}(x) = x h_{j:n}(x) - H_{j:n}(x) \tag{2.11}$$

consists of positive and nonnegative terms there. The minimum over $[0,\frac{j-1}{n}]$ is attained at the right end, because each point of the graph of the concave part of $H_{j:n}$ lies above the line secant at 0 and $\frac{j-1}{n}$. Likewise, geometrical arguments lead us to the conclusion that the minimization problem for $x \in [\frac{j-1}{n}, \frac{j-1}{n-1}]$ has the solution at either of the end-points. Accordingly, we are reduced to considering points of $\{\frac{j-1}{n}\} \cup [\frac{j-1}{n-1}, r)$, and we now analyze behavior of $S_{j:n}$ in the interval. Differentiating (2.11), we obtain $xh'_{j:n}(x)$, which is positive in $(\frac{j-1}{n-1}, r]$ and implies strict increase of $S'_{j:n}$ there. Since $S'_{j:n}(r) = -H_{j:n}(r)/r^2 > 0$, we have $\min_{x\in[(j-1)/n,r)} S_{j:n}(x) = S_{j:n}(s)$ for s defined in the first assertion of Theorem 1. This proves (2.10).

Suppose that $S_{j:n}(\frac{j-1}{n}) \geq S_{j:n}(s)$. Since $H_{j:n}$ is convex on $[s,1]$, we conclude that

$$\bar{H}_{j:n}(x) = \begin{cases} S_{j:n}(s)x, & \text{if } \ 0 \leq x < s, \\ H_{j:n}(x), & \text{if } \ s \leq x \leq 1. \end{cases}$$

is the greatest convex minorant of $H_{j:n}$, and (2.7) is the respective derivative.

If $S_{j:n}(\frac{j-1}{n}) < S_{j:n}(s)$, we have only found $\bar{H}_{j:n}(x) = S_{j:n}(\frac{j-1}{n})x$ for $x \in [0, \frac{j-1}{n})$. The next piece will be derived by minimizing (2.5). We merely sketch the solution, because arguments similar to the above are used. By concavity, we exclude points of $[\frac{j-1}{n}, \frac{j-1}{n-1})$. Relations

$$\begin{aligned} \left[\left(x-\frac{j-1}{n}\right)^2 T'_{j:n}(x)\right]' &= \left[h_{j:n}(x)\left(x-\frac{j-1}{n}\right) + H_{j:n}\left(\frac{j-1}{n}\right) - H_{j:n}(x)\right]' \\ &= -f'_{j:n}(x)\left(x-\frac{j-1}{n}\right) > 0, \qquad \frac{j-1}{n-1} < x < 1, \end{aligned} \tag{2.12}$$

imply increase of $T'_{j:n}$ in $[\frac{j-1}{n-1}, 1]$. Since $\min_{x\in[\frac{j-1}{n},\frac{j-1}{n-1}]} h_{j:n}(x) = h_{j:n}(\frac{j-1}{n-1})$, we have

$$\begin{aligned} h_{j:n}\left(\frac{j-1}{n-1}\right)\left(\frac{j-1}{n-1} - \frac{j-1}{n}\right) &< \int_{(j-1)/n}^{(j-1)/(n-1)} h_{j:n}(x)dx \\ &= H_{j:n}\left(\frac{j-1}{n-1}\right) - H_{j:n}\left(\frac{j-1}{n}\right), \end{aligned}$$

which in combination with the first line of (2.12) gives $T'_{j:n}(\frac{j-1}{n-1}) < 0$. Putting $x = 1$, we obtain

$$\left(1-\frac{j-1}{n}\right)^2 T'_{j:n}(1) = 1 - F_{j:n}\left(\frac{j-1}{n}\right) > 0.$$

Therefore $T'_{j:n}$ changes the sign once in $[\frac{j-1}{n-1}, 1]$ (from $-$ to $+$), and $T_{j:n}$ is minimized at the change point t, say, for which $T_{j:n}(t) = h_{j:n}(t)$ holds (cf. (2.12) again). Observe finally that $H_{j:n}$ is convex on $[t, 1]$. Summing up, we can construct the greatest convex minorant as follows

$$\bar{H}_{j:n}(x) = \begin{cases} S_{j:n}\left(\frac{j-1}{n}\right)x, & \text{if } 0 \le x < \frac{j-1}{n}, \\ h_{j:n}(t)\left(x - \frac{j-1}{n}\right) - F_{j:n}\left(\frac{j-1}{n}\right), & \text{if } \frac{j-1}{n} \le x \le t, \\ H_{j:n}(x), & \text{if } t \le x \le 1. \end{cases}$$

It remains to deduce the same formula in the case $h_{j:n}(\frac{j-1}{n-1}) \ge 0$. Then $h_{j:n}$ is negative decreasing in $(0, \frac{j-1}{n})$, positive decreasing in $[\frac{j-1}{n}, \frac{j-1}{n-1})$, and positive increasing in $(\frac{j-1}{n-1}, 1)$. The antiderivative $H_{j:n}$ is concave decreasing in $[0, \frac{j-1}{n}]$, concave increasing in $[\frac{j-1}{n}, \frac{j-1}{n-1}]$, and convex increasing in $[\frac{j-1}{n-1}, 1]$. It is clear that $S_{j:n}$ is minimized at $\frac{j-1}{n}$. Repeating arguments of the previous paragraph, we conclude that $T_{j:n}$ has the minimum at t, and the final form of the Moriguti construction. □

3. Main results

Below we present bounds on expectation increase for order statistics under violation of independence assumption. The bounds are expressed in the scale units determined by p-th central absolute moments, $1 \le p \le \infty$. If bounds are attainable by some distributions of samples X_j, Y_j, $1 \le j \le n$, we describe respective common marginal distribution functions. We shall tacitly assume that the joint distribution of the dependent sample is one that provides the extreme expectation of the order statistic for the given marginal (see (1.6)). If a bound is not attained, we present an example of a sequence of marginals for which expected order statistics approximate the bound. In three subsequent theorems we analyze bounds for nonextreme order statistics measured in σ_p-units for $p = 1$, $1 < p < \infty$, and $p = \infty$, respectively.

THEOREM 2. *Let $2 \le j \le n-1 < \infty$ and s^* be equal to either s or $\frac{j-1}{n}$ if either (2.6) or (2.8) holds. Then*

$$\sup_{F \in \mathcal{F}_1} (E_F Y_{j:n} - E_F X_{j:n})/\sigma_1 = D_1(j,n) = \frac{1}{2s^*}\left[\frac{j-1}{n+1-j} + F_{j:n}(s^*)\right]. \tag{3.1}$$

For the sequence of three-point distribution functions F_k, $k \ge 1/(1-s^*)$, assigning probabilities s^*, $1-s^*-1/k$ and $1/k$ to values $\mu - \sigma_1/(2s^*)$, μ and $\mu + k\sigma_1/2$, respectively, we have $E_{F_k} X_1 = \mu$, $E_{F_k}|X_1 - \mu| = \sigma_1$, and

$$D_1(j,n) - \sup(E_{F_k} Y_{j:n} - E_{F_k} X_{j:n})/\sigma_1 < f_{j:n}(1-1/k)/2 \to 0. \tag{3.2}$$

PROOF. By (1.4) and Lemma 1,

$$\begin{aligned} \sup(E_F Y_{j:n} - E_F X_{j:n}) &\le \int_0^1 [F^{-1}(x) - \mu][\bar{h}_{j:n}(x) - c]dx \\ &\le \sup|\bar{h}_{j:n} - c| \int_0^1 |F^{-1}(x) - \mu|dx \\ &= \|\bar{h}_{j:n} - c\|_\infty \sigma_1 \end{aligned} \tag{3.3}$$

for arbitrary real c. Since $\bar{h}_{j:n}$ is nondecreasing, the maximal value of the supremum is provided by $c = \frac{1}{2}[\bar{h}_{j:n}(0) + \bar{h}_{j:n}(1)]$ for which

$$\|\bar{h}_{j:n} - c\|_\infty = \frac{1}{2}[\bar{h}_{j:n}(1) - \bar{h}_{j:n}(0)] = D_1(j,n).$$

The first relation in (3.3) becomes equality iff $F^{-1} - \mu$ is constant on $\{\bar{h}_{j:n} = \bar{h}_{j:n}(0)\}$. The condition for equality in the latter one is that $F^{-1} - \mu$ is almost surely negative, zero, and positive on the sets $\{\bar{h}_{j:n} = \bar{h}_{j:n}(0)\}$, $\{\bar{h}_{j:n}(0) < \bar{h}_{j:n} < \bar{h}_{j:n}(1)\}$, and $\{\bar{h}_{j:n} = \bar{h}_{j:n}(1)\}$, respectively. Since $\{\bar{h}_{j:n} = \bar{h}_{j:n}(1)\}$ has zero measure, the conditions contradict $E_F X_1 - \mu = \int_0^1 [F^{-1}(x) - \mu]dx = 0$. However, for sequence F_k, we easily verify both moment assumptions, and the equality in the first line of (3.3). Finally,

$$\begin{aligned}\int_0^1 [F_k^{-1}(x) - \mu][\bar{h}_{j:n}(x) - c]dx &= D_1(j,n)\sigma_1 - \frac{k\sigma_1}{2}\int_{1-\frac{1}{k}}^1 [h_{j:n}(1) - h_{j:n}(x)]dx \\ &> D_1(j,n)\sigma_1 - \left[h_{j:n}(1) - h_{j:n}\left(1 - \frac{1}{k}\right)\right]\sigma_1/2,\end{aligned}$$

which is identical with the inequality in (3.2). □

For $1 < p < \infty$ our solution has complicated representations. Altogether, we have four cases. Under either of conditions (2.6) and (2.8) we have two cases, when each bound $D_p(j,n)$ depends on a parameter, denoted below by either c or d, which is implicitely defined by an equation. Generally, the distributions for which the bounds are attained have the form

$$(3.4)\; F(x) = F(x;\alpha,\beta,\gamma)$$
$$= \begin{cases} 0, & \text{if } \dfrac{x-\mu}{\sigma_p} < -\left[\dfrac{\gamma - S_{j:n}(\alpha)}{D}\right]^{q/p}, \\ \alpha, & \text{if } -\left[\dfrac{\gamma - S_{j:n}(\alpha)}{D}\right]^{q/p} \le \dfrac{x-\mu}{\sigma_p} \\ & \le \left|\dfrac{h_{j:n}(\beta) - \gamma}{D}\right|^{q/p} \operatorname{sgn}(h_{j:n}(\beta) - \gamma), \\ h_{j:n}^{-1}\left(\gamma + D\left|\dfrac{x-\mu}{\sigma_p}\right|^{p/q} \operatorname{sgn}(x-\mu)\right), & \text{if } \left|\dfrac{h_{j:n}(\beta) - \gamma}{D}\right|^{q/p} \operatorname{sgn}(h_{j:n}(\beta) - \gamma) \\ & \le \dfrac{x-\mu}{\sigma_p} \le \left[\dfrac{\frac{n}{n+1-j} - \gamma}{D}\right]^{q/p}, \\ 1, & \text{if } \dfrac{x-\mu}{\sigma_p} \ge \left[\dfrac{\frac{n}{n+1-j} - \gamma}{D}\right]^{q/p}, \end{cases}$$

for $q = p/(p-1)$ and $D = D_p(j,n)$. Parameters $0 < \alpha < \beta < 1$ and $S_{j:n}(\alpha) < \gamma < \frac{n}{n+1-j}$ will be precisely defined in Theorem 3. Distribution function (3.4) has a finite support with one (if $S_{j:n}(\alpha) = h_{j:n}(\beta)$) or two jumps (otherwise) at the left-end of its support.

On the right, it has a smooth density which cannot be written explicitely, because the third formula in (3.4) is actually the inverse of a linear modification of a Bernstein polynomial of degree $n-1$ of a linear function of some, generally noninteger, power of the standarized argument. In the specific cases, representation (3.4) can be slightly simplified.

THEOREM 3. (i) *If* (2.6) *and*

$$S_{j:n}\left(\frac{j-1}{n-1}\right) < h_{j:n}\left(\frac{j-1}{n-1}\right), \tag{3.5}$$

$$\frac{j-1}{n-1}\left[h_{j:n}\left(\frac{j-1}{n-1}\right) - S_{j:n}\left(\frac{j-1}{n-1}\right)\right]^{q/p} > \int_{\frac{j-1}{n-1}}^{1}\left[f_{j:n}\left(\frac{j-1}{n-1}\right) - f_{j:n}(x)\right]^{q/p} dx, \tag{3.6}$$

hold, then there exists $S_{j:n}(\frac{j-1}{n-1}) < c < h_{j:n}(\frac{j-1}{n-1})$ *such that*

$$\frac{j-1}{n-1}\left[c - S_{j:n}\left(\frac{j-1}{n-1}\right)\right]^{q/p} = \int_{\frac{j-1}{n-1}}^{1} [h_{j:n}(x) - c]^{q/p} dx, \tag{3.7}$$

and

$$\sup_{F\in\mathcal{F}_p} (E_F Y_{j:n} - E_F X_{j:n})/\sigma_p = D_p(j,n) \tag{3.8}$$

with

$$D_p^q(j,n) = \frac{j-1}{n-1}\left[c - S_{j:n}\left(\frac{j-1}{n-1}\right)\right]^{q} + \int_{\frac{j-1}{n-1}}^{1} [h_{j:n}(x) - c]^{q} dx. \tag{3.9}$$

Bound (3.8) *is attained by* $F(x) = F(x; \frac{j-1}{n-1}, \frac{j-1}{n-1}, c)$ *defined in* (3.4).

(ii) *If* (2.6) *is true and either of* (3.5), (3.6) *is false, then there exists* $s \le d < 1$ *such that*

$$\begin{aligned} s[h_{j:n}(d) - S_{j:n}(s)]^{q/p} &+ \int_s^d [f_{j:n}(x) - f_{j:n}(d)]^{q/p} dx \\ &= \int_d^1 [f_{j:n}(d) - f_{j:n}(x)]^{q/p} dx, \end{aligned} \tag{3.10}$$

and then we have (3.8) *with*

$$D_p^q(j,n) = s[h_{j:n}(d) - S_{j:n}(s)]^q + \int_s^1 |f_{j:n}(x) - f_{j:n}(d)|^q dx. \tag{3.11}$$

The bound is attained by $F(x; s, s, h_{j:n}(d))$ *defined in* (3.4).

(iii) *If* (2.8) *and*

$$\frac{j-1}{n}\left[h_{j:n}(t) - S_{j:n}\left(\frac{j-1}{n}\right)\right]^{q/p} > \int_t^1 [f_{j:n}(t) - f_{j:n}(x)]^{q/p} dx \tag{3.12}$$

hold, then there exists $S_{j:n}(\frac{j-1}{n}) < c < h_{j:n}(t)$ *such that*

$$\frac{j-1}{n}\left[c - S_{j:n}\left(\frac{j-1}{n}\right)\right]^{q/p} = \left(t - \frac{j-1}{n}\right)[h_{j:n}(t) - c]^{q/p} + \int_t^1 [h_{j:n}(x) - c]^{q/p} dx, \tag{3.13}$$

and then

$$(3.14)\qquad D_p^q(j,n) = \frac{j-1}{n}\left[c - S_{j:n}\left(\frac{j-1}{n}\right)\right]^q + \left(t - \frac{j-1}{n}\right)[h_{j:n}(t) - c]^q + \int_t^1 [h_{j:n}(x) - c]^q dx.$$

The bound in (3.8) *is attained by* $F(x; \frac{j-1}{n}, t, c)$.

(iv) *If* (2.8) *is true and* (3.6) *is false, then there exists* $t \le d < 1$ *such that*

$$\frac{j-1}{n}\left[h_{j:n}(d) - S_{j:n}\left(\frac{j-1}{n}\right)\right]^{q/p} + \left(t - \frac{j-1}{n}\right)[f_{j:n}(t) - f_{j:n}(d)]^{q/p} + \int_t^d [f_{j:n}(x) - f_{j:n}(d)]^{q/p} dx = \int_d^1 [f_{j:n}(d) - f_{j:n}(x)]^{q/p} dx,$$

and

$$(3.15)\, D_p^q(j,n) = \frac{j-1}{n}\left[h_{j:n}(d) - S_{j:n}\left(\frac{j-1}{n}\right)\right]^q + \left(t - \frac{j-1}{n}\right)[f_{j:n}(t) - f_{j:n}(d)]^q + \int_t^1 |f_{j:n}(x) - f_{j:n}(d)|^q dx.$$

This bound is attained by $F(x; \frac{j-1}{n}, t, h_{j:n}(d))$.

PROOF. Proof is based on the Hölder inequality

$$\int_0^1 [F^{-1}(x) - \mu][\bar{h}_{j:n}(x) - c]dx \le \left[\int_0^1 |F^{-1}(x) - \mu|^p dx\right]^{1/p}\left[\int_0^1 |\bar{h}_{j:n}(x) - c|^q dx\right]^{1/q} = \|\bar{h}_{j:n} - c\|_q \sigma_p,$$

where the equality holds true iff

$$(3.16)\qquad F^{-1}(x) - \mu = \alpha|\bar{h}_{j:n}(x) - c|^{q/p}\mathrm{sgn}(\bar{h}_{j:n}(x) - c)$$

for some $\alpha \ge 0$. Condition $\int_0^1 |F^{-1}(x) - \mu|^p dx = \sigma_p^p$ forces

$$(3.17)\qquad \alpha = \sigma_p / \|\bar{h}_{j:n} - c\|_q^{q/p}.$$

Constant c is uniquely determined by condition

$$(3.18)\qquad \int_0^1 [F^{-1}(x) - \mu]dx = \alpha \int_0^1 |\bar{h}_{j:n}(x) - c|^{q/p}\mathrm{sgn}(\bar{h}_{j:n}(x) - c)dx = 0.$$

Clearly, $\bar{h}_{j:n}(0) < c < \bar{h}_{j:n}(1)$, because otherwise the latter integrand in (3.18) has one sign. One can easily verify that the integral is stricly decreasing continuous function of c, and so the solution to (3.18) is uniquely determined. Also, (3.16) is nondecreasing, and constant on all intervals of $\{\bar{H}_{j:n} < H_{j:n}\}$, which are contained in ones where $\bar{h}_{j:n}$ is constant. Summing up, (3.16) with constants defined in (3.17), (3.18) determines the unique distribution function for which

$$\sup(E_F Y_{j:n} - E_F X_{j:n}) = \int_0^1 [F^{-1}(x) - \mu][h_{j:n}(x) - c]dx = \int_0^1 [F^{-1}(x) - \mu][\bar{h}_{j:n}(x) - c]dx = \|\bar{h}_{j:n} - c\|_q \sigma_p.$$

It suffices now to determine values of $D_p(j,n) = \|\bar{h}_{j:n} - c\|_q$ and functions F for various forms of $\bar{h}_{j:n}$ described in Theorem 1.

Suppose first that (2.6) holds. Condition (3.5) implies that $s = \frac{j-1}{n-1}$ and discontinuity of $\bar{h}_{j:n}$ at $\frac{j-1}{n-1}$. By (3.6), function (3.16) is negative for $c = \bar{h}_{j:n}(\frac{j-1}{n-1}) = h_{j:n}(\frac{j-1}{n-1})$, and so $c \in (S_{j:n}(\frac{j-1}{n-1}), h_{j:n}(\frac{j-1}{n-1}))$. The constant is determined by (3.7). Consequently, $D_p^q(j,n) = \int_0^1 |\bar{h}_{j:n}(x) - c|^q dx$ can be written as (3.9). Distribution function (3.4) with parameters $\alpha = \beta = \frac{j-1}{n-1}$ and $\gamma = c$ is derived from (3.16) combined with (3.17)–(3.18).

If either (3.6) is false, or $S_{j:n}(\frac{j-1}{n-1}) = h_{j:n}(\frac{j-1}{n-1})$, or $s > \frac{j-1}{n-1}$, then $c \in [h_{j:n}(s), h_{j:n}(1))$, and hence $c = h_{j:n}(d)$ for some $d \in [s,1)$. The new parameter, defined by (3.10), provides simpler formulae (3.11) and (3.4) with $\alpha = \beta = s$, $\gamma = h_{j:n}(d)$ for the bounds and extreme distribution than the original one.

Under condition (2.8), we can apply similar arguments. Then $\bar{h}_{j:n}$ has a single jump at $\frac{j-1}{n}$. If $c \in (\bar{h}_{j:n}(\frac{j-1}{n}-), \bar{h}_{j:n}(\frac{j-1}{n})) = (S_{j:n}(\frac{j-1}{n}), h_{j:n}(\frac{j-1}{n}))$ (cf. (3.12)), then this is determined by (3.13). Otherwise $c = \bar{h}_{j:n}(d) = h_{j:n}(d)$ for some $d \in [t,1)$. In both the cases, we obtain bounds (3.14) and (3.15), and the respective extreme marginals by elementary calculations. □

THEOREM 4. *If $s^* \geq 1/2$ (see Theorem 2 for definition), then*

$$D_\infty(j,n) = -S_{j:n}(s^*), \qquad 2 \leq j \leq n-1. \tag{3.19}$$

The bound is attained for the two-point marginal distribution on $\mu - \frac{1-s^*}{s^*}\sigma_\infty$ and $\mu + \sigma_\infty$ with probabilities s^* and $1-s^*$, respectively.

If either $s \leq 1/2$ with (2.6) or $t \leq 1/2$ with (2.8) hold, then

$$D_\infty(j,n) = -S_{j:n}(1/2), \qquad 2 \leq j \leq n-1, \tag{3.20}$$

which is attained for the symmetric two-point distribution on $\mu \mp \sigma_\infty$.

In the remaining case for which (2.8) with $\frac{j-1}{n} < 1/2 < t$ hold, we have

$$D_\infty(j,n) = (2t-1)h_{j:n}(t) - 2H_{j:n}(t). \tag{3.21}$$

This bound is attained by the three-point marginal distribution assigning probabilities $\frac{j-1}{n}$, $t - \frac{j-1}{n}$ and $1-t$ to $\mu - \sigma_\infty$, $\mu + (t - 1 + \frac{j-1}{n})\sigma_\infty/(t - \frac{j-1}{n})$ and $\mu + \sigma_\infty$, respectively.

PROOF. We here use evaluations

$$\begin{aligned}\int_0^1 [F^{-1}(x) - \mu][h_{j:n}(x) - c]dx &\leq \int_0^1 [F^{-1} - \mu][\bar{h}_{j:n}(x) - c]dx \\ &\leq \sup |F^{-1}(x) - \mu| \int_0^1 |\bar{h}_{j:n}(x) - c|dx \\ &= \|\bar{h}_{j:n} - c\|_1 \sigma_\infty.\end{aligned} \tag{3.22}$$

It is well known that for arbitrary nondecreasing $\bar{h}_{j:n}$, function $c \mapsto \|\bar{h}_{j:n} - c\|_1$ is minimized by any $c \in [\bar{h}_{j:n}(1/2-), \bar{h}_{j:n}(1/2+)]$. The latter inequality in (3.22) becomes equality iff

$$F^{-1}(x) - \mu = \operatorname{sgn}(\bar{h}_{j:n}(x) - c)\sigma_\infty \tag{3.23}$$

on the intervals where $\bar{h}_{j:n} - c$ does not vanish. This holds for the former one if the intervals where $F^{-1} - \mu$ is constant contain those of $\bar{h}_{j:n} - c$.

Assume first that $s^* \geq 1/2$, which implies that $c = \bar{h}_{j:n}(0)$, and

$$D_\infty(j,n) = \|\bar{h}_{j:n} - c\|_1 = \int_{s^*}^1 [h_{j:n}(x) - S_{j:n}(s^*)]dx$$
$$= -H_{j:n}(s^*) - (1-s^*)S_{j:n}(s^*) = -S_{j:n}(s^*).$$

Condition (3.23) implies that $F^{-1} - \mu = \sigma_\infty$ on $(s^*, 1)$. Since $F^{-1} - \mu$ is required to be constant on $(0, s^*)$ (see Lemma 1), and integrate to 1, we conclude that $F^{-1} - \mu = -\frac{1-s^*}{s^*}\sigma_\infty$ on $(0, s^*)$.

In the second case $\bar{h}_{j:n} - c$ has a single zero at 1/2, and (3.23) implies that

$$F^{-1}(x) - \mu = \begin{cases} -\sigma_\infty, & \text{if} \quad 0 < x < 1/2, \\ +\sigma_\infty, & \text{if} \quad 1/2 < x < 1. \end{cases}$$

This is clearly constant on $(0, s)$, and $(0, \frac{j-1}{n})$ and $(\frac{j-1}{n}, t)$ in cases (2.6) and (2.8), respectively. In the former,

$$D_\infty(j,n) = s[h_{j:n}(1/2) - S_{j:n}(s)] + \int_s^{1/2} [h_{j:n}(1/2) - h_{j:n}(x)]dx$$
$$+ \int_{1/2}^1 [h_{j:n}(x) - h_{j:n}(1/2)]dx = -2H_{j:n}(1/2).$$

Similar calculations in the latter one lead us to the same conclusion.

The last statement is proved as follows

$$D_\infty(j,n) = \frac{j-1}{n}\left[h_{j:n}(t) - S_{j:n}\left(\frac{j-1}{n}\right)\right] + \int_t^1 [h_{j:n}(x) - h_{j:n}(t)]dx$$
$$= \frac{j-1}{n}\left[\frac{H_{j:n}(t) - H_{j:n}\left(\frac{j-1}{n}\right)}{t - \frac{j-1}{n}} - \frac{H_{j:n}\left(\frac{j-1}{n}\right)}{\frac{j-1}{n}}\right]$$
$$-H_{j:n}(t) - (1-t)\frac{H_{j:n}(t) - H_{j:n}\left(\frac{j-1}{n}\right)}{t - \frac{j-1}{n}},$$

which is identical with (3.21). By (3.23), $F^{-1} - \mu = -\sigma_\infty$ and $+\sigma_\infty$ on $[0, \frac{j-1}{n})$ and $[t, 1]$, respectively. By Lemma 1, this is constant on $[\frac{j-1}{n}, t)$. The value is determined by the first moment condition, and belongs to $(\mu - \sigma_\infty, \mu + \sigma_\infty)$. □

In the same manner we can obtain analogous results for extreme order statistics. We only need to replace the Moriguti approximations $\bar{h}_{j:n}$, $2 \leq j \leq n-1$, by $h_{1:n}$ and (2.3). The details of proofs are left to the reader.

THEOREM 5. *For $j = 1$ and $p = 1$ we have*

$$\sup_{F \in \mathcal{F}_1} (E_F Y_{1:n} - E_F X_{1:n})/\sigma_1 = n/2 \leftarrow \sup(E_{F_k} Y_{1:n} - E_{F_k} X_{1:n})/\sigma_1, \qquad as \quad k \to \infty,$$

where F_k is the three-point distribution function with atoms at $\mu - k\sigma_1/2$, μ and $\mu + k\sigma_1/2$ and respective probabilities $1/k$, $1-2/k$ and $1/k$.

For $1 < p < \infty$, and $0 < d < 1$ uniquely defined by the equation

$$\int_0^d (d^{n-1} - x^{n-1})^{q/p} dx = \int_d^1 (x^{n-1} - d^{n-1})^{q/p} dx$$

we have

$$\sup_{F \in \mathcal{F}_p} (E_F Y_{1:n} - E_F X_{1:n})/\sigma_p = D_p(1,n)$$
$$= \int_0^d (d^{n-1} - x^{n-1})^q dx + \int_d^1 (x^{n-1} - d^{n-1})^q dx.$$

The supremum is attained by

$$F(x) = 1 - \left[d^{n-1} - \frac{D_p(1,n)}{n} \left| \frac{x-\mu}{\sigma_p} \right|^{p/q} \operatorname{sgn}(x-\mu) \right]^{1/(n-1)},$$
$$-\left[\frac{n(1-d^{n-1})}{D_p(1,n)} \right]^{q/p} \le \frac{x-\mu}{\sigma_p} \le \left[\frac{nd^{n-1}}{D_p(1,n)} \right]^{q/p}.$$

Finally, for $p = \infty$ the bound

$$\sup_{F \in \mathcal{F}_\infty} (E_F Y_{1:n} - E_F X_{1:n})/\sigma_\infty = 1 - 2^{1-n}$$

is attained by F symmetrically distributed at $\mu \mp \sigma_\infty$.

THEOREM 6. *Set*

$$d_p(n) = \left[\frac{n(n-1)^{p-1}}{(n-1)^{p-1} + 1} \right]^{1/p}, \qquad 1 \le p < \infty,$$
$$d_\infty(n) = 1.$$

Then for $1 \le p \le \infty$ yields

$$\sup_{F \in \mathcal{F}_p} (E_F Y_{n:n} - E_F X_{n:n})/\sigma_p = (1 - 1/n)^{n-1} d_p(n),$$

which is attained by the two-point marginal distribution concentrated on $\mu - d_p(n)\sigma_p/(n-1)$ and $\mu + d_p(n)\sigma_p$ with respective probabilities $1 - 1/n$ and $1/n$.

Observe that $d_p(n) \to d_\infty(n)$, as $p \to \infty$. Arnold (1985) proved that

$$\sup_{F \in \mathcal{F}_p} (E_F Y_{n:n} - \mu)/\sigma_p = d_p(n), \qquad 1 \le p \le \infty,$$

and the bound is attained for the distribution defined in Theorem 6. Therefore

$$\frac{\sup_{F \in \mathcal{F}_p} (E_F Y_{n:n} - E_F X_{n:n})}{\sup_{F \in \mathcal{F}_p} (E_F Y_{n:n} - \mu)} = (1 - 1/n)^{n-1} = \frac{1}{\xi(n-1)} \searrow e^{-1} \approx 0.3679$$

holds for arbitrary $1 \le p \le \infty$.

It is worth mentioning that our formulae are significantly simplified in the case $p = 2$. Then, referring to (3.18), we conclude that

$$c = \int_0^1 \bar{h}_{j:n}(x)dx = H_{j:n}(1) = 0,$$
$$D_2(j,n) = \|\bar{h}_{j:n}\|_2,$$
$$F^{-1}(x) - \mu = \bar{h}_{j:n}(x)\sigma_2 / D_2(j,n).$$

We specify results of Theorems 3 and 5 for this important case in Corollary 1. For the sample maximum it suffices to recall the statements of Theorem 6 with $d_2(n) = (n-1)^{1/2}$.

COROLLARY 1. *The bounds*

$$D_2(1,n) = (n-1)/(2n-1)^{1/2}$$

are attained by the marginal distribution functions

$$F(x) = 1 - \left[\frac{1}{n}\left(1 - \frac{n-1}{(2n-1)^{1/2}}\frac{x-\mu}{\sigma_2}\right)\right]^{1/(n-1)}, \quad -(2n-1)^{1/2} \le \frac{x-\mu}{\sigma_2} \le \frac{(2n-1)^{1/2}}{n-1}.$$

If (2.6) *holds for* $2 \le j \le n-1$, *then*

$$\sup_{F \in \mathcal{F}_2} (E_F Y_{j:n} - E_F X_{j:n})/\sigma_2 = D_2(j,n) \tag{3.24}$$

for

$$D_2^2(j,n) = \frac{1}{s}\left[\frac{ns+1-j}{n+1-j} - F_{j:n}(s)\right]^2 + \frac{n^2(1-s)}{(n+1-j)^2} - \frac{2n}{n+1-j}[1 - F_{j:n}(s)] + n\binom{2j-2}{j-1}\binom{2n-2j}{n-j}\binom{2n-1}{n}^{-1}[1 - F_{2j-1:2n-1}(s)]. \tag{3.25}$$

Bound (3.24) *is attained by*

$$F(x) = \begin{cases} 0, & \text{if } \dfrac{x-\mu}{\sigma_2} < \dfrac{S_{j:n}(s)}{D_2(j,n)}, \\ s, & \text{if } \dfrac{S_{j:n}(s)}{D_2(j,n)} \le \dfrac{x-\mu}{\sigma_2} \le \dfrac{h_{j:n}(s)}{D_2(j,n)}, \\ h_{j:n}^{-1}\left(D_2(j,n)\dfrac{x-\mu}{\sigma_2}\right), & \text{if } \dfrac{h_{j:n}(s)}{D_2(j,n)} \le \dfrac{x-\mu}{\sigma_2} \le \dfrac{n/(n+1-j)}{D_2(j,n)}, \\ 1, & \text{if } \dfrac{x-\mu}{\sigma_2} \ge \dfrac{n/(n+1-j)}{D_2(j,n)}. \end{cases} \tag{3.26}$$

Under conditions (2.8) *we have* (3.24) *with*

$$D_2^2(j,n) = \frac{n}{j-1}F_{j:n}^2\left(\frac{j-1}{n}\right) + \left(t - \frac{j-1}{n}\right)\left[\frac{n}{n+1-j} - f_{j:n}(t)\right]^2 + \frac{n^2(1-t)}{(n+1-j)^2} - \frac{2n}{n+1-j}[1 - F_{j:n}(t)] + n\binom{2j-2}{j-1}\binom{2n-2j}{n-j}\binom{2n-1}{n}^{-1}[1 - F_{2j-1:2n-1}(t)]. \tag{3.27}$$

Table 1. Bounds on deviations of expectations of order statistics of samples of size $n = 20$ under violation of independence.

j	s/t^*	$D_1(j,20)$	$D_2(j,20)$	$D_\infty(j,20)$
1	—	10.0	3.04243	0.999998
2	0.081632	3.34901	2.12540	1.05259
3	0.154426	2.35514	1.80169	1.11071
4	0.220794	1.92629	1.62166	1.17389
5	0.282865	1.69323	1.50515	1.23818
6	0.341654	1.55523	1.42409	1.29194
7	0.397736	1.47326	1.36564	1.31325
8	0.451461	1.42942	1.32301	1.27529
9	0.503044	1.41503	1.29228	1.16338
10	0.552604	1.42632	1.27102	1.03446
11	0.600174	1.46278	1.25775	0.92556
12	0.645684	1.52674	1.25148	0.83126
13	0.688884	1.62382	1.25162	0.74763
14	0.729075	1.76447	1.25770	0.67180
15	0.763548	1.96757	1.26921	0.60180
16	0.808151*	2.27656	1.28635	0.55312
17	0.861456*	2.75716	1.31285	0.51431
18	0.914198*	3.57151	1.35206	0.47635
19	0.965194*	5.21764	1.41421	0.43527
20	—	3.77354	1.64485	0.37735

The supremum is attained by (3.26) *with* s, $S_{j:n}(s)$ *and* $h_{j:n}(s)$ *replaced by* $\frac{j-1}{n}$, $S_{j:n}(\frac{j-1}{n})$ *and* $h_{j:n}(t)$, *respectively.*

Table 1 contains numerical values of bounds $D_p(j,20)$, $p = 1,2,\infty$, $1 \le j \le 20$ for samples of size $n = 20$. We recall that $D_p(j,n) = D_p^-(n+1-j,n)$ are both the upper and lower bounds for deviations of the j-th smallest and largest order statistics, respectively. Parameters $p = 1,2,\infty$ refer to the most popular scale units: the mean absolute, standard and maximal absolute deviations, respectively. For $2 \le j \le 19$, in the second column we provide values of parameters s and t which allow us to determine the Moriguti approximations $\bar{h}_{j:20}$, bounds $D_p(j,20)$, and distributions attaining them, when conditions (2.6) and (2.8) are satisfied, respectively (see Theorem 1). The former holds for $2 \le j \le 15$, and then numerical values of parameter s are presented. Accordingly, $\bar{h}_{j:20}$ is defined by (2.7), and then we calculate $D_1(j,20)$ and $D_2(j,20)$ using (3.1) with $s^* = s$ and (3.25), respectively. Conditions $h_{j:n}(\frac{j-1}{n-1}) < 0$ with $S_{j:n}(s) > S_{j:n}(\frac{j-1}{n})$ of (2.8) are satisfied for $j = 16$, and $h_{j:n}(\frac{j-1}{n-1}) \ge 0$ holds for $17 \le j \le 19$. Therefore $\bar{h}_{j:20}$, $16 \le j \le 19$, have form (2.9), dependent on parameter t, whose numerical values are marked with asterisks. Then we use (3.1) with $s^* = \frac{j-1}{n}$ and (3.27) for determining values of $D_p(j,20)$, $p = 1,2$, respectively. In the last column, we evaluate (3.20) and (3.19) for $2 \le j \le 8$ and $9 \le j \le 19$, respectively.

We see that the central order statistics are more stable under departures from independence than the extreme ones when measured in terms of the first two central absolute moments. The upward inclination of the small (large) order statistics is greater (smaller) than the downward one. The opposite tendencies can be observed for $p = \infty$. Relations $D_1(j,n) > D_2(j,n) > D_\infty(j,n)$ are certainly forced by the reversed ones for the respective gauge units σ_p. Applying Theorems 4, 5, and Corollary 1, we can immediately conclude that asymptotic deviations for sample extremes $D_p(j,n)$, $j = 1, n$, $n \to \infty$, have orders $\mathcal{O}(n)$, $\mathcal{O}(n^{1/2})$, and $\mathcal{O}(1)$ for $p = 1, 2, \infty$, respectively. More sophisticated arguments are needed for showing that all the $D_p(j,n)$ remain asymptotically bounded for the central order statistics.

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