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THE HAZARD RATE AND THE REVERSED HAZARD RATE ORDERS, WITH APPLICATIONS TO ORDER STATISTICS

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Abstract. In this paper we first point out a simple observation that can be used successfully in order to translate results about the hazard rate order into results about the reversed hazard rate order. Using it, we derive some interesting new results which compare order statistics in the hazard and in the reversed hazard rate orders; as well as in the usual stochastic order. We also simplify proofs of some known results involving the reversed hazard rate order. Finally, a few further applications of the observation are given.

Key words and phrases: Stochastic orders, k-out-of-n systems.

1. Introduction

The hazard rate order is a well-known and useful tool in reliability theory and in other areas of applied probability and statistics. A little less well known is the reversed hazard rate order. The latter has received less attention, mainly because it has a somewhat less obvious intuitive meaning in reliability theory. However, there are quite a few instances in applications where the reversed hazard rate order appears genuinely as a natural condition which implies useful inequalities, and which leads to optimal strategies. For example, Kijima (1998) has shown that certain first-passage times, associated with some continuous time Markov chains, have increasing failure rates *if the underlying chain is monotone in the sense of the reversed hazard rate order*; see also Shaked and Shanthikumar (1988). Shanthikumar *et al.* (1991) have shown that *if the service times of servers in a tandem queue with blocking are comparable in the reversed hazard rate order*, then there exists an optimal allocation where the server allocated to the first stage has a larger mean service time than that assigned to the second server. Cheng and Zhu (1993) and Cheng and Righter (1995) have extended the results of Shanthikumar *et al.* (1991) to more versatile queueing systems, and obtained further optimality results, still *under the condition* (which arises naturally) *that the service times of different servers are ordered according to the reversed hazard rate order*. Shaked *et al.* (1995) showed that *if a family of random variables is increasing in the reversed hazard rate order* then a corresponding family of partial sums has a useful stochastic transposition increasing property. Shaked and Wong (1997*a*, 1997*b*) noticed that *the reversed hazard rate order implies* the Laplace transform ratio order, which in turn, yields some useful inequalities. They also showed that comparability of some random counters in the reversed hazard rate order implies the hazard rate ordering of some related random sums and random minima and maxima. Finally, it is worthwhile to mention that the reversed hazard rate order arises naturally also in economics and risk theory. Eeckhoudt and Gollier (1995) considered an agent that has an increasing concave utility function, and a choice among several random risks. If the agent maximizes his expected utility by selecting an optimal level of exposure to a specified risk, then this optimal level of exposure decreases *when the random risks are increasing in the reversed hazard rate order* (but, interestingly enough, not necessarily when they are merely increasing according to the usual (first-order) stochastic dominance); further results in this vein can be found in Kijima and Ohnishi (1999).

It is thus seen that mathematical results which constitute the reversed hazard rate ordering of pairs of random variables can be very useful. Therefore it is not surprising that recently several researchers have devoted some effort in order to obtain results of this kind. However, in some recent papers, the proofs that have been utilized in order to establish the reversed hazard rate ordering of some pairs of random variables are quite complicated; see, for instance, Block *et al.* (1998), Nanda *et al.* (1998), Ma (1999), and Hu and He (2000). The purpose of the present paper is to point out a simple observation that can be used successfully in order to translate results about the hazard rate order into results about the reversed hazard rate order. Thus, the effort that has been made in some recent instances in the literature, in establishing the reversed hazard rate order, is often unnecessary once an analogous result exists for the hazard rate order. The above observation is also useful in deriving new comparisons of random variables in the hazard and the reversed hazard rate orders.

The paper is organized as follows. The observation, mentioned above, is stated and proven in Section 2 as Theorem 2.1. In Section 3 we derive some interesting new results which compare order statistics in the hazard and in the reversed hazard rate orders; as well as in the usual stochastic order. We also give there a short overview of the current literature on comparisons of order statistics, and, using Theorem 2.1, we point out simple proofs of many known results. It should be mentioned that order statistics have a useful interpretation in reliability theory as the lifetimes of k-out-of-n systems, and in nonparametric statistics as the values that determine the empirical distribution function. Finally, a few further applications of Theorem 2.1 are described in Section 4.

The definitions of the hazard rate and the reversed hazard rate orders, in a general setting (that is, without necessarily assuming that the compared random variables are nonnegative), are the following.

Let X and Y be two continuous (not necessarily nonnegative) random variables, each with an interval support which we denote by (l_X, u_X) and (l_Y, u_Y), respectively; l_X and l_Y may be $-\infty$, and u_X and u_Y may be ∞. Let F and G be the distribution functions of X and Y, respectively, and let $\bar{F} \equiv 1-F$ and $\bar{G} \equiv 1-G$ be the corresponding survival functions.

DEFINITION 1.1. The random variable X is said to be smaller than the random variable Y in the hazard rate order (denoted as $X \leq_{\mathrm{hr}} Y$) if

$$\frac{\bar{G}(t)}{\bar{F}(t)} \quad \text{is increasing in } t \in (-\infty, \max(u_X, u_Y)). \tag{1.1}$$

Note that in (1.1), when $u_X < u_Y$, we use the convention $a/0 = \infty$ when $a > 0$. In particular, it is seen that if $u_X < l_Y$, then $X \leq_{\mathrm{hr}} Y$. From the definition of $\leq_{\mathrm{hr}}$ it follows that

(1.2) $$X \leq_{\mathrm{hr}} Y \Rightarrow \big(l_X \leq l_Y \text{ and } u_X \leq u_Y\big).$$

The hazard rate order is useful in many areas of probability and statistics; in particular, in reliability theory. Various results that establish the hazard rate ordering of various pairs of random variables can be found in the literature (see, for example, Section 1.B in Shaked and Shanthikumar (1994)). Often it is assumed that the random variables X and Y that are compared are nonnegative, but using the definition (1.1) this is not necessary.

DEFINITION 1.2. The random variable X is said to be smaller than the random variable Y in the reversed hazard rate order (denoted as $X \leq_{\mathrm{rh}} Y$) if

$$\frac{G(t)}{F(t)} \quad \text{is increasing in } t \in (\min(l_X, l_Y), \infty).$$

In particular, it is seen that if $u_X < l_Y$, then $X \leq_{\mathrm{rh}} Y$. From the definition of $\leq_{\mathrm{rh}}$ it follows that if $X \leq_{\mathrm{rh}} Y$ then $l_X \leq l_Y$ and $u_X \leq u_Y$. The reversed hazard rate order also has recently received a lot of attention (again, see, for example, Section 1.B in Shaked and Shanthikumar (1994)).

2. An observation

A simple useful observation is the following result.

THEOREM 2.1. *Let X and Y be two continuous random variables with supports (l_X, u_X) and (l_Y, u_Y), respectively. Then*

$$X \leq_{\mathrm{hr}} Y \Rightarrow \phi(X) \geq_{\mathrm{rh}} \phi(Y)$$

for any continuous function ϕ which is strictly decreasing on (l_X, u_Y). Also,

$$X \leq_{\mathrm{rh}} Y \Rightarrow \phi(X) \geq_{\mathrm{hr}} \phi(Y)$$

for any such function ϕ.

PROOF. Let ϕ be a continuous strictly decreasing function on (l_X, u_Y). Denote its range by (l_ϕ, u_ϕ). Let $\bar{F}$ and $\bar{G}$ be the survival functions of X and Y, respectively, and denote the distribution functions of $\phi(X)$ and $\phi(Y)$ by F_ϕ and G_ϕ, respectively. Since ϕ is continuous and strictly decreasing, its inverse, ϕ^{-1}, is uniquely defined on (l_ϕ, u_ϕ). For $t \in (l_\phi, u_\phi)$ we have

$$\frac{F_\phi(t)}{G_\phi(t)} = \frac{\bar{F}(\phi^{-1}(t))}{\bar{G}(\phi^{-1}(t))}.$$

If $X \leq_{\mathrm{hr}} Y$ then $\bar{F}/\bar{G}$ is decreasing on (l_X, u_Y). Also, the decreasingness of ϕ implies that ϕ^{-1} is decreasing on (l_ϕ, u_ϕ). Therefore $\frac{F_\phi(t)}{G_\phi(t)}$ is increasing in $t \in (l_\phi, u_\phi)$; that is, $\phi(Y) \leq_{\mathrm{rh}} \phi(X)$.

The proof of the second part of the theorem is similar. □

Sengupta *et al.* (1999) have used a special case of Theorem 2.1 (that for nonnegative random variables X and Y we have $X \leq_{\mathrm{rh}} Y \Leftrightarrow X^{-1} \geq_{\mathrm{hr}} Y^{-1}$) in some derivations of their results. Kijima and Ohnishi ((1999), p. 358) have pointed out another special case of Theorem 2.1; that is, that for any random variables X and Y we have $X \leq_{\rm rh} Y \Leftrightarrow -X \geq_{\rm hr} -Y$.

Theorem 2.1 highlights a mistake in Theorems 1.B.2 and 1.B.22 in Shaked and Shanthikumar (1994)—the parenthetical statements there are incorrect. Theorem 2.1 straightens out these errors. For completeness we state here the correct results which are slightly more general than those stated in Shaked and Shanthikumar (1994) (because here we do not require the random variables to be nonnegative); these results will be used in the sequel. The proof of the following results is similar to the proof of Theorem 2.1 and is therefore omitted.

THEOREM 2.2. *Let X and Y be two continuous random variables with supports (l_X, u_X) and (l_Y, u_Y), respectively. Then*

$$X \leq_{\rm hr} Y \Rightarrow \phi(X) \leq_{\rm hr} \phi(Y)$$

for any continuous function ϕ which is strictly increasing on (l_X, u_Y). Also,

$$X \leq_{\rm rh} Y \Rightarrow \phi(X) \leq_{\rm rh} \phi(Y)$$

for any such function ϕ.

A referee has pointed out that the statements in Theorems 2.1 and 2.2 hold also in the reversed direction. For example, in Theorem 2.1, if $\phi(X) \geq_{\rm rh} \phi(Y)$ for some continuous function ϕ which is strictly decreasing then $X \leq_{\rm hr} Y$. This can be seen by applying the second part of Theorem 2.1 with the function ϕ^{-1} (which is also continuous and strictly decreasing).

3. Ordering order statistics

A particular set of results in the literature, which establish the hazard or the reversed hazard rate ordering, involves order statistics; these results have useful applications in reliability theory—see, for example, the references that are mentioned throughout this section. We describe below some new results, and we also provide, with the aid of Theorem 2.1, simple proofs to some known results.

The notation that we use in this section is the following. For any set $\{X_1, X_2, \ldots, X_m\}$ [$\{Y_1, Y_2, \ldots, Y_n\}$, etc.] of random variables, the corresponding order statistics will be denoted by $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ [$Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$, etc.]. When we want to indicate the sample size in the notation, we denote the order statistics by $X_{(i:m)}$, $1 \leq i \leq m$ [$Y_{(j:n)}$, $1 \leq j \leq n$, etc.].

Khaledi and Kochar (1999) have proven recently that if $X_1, X_2, \ldots, X_m$ [respectively, $Y_1, Y_2, \ldots, Y_n$] are independent and identically distributed (i.i.d.), then for any m and n we have that

$$\text{(3.1)} \qquad X_1 \leq_{\rm st} Y_1 \Rightarrow X_{(i:m)} \leq_{\rm st} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \text{ and } m - i \geq n - j,$$

where $\leq_{\rm st}$ denotes the usual stochastic order (see, for example, Shaked and Shanthikumar (1994)). Lillo *et al.* (2001) have proven, among other things, that in this (i.i.d.) case, for any m and n, we have that

$$\text{(3.2)} \qquad X_1 \leq_{\rm lr} Y_1 \Rightarrow X_{(i:m)} \leq_{\rm lr} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \text{ and } m - i \geq n - j,$$

where $\leq_{\rm lr}$ denotes the likelihood ratio stochastic order (again, see, for example, Shaked and Shanthikumar (1994)). The next two results show that the above implications hold also for the hazard rate and the reversed hazard rate orders. The first result extends a result of Singh and Vijayasree (1991), which is also given as Theorem 1.B.4 in Shaked and Shanthikumar (1994).

THEOREM 3.1. *Let $X_1, X_2, \ldots, X_m$ [respectively, $Y_1, Y_2, \ldots, Y_n$] be i.i.d. absolutely continuous random variables with support (l_X, u_X) [respectively, (l_Y, u_Y)]. Then*

$$X_1 \leq_{\rm hr} Y_1 \Rightarrow X_{(i:m)} \leq_{\rm hr} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \text{ and } m - i \geq n - j.$$

PROOF. The supports of $X_{(i:m)}$ and $Y_{(j:n)}$ are (l_X, u_X) and (l_Y, u_Y), respectively. From (1.2) it follows that $l_X \leq l_Y$ and $u_X \leq u_Y$. If $u_X \leq l_Y$ then the stated result is obvious. Thus, let us assume that $l_Y < u_X$. Denote by f, F, $\bar{F}$, and $r_X \equiv f/\bar{F}$, the density, distribution, survival, and hazard rate functions of X_1, respectively. Similarly, denote by g, G, $\bar{G}$, and $r_Y \equiv g/\bar{G}$, the density, distribution, survival, and hazard rate functions of Y_1, respectively. The condition $X_1 \leq_{\rm hr} Y_1$ is equivalent to

$$r_X(t) \geq r_Y(t), \qquad t \in (l_Y, u_X). \tag{3.3}$$

Let us now compute the hazard rate function $r_{X_{(i:m)}}$ of $X_{(i:m)}$. For $t \in (l_X, u_X)$ we have

$$\begin{aligned} r_{X_{(i:m)}}(t) &= \frac{F^{i-1}(t)\bar{F}^{m-i+1}(t) r_X(t)}{\int_t^\infty F^{i-1}(s) f(s) \bar{F}^{m-i}(s) ds} \\ &= \frac{F^{i-1}(t)\bar{F}^{m-i+1}(t)}{\int_{F(t)}^1 u^{i-1}(1-u)^{m-i} du} \cdot r_X(t) \\ &= \left[\int_{F(t)}^1 \left(\frac{u}{F(t)}\right)^{i-1} \left(\frac{1-u}{1-F(t)}\right)^{m-i} \frac{du}{\bar{F}(t)}\right]^{-1} r_X(t) \\ &= \left[\int_0^1 \left(\frac{1-v\bar{F}(t)}{F(t)}\right)^{i-1} v^{m-i} dv\right]^{-1} r_X(t). \end{aligned}$$

Similarly, for $t \in (l_Y, u_Y)$ we have

$$r_{Y_{(j:n)}}(t) = \left[\int_0^1 \left(\frac{1-v\bar{G}(t)}{G(t)}\right)^{j-1} v^{n-j} dv\right]^{-1} r_Y(t).$$

Now, since $v \leq 1$ and $m - i \geq n - j$ we have $v^{m-i} \leq v^{n-j}$. Also, $X_1 \leq_{\rm hr} Y_1$ implies that $\bar{F}(t) \leq \bar{G}(t)$ for $t \in (l_Y, u_X)$. Therefore

$$\left(\frac{1-v\bar{F}(t)}{F(t)}\right)^{i-1} \leq \left(\frac{1-v\bar{G}(t)}{G(t)}\right)^{i-1} \leq \left(\frac{1-v\bar{G}(t)}{G(t)}\right)^{j-1},$$

where the second inequality follows from $i \leq j$ and the easily verified fact that $(1 - v\bar{G}(t))/G(t) \geq 1$. Combining these inequalities with (3.3) we obtain $r_{X_{(i:m)}}(t) \geq r_{Y_{(j:n)}}(t)$ for $t \in (l_Y, u_X)$; that is, $X_{(i:m)} \leq_{\rm hr} Y_{(j:n)}$. □

Using Theorems 2.1 and 3.1 we obtain the following result.

THEOREM 3.2. *Let $\{X_1, X_2, \ldots, X_m\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ be two sets of i.i.d. random variables as in Theorem* 3.1. *Then*

$$X_1 \leq_{\rm rh} Y_1 \Rightarrow X_{(i:m)} \leq_{\rm rh} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \text{ and } m-i \geq n-j. \tag{3.4}$$

PROOF. Note that from $X_1 \leq_{\rm rh} Y_1$ it follows that $l_X \leq l_Y$ and $u_X \leq u_Y$. Let ϕ be some strictly decreasing continuous function defined on (l_X, u_Y), and denote $W_i = \phi(X_i)$, $i = 1, 2, \ldots, m$, and $Z_j = \phi(Y_j)$, $j = 1, 2, \ldots, n$. If $X_1 \leq_{\rm rh} Y_1$ then, by Theorem 2.1, $Z_1 \leq_{\rm hr} W_1$. Therefore, by Theorem 3.1,

$$Z_{(j:n)} \leq_{\rm hr} W_{(i:m)} \quad \text{whenever} \quad j \leq i \text{ and } n-j \geq m-i.$$

Note that ϕ^{-1} is also a strictly decreasing continuous function. Using Theorem 2.1 again, we get

$$\phi^{-1}(W_{(i:m)}) \leq_{\rm rh} \phi^{-1}(Z_{(j:n)}) \quad \text{whenever} \quad j \leq i \text{ and } n-j \geq m-i.$$

Since $\phi^{-1}(W_{(i:m)}) =_{\rm st} X_{(m-i+1:m)}$, $i = 1, 2, \ldots, m$, and $\phi^{-1}(Z_{(j:n)}) =_{\rm st} Y_{(n-j+1:n)}$, $j = 1, 2, \ldots, n$, we thus have

$$X_{(m-i+1:m)} \leq_{\rm rh} Y_{(n-j+1:n)} \quad \text{whenever} \quad j \leq i \text{ and } n-j \geq m-i,$$

which is easily seen to be equivalent to the right hand side of (3.4). □

In the next four theorems we give a short up-to-date review of the literature on comparisons of order statistics from non-i.i.d. samples in the hazard and the reversed hazard rate orders. The purpose of this short review is two-fold. (a) Most of the results mentioned in it will be used later in the section. (b) We also point out in passing proofs, involving the reversed hazard order, that are much simpler than the original proofs in the literature; and, using Theorem 2.2, we slightly extend some known results.

THEOREM 3.3. *Let $X_1, X_2, \ldots, X_m$ be independent (not necessarily i.i.d.) absolutely continuous random variables, all with support (a, b) for some $a < b$. Then*

$$X_{(k:m)} \leq_{\rm hr} X_{(k+1:m)}, \quad k = 1, 2, \ldots, m-1, \tag{3.5}$$
$$X_{(k:m)} \leq_{\rm rh} X_{(k+1:m)}, \quad k = 1, 2, \ldots, m-1. \tag{3.6}$$

Remarks on Theorem 3.3. Boland *et al.* (1994) proved (3.5) for nonnegative random variables; however, by Theorem 2.2, the inequality (3.5) is valid under the weaker assumptions of Theorem 3.3. The result (3.6) is Theorem 4.1 of Block *et al.* (1998); in that paper it has a lengthy proof, however, using Theorem 2.1 and the ideas in the proof of Theorem 3.2 it is seen that (3.6) is actually equivalent to (3.5).

THEOREM 3.4. *Let $X_1, X_2, \ldots, X_m$ be independent (not necessarily i.i.d.) absolutely continuous random variables, all with support (a, b) for some $a < b$. Then*

$$X_{(k:m)} \leq_{\rm hr} X_{(k:m-1)}, \quad k = 1, 2, \ldots, m-1, \tag{3.7}$$
$$X_{(k:m-1)} \leq_{\rm rh} X_{(k+1:m)}, \quad k = 1, 2, \ldots, m-1. \tag{3.8}$$

Remarks on Theorem 3.4. Both (3.7) and (3.8) were proved in Hu and He (2000). They provided quite lengthy proofs to each of these. However, again, using Theorem 2.1 and the ideas in the proof of Theorem 3.2 it is seen that (3.8) is actually equivalent to (3.7). Thus, the lengthy proof of (3.8) in Hu and He (2000) is unnecessary.

THEOREM 3.5. *Let $X_1, X_2, \ldots, X_m$ be independent (not necessarily i.i.d.) absolutely continuous random variables, all with support (a, b) for some $a < b$.*

(i) *If $X_i \le_{\rm hr} X_m$, $i = 1, 2, \ldots, m-1$, then $X_{(k:m-1)} \le_{\rm hr} X_{(k+1:m)}$, $k = 1, 2, \ldots, m-1$.*

(ii) *If $X_m \le_{\rm rh} X_i$, $i = 1, 2, \ldots, m-1$, then $X_{(k:m)} \le_{\rm rh} X_{(k:m-1)}$, $k = 1, 2, \ldots, m-1$.*

Remarks on Theorem 3.5. Boland *et al.* (1994) proved part (i) for nonnegative random variables; however, by Theorem 2.2, the inequality in part (i) is valid under the weaker assumptions of Theorem 3.5. Part (ii) is Theorem 4.2 of Block *et al.* (1998); however, using once more Theorem 2.1 and the ideas in the proof of Theorem 3.2 it is seen that part (ii) is actually equivalent to part (i).

In analogy to (3.7), one may wonder whether the condition $X_i \le_{\rm hr} X_m$ in Theorem 3.5(i) can be disposed of. To see that this is not the case, let X_i, $i = 1, 2, 3$, be independent exponential random variables with rates $\lambda_i = i$, $i = 1, 2, 3$. Then it is easy to verify that $X_{(2:2)}$ is not smaller than $X_{(3:3)}$ in the hazard rate order. Similarly, the condition $X_m \le_{\rm rh} X_i$ in Theorem 3.5(ii) cannot be disposed of.

THEOREM 3.6. *Let $X_1, X_2, \ldots, X_m$ [respectively, $Y_1, Y_2, \ldots, Y_m$] be m independent (not necessarily i.i.d.) absolutely continuous random variables, all with support (a, b) for some $a < b$.*

(i) *If $X_i \le_{\rm hr} Y_j$ for all i and j, then $X_{(k:m)} \le_{\rm hr} Y_{(k:m)}$, $k = 1, 2, \ldots, m$.*

(ii) *If $X_i \le_{\rm rh} Y_j$ for all i and j, then $X_{(k:m)} \le_{\rm rh} Y_{(k:m)}$, $k = 1, 2, \ldots, m$.*

Remarks on Theorem 3.6. Boland and Proschan (1994) proved part (i) for nonnegative random variables; however, by Theorem 2.2, the inequality in part (i) is valid under the weaker assumptions of Theorem 3.6. Part (ii) strengthens Corollary 3.1 of Nanda *et al.* (1998). Note again that by using Theorem 2.1 and the ideas in the proof of Theorem 3.2 it is seen that part (ii) is actually equivalent to part (i).

We now proceed to derive some additional new results. To motivate the first result, we recall that Lillo *et al.* (2001) have extended (3.2) as follows. Let $X_1, X_2, \ldots, X_m$ [respectively, $Y_1, Y_2, \ldots, Y_n$] be independent (not necessarily i.i.d.) random variables. Then, for any m and n we have that

$$X_i \le_{\rm lr} Y_j \quad \text{for all} \quad i, j \Rightarrow X_{(i:m)} \le_{\rm lr} Y_{(j:n)} \text{ whenever } i \le j,\ m - i \ge n - j.$$

The next result shows that in the above, the order $\le_{\rm st}$ can replace the order $\le_{\rm lr}$. Of course, this next result also strengthens the result of Khaledi and Kochar (1999) given as (3.1) earlier in this section.

THEOREM 3.7. *Let $X_1, X_2, \ldots, X_m$ [respectively, $Y_1, Y_2, \ldots, Y_n$] be independent (not necessarily i.i.d.) absolutely continuous random variables, all with support (a, b) for some $a < b$. Then for any m and n we have that*

$$X_i \le_{\rm st} Y_i \quad \textit{for all} \quad i \Rightarrow X_{(i:m)} \le_{\rm st} Y_{(j:n)} \textit{ whenever } i \le j,\ m - i \ge n - j. \tag{3.9}$$

PROOF. First suppose that $m \le n$. Then

$$\begin{aligned} X_{(i:m)} &\le_{\text{rh}} X_{(m-n+j:m)} && \text{(by (3.6) and } i \le m-n+j) \\ &\le_{\text{st}} Y_{(m-n+j:m)} && \text{(since } X_i \le_{\text{st}} Y_i \text{ for all } i) \\ &\le_{\text{rh}} Y_{(j:n)} && \text{(by (3.8) and } m \le n). \end{aligned}$$

Since the order $\le_{\text{rh}}$ implies the order $\le_{\text{st}}$ we obtain (3.9) when $m \le n$.

Next suppose that $m \ge n$. Then

$$\begin{aligned} X_{(i:m)} &\le_{\text{hr}} X_{(i:n)} && \text{(by (3.7) and } m \ge n) \\ &\le_{\text{st}} Y_{(i:n)} && \text{(since } X_i \le_{\text{st}} Y_i \text{ for all } i) \\ &\le_{\text{hr}} Y_{(j:n)} && \text{(by (3.5) and } i \le j). \end{aligned}$$

Since the order $\le_{\text{hr}}$ implies the order $\le_{\text{st}}$ we obtain (3.9) when $m \ge n$. □

We have not been able to obtain a complete analog of (3.9) for the hazard and the reversed hazard orders. However, some partial results in this direction are given in the following two theorems. For example, the conclusions in the first theorem below are not complete analogs of the conclusion in (3.9) because each of the conclusions $X_{(i:m)} \le_{\text{hr}} Y_{(j:n)}$ and $X_{(i:m)} \le_{\text{rh}} Y_{(j:n)}$ holds for fewer choices of i, m, j and n than in (3.9).

THEOREM 3.8. *Let $X_1, X_2, \ldots, X_m$ [respectively, $Y_1, Y_2, \ldots, Y_n$] be independent (not necessarily i.i.d.) random variables as in Theorem 3.7. Then*

(3.10) $X_i \le_{\text{hr}} Y_j$ *for all* $i, j \Rightarrow X_{(i:m)} \le_{\text{hr}} Y_{(j:n)}$ *whenever* $i \le j$, $m \ge n$,

and

(3.11) $X_i \le_{\text{rh}} Y_j$ *for all* $i, j \Rightarrow X_{(i:m)} \le_{\text{rh}} Y_{(j:n)}$ *whenever* $m - i \ge n - j$, $m \le n$.

PROOF. The proof uses the ideas of the proof of Theorem 3.7. In order to prove the first statement we see that

$$\begin{aligned} X_{(i:m)} &\le_{\text{hr}} X_{(i:n)} && \text{(by (3.7) and } m \ge n) \\ &\le_{\text{hr}} Y_{(i:n)} && \text{(by Theorem 3.6(i))} \\ &\le_{\text{hr}} Y_{(j:n)} && \text{(by (3.5) and } i \le j). \end{aligned}$$

The proof of the second statement is similar to the first part of the proof of Theorem 3.7, using, respectively, (3.6), Theorem 3.6(ii), and (3.8). □

Using Theorem 3.5 we obtain the following results. We omit the proofs.

THEOREM 3.9. *Let $X_1, X_2, \ldots, X_m$ [respectively, $Y_1, Y_2, \ldots, Y_n$] be independent (not necessarily i.i.d.) random variables as in Theorem 3.7. Then for all m and n we have*

(3.12) $X_i \le_{\text{hr}} Y_j \le_{\text{hr}} Y_1$ *for all* $i, j \Rightarrow X_{(i:m)} \le_{\text{hr}} Y_{(j:n)}$ *whenever* $i \le j$, $m - i \ge n - j$,

and

(3.13) $X_1 \le_{\text{rh}} X_i \le_{\text{rh}} Y_j$ *for all* $i, j \Rightarrow X_{(i:m)} \le_{\text{rh}} Y_{(j:n)}$ *whenever* $i \le j$, $m - i \ge n - j$.

Comparing Theorems 3.8 and 3.9 it is seen that whereas the assumption in (3.12) [respectively, (3.13)] is stronger than the assumption in (3.10) [respectively, (3.11)], the conclusion $X_{(i:m)} \leq_{\rm hr} Y_{(j:n)}$ [respectively, $X_{(i:m)} \leq_{\rm rh} Y_{(j:n)}$] in (3.12) [respectively, (3.13)] holds for more choices of i, m, j and n than in (3.10) [respectively, (3.11)].

To end this section we mention that using Theorem 2.1, some results of Kochar and Kirmani (1995) and of Khaledi and Kochar (1999) which compare normalized spacings involving the hazard rate order, can be recast as new results that give comparisons involving the reversed hazard rate order. We will not detail these results here.

4. Some other applications

In this section we describe a few further instances in which, using Theorems 2.1 and 2.2, we provide simple proofs for known results which in the literature have complicated or incomplete proofs. We also derive some new results.

In Shaked and Shanthikumar ((1994), Subsection 1.B.6) some results about the hazard rate order have been translated into results about the reversed hazard rate order. Shaked and Shanthikumar noted the special case of Theorem 2.1 that if $X < a$ and $Y < a$ a.s. for some finite a, then $X \leq_{\rm rh} Y$ if, and only if, $a - X \geq_{\rm hr} a - Y$, and they used this equivalence as the main tool for their translations. However, they encountered some technical difficulties in situations in which X and Y were not bounded from below or from above. With the use of Theorem 2.1 these difficulties disappear.

As a first simple example, consider the following result which is stated in Shaked and Shanthikumar (1994) (without an explicit proof) as Theorem 1.B.25. Using Theorem 2.1 we will give here a simple formal proof of it. In the following theorem and proof we use the convention $\log 0 = -\infty$.

THEOREM 4.1. *Let (X_i, Y_i), $i = 1, 2, \ldots, m$, be independent pairs of random variables such that $X_i \leq_{\rm rh} Y_i$, $i = 1, 2, \ldots, m$. If X_i, Y_i, $i = 1, 2, \ldots, m$, all have decreasing reversed hazard rate (that is, have logconcave distribution functions over $(-\infty, \infty)$), then*

$$\sum_{i=1}^{m} X_i \leq_{\rm rh} \sum_{i=1}^{m} Y_i.$$

PROOF. It is well known (see, for example, Theorem 1.B.6 in Shaked and Shanthikumar (1994)) that if (Z_i, W_i), $i = 1, 2, \ldots, m$, are independent pairs of random variables such that $Z_i \leq_{\rm hr} W_i$, $i = 1, 2, \ldots, m$, and if Z_i, W_i, $i = 1, 2, \ldots, m$, all have increasing hazard rate (that is, have logconcave survival functions over $(-\infty, \infty)$), then $\sum_{i=1}^{m} Z_i \leq_{\rm hr} \sum_{i=1}^{m} W_i$.

Now, since X_i and Y_i have logconcave distribution functions, it follows that $-X_i$ and $-Y_i$ have logconcave survival functions. From Theorem 2.1 we have that $-X_i \geq_{\rm hr} -Y_i$. Therefore, by the above result (that is, by Theorem 1.B.6 in Shaked and Shanthikumar (1994)) we have that $-\sum_{i=1}^{m} X_i \geq_{\rm hr} -\sum_{i=1}^{m} Y_i$, and, using Theorem 2.1 again, we get the stated inequality. □

The transformations described in Theorems 2.1 and 2.2 can be used to derive new results from known results, when the known results hold for random variables with specific supports. For example, using Theorem 2.2, a result which establishes a comparison of two nonnegative random variables in the hazard rate order, can be translated into a similar result for two random variables with support $(-\infty,\infty)$, by using a strictly increasing $\phi:(0,\infty)\to(-\infty,\infty)$ (for example, $\phi(t)=\log t$, $t>0$). Similarly, a result which compares nonnegative random variables in the hazard rate order, can be translated into a result which compares nonnegative random variables in the reversed hazard rate order, by using a strictly decreasing $\phi:(0,\infty)\to(0,\infty)$ (for example, $\phi(t)=1/t$, $t>0$).

As an example of the use of the latter transformation, let us recall some results of Ma (1999). Let $X_1,X_2,\ldots,X_n$, Λ_1 and Λ_2 be independent nonnegative random variables. Define

$$T_{j,i}=\frac{X_i}{\Lambda_j},\qquad i=1,2,\ldots,n,\quad j=1,2. \tag{4.1}$$

For each $j=1,2$, Ma (1999) interpreted the $T_{j,i}$'s as the lifetimes of n components which perform under a random environment (or stress) Λ_j which affects the scale of the lifetimes of the components. Let

$$N_j(t)=\sum_{i=1}^{n}I_{T_{j,i}}(t),\qquad j=1,2,\quad t\ge 0,$$

where $I_{T_{j,i}}(t)=1$ if $T_{j,i}>t$, and $I_{T_{j,i}}(t)=0$ otherwise. That is, $N_j(t)$ denote the number of components that are still functioning at time t in each of the systems $j=1,2$. Ma (1999) proved

$$\Lambda_1\le_{\rm rh}\Lambda_2\Rightarrow N_1(t)\ge_{\rm hr}N_2(t),\qquad t\ge 0, \tag{4.2}$$

and

$$\Lambda_1\le_{\rm hr}\Lambda_2\Rightarrow N_1(t)\ge_{\rm rh}N_2(t),\qquad t\ge 0. \tag{4.3}$$

He provided detailed proofs for both (4.2) and (4.3). However, using Theorem 2.1 it is seen that (4.2) and (4.3) are, in fact, equivalent statements (in fact, for the validity of this claim we need to use, in addition to Theorem 2.1, a version of that theorem, involving discrete random variables, which we have not stated and proved in the present paper—such a version can be obtained by a straightforward modification of Theorem 2.1 and its proof). Thus it is sufficient to provide a detailed proof for only one of these two statements.

Model (4.1) may be modified to the model

$$S_{j,i}=\Theta_j X_i,\qquad i=1,2,\ldots,n,\quad j=1,2, \tag{4.4}$$

where $X_1,X_2,\ldots,X_n$, Θ_1 and Θ_2 are independent nonnegative random variables. Model (4.4) may be more intuitive than Model (4.1) in some applications. Let here $M_j(t)$ be the number of components that are still alive at time $t\ge 0$ in system j, $j=1,2$. Using the fact that we may set $\Theta_j=\Lambda_j^{-1}$, $j=1,2$, and that the transformation $\phi:(0,\infty)\to(0,\infty)$, defined by $\phi(t)=1/t$, is continuous and strictly decreasing, it is seen from Theorem 2.1 and (4.2) [respectively, (4.3)] that

$$\Theta_1\le_{\rm hr}[\le_{\rm rh}]\,\Theta_2\Rightarrow M_1(t)\le_{\rm hr}[\le_{\rm rh}]\,M_2(t),\qquad t\ge 0.$$

In a similar vein, the last parts in Theorems 2, 3 and 4 of Ma (1999) need not be proven because, by Theorem 2.1, each of those statements is equivalent to the statement preceding it in Ma (1999).

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References

Belzunce, F., Franco, M., Ruiz, J. M. and Ruiz, M. C. (2001), On partial orderings between coherent systems with different structure, *Probab. Engrg. Inform. Sci.*, **15**, 273–293.

Block, H. W., Savits, T. H. and Singh, H. (1998). The reversed hazard rate function, *Probab. Engrg. Inform. Sci.*, **12**, 69–90.

Boland, P. J. and Proschan, F. (1994). Stochastic order in system reliability theory, *Stochastic Orders and Their Applications* (eds. M. Shaked and J. G. Shanthikumar), 485–508, Academic Press, San Diego.

Boland, P. J., El-Neweihi, E. and Proschan, F. (1994). Applications of the hazard rate ordering in reliability and order statistics, *J. Appl. Probab.*, **31**, 180–192.

Boland, P. J., Hu, T., Shaked, M. and Shanthikumar, J. G. (2001). Stochastic ordering of order statistics, *Essays on Uncertainty* (eds. M. Dror, P. L'Ecuyer and F. Szidarovsky) (to appear).

Cheng, D. W. and Righter, R. (1995). On the order of tandem queues, *Queueing Systems*, **21**, 143–160.

Cheng, D. W. and Zhu, Y. (1993). Optimal order of servers in a tandem queue with general blocking, *Queueing Systems*, **14**, 427–437.

Eeckhoudt, L. and Gollier, C. (1995). Demand for risky assets and the monotone probability ratio order, *Journal of Risk and Uncertainty*, **11**, 113–122.

Hu, T. and He, F. (2000). A note on comparisons of k-out-of-n systems with respect to the hazard and reversed hazard rate orders, *Probab. Engrg. Inform. Sci.*, **14**, 27–32.

Khaledi, B.-H. and Kochar, S. (1999). Stochastic orderings between distributions and their sample spacings—II, *Statist. Probab. Lett.*, **44**, 161–166.

Kijima, M. (1998). Hazard rate and reversed hazard rate monotonicities in continuous-time Markov chains, *J. Appl. Probab.*, **35**, 545–556.

Kijima, M. and Ohnishi, M. (1999). Stochastic orders and their applications in financial optimization, *Math. Methods Oper. Res.*, **50**, 351–372.

Kochar, S. C. and Kirmani, S. N. U. A. (1995). Some results on normalized spacings from restricted families of distributions, *J. Statist. Plann. Inference*, **46**, 47–57.

Lillo, R. E., Nanda, A. K. and Shaked, M. (2001). Preservation of some likelihood ratio stochastic orders by order statistics, *Statist. Probab. Lett.*, **51**, 111–119.

Ma, C. (1999). Uniform stochastic ordering on a system of components with dependent lifetimes induced by a common environment, *Sankhyā Ser. A*, **61**, 218–228.

Nanda, A. K., Jain, K. and Singh, H. (1998). Preservation of some partial orderings under the formation of coherent systems, *Statist. Probab. Lett.*, **39**, 123–131.

Sengupta, D., Singh, H. and Nanda, A. K. (1999). The proportional reversed hazards model, Tech. Report, Applied Statistics Division, Indian Statistical Institute, Calcutta.

Shaked, M. and Shanthikumar, J. G. (1988). On the first-passage times of pure jump processes, *J. Appl. Probab.*, **25**, 501–509.

Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic Orders and Their Applications*, Academic Press, San Diego.

Shaked, M. and Wong, T. (1997*a*). Stochastic orders based on ratios of Laplace transforms, *J. Appl. Probab.*, **34**, 404–419.

Shaked, M. and Wong, T. (1997*b*). Stochastic comparisons of random minima and maxima, *J. Appl. Probab.*, **34**, 420–425.

Shaked, M., Shanthikumar, J. G. and Tong, Y. L. (1995). Parametric Schur convexity and arrangement monotonicity properties of partial sums, *J. Multivariate Anal.*, **53**, 293–310.

Shanthikumar, J. G., Yamazaki, G. and Sakasegawa, H. (1991). Characterization of optimal order of servers in a tandem queue with blocking, *Oper. Res. Lett.*, **10**, 17–22.

Singh, H. and Vijayasree, G. (1991). Preservation of partial orderings under the formation of k-out-of-n:G systems of i.i.d. components, *IEEE Transactions on Reliability*, **40**, 273–276.