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INTERMEDIATE APPROACH TO COMPARISON OF SOME GOODNESS-OF-FIT TESTS

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Abstract. In this paper we present the intermediate approach to investigating asymptotic power and measuring the efficiency of nonparametric goodness-of-fit tests for testing uniformity. Contrary to the classical Pitman approach, the intermediate approach allows the explicit quantitative comparison of powers and calculation of efficiencies. For standard tests, like the Cramér-von Mises test, an intermediate approach gives conclusions consistent with qualitative results obtained using the Pitman approach. For other more complicated cases the Pitman approach does not give the right picture of power behaviour. An example is the data driven Neyman test we present in this paper. In this case the intermediate approach gives results consistent with finite sample results. Moreover, using this setting, we prove that the data driven Neyman test is asymptotically the most powerful and efficient under any smooth departures from uniformity. This result shows that, contrary to classical tests being efficient and the most powerful under one particular type of departure from uniformity, the new test is an adaptive one.

Key words and phrases: Intermediate efficiency, vanishing shortcoming, goodness-of-fit test, data driven test, smooth test, Schwarz rule.

1. Introduction

The basic classical approach to the comparison of tests consists of the calculation of powers and efficiencies under contiguous alternatives. This approach works nicely for statistics which are asymptotically normal. In this case, in fact, many approaches to measuring the efficiency of these tests give the same results. For an explanation see Kallenberg (1983) and Kallenberg and Ledwina (1987). In more complicated cases a simple and explicit determination of the asymptotic power and efficiency by applying contiguity is not feasible. The Cramér-von Mises (CvM) test for uniformity, throughly studied by Neuhaus (1976, 1986), serves as an illustration of this. He investigated the asymptotic power of the CvM test under a sequence of alternative densities

$$p_n(x) = 1 + n^{-1/2}\rho a(x), \tag{1.1}$$

where $\int_0^1 a(x)dx = 0$, $\int_0^1 a^2(x)dx = 1$, $\rho > 0$. The constant ρ represents the distance from $p_0(x) \equiv 1$ and $a(x)$ the direction of the alternative (1.1). Let α denote a given significance level and $\beta(\alpha, a, \rho)$ denote the asymptotic power of the α-level CvM test under the alternative (1.1). $\beta(\alpha, a, \rho)$ coincides with the tail probability of an infinite series of weighted non-central chi-squared random variables. To evaluate this function Neuhaus (1976) used the Hájek-Šidák (1967) approach based on a comparison with the power of the best test for a related asymptotic testing problem. The power $\beta(\alpha, \rho)$ of this best test does not depend on the direction a. The function $e = e(\alpha, a, \rho)$ defined by

$$\beta(\alpha, a, \rho) = \beta(\alpha, \rho\sqrt{e}), \tag{1.2}$$

is taken to be asymptotic efficiency of the CvM test in the direction a. Finally, the asymptotic local efficiency is defined as

$$e(\alpha, a) = \lim_{\rho \to 0} e(\alpha, a, \rho). \tag{1.3}$$

There is no closed expression for $e(\alpha, a)$. Also, according to the best of our knowledge, the finite sample interpretation of this notion has not been formally discussed (cf. however Chapter VII.2.3. in Hájek and Šidák (1967)). For further discussion and developments of this approach we refer to Milbrodt and Strasser (1990), Strasser (1990), Janssen (1995) and references therein. An alternative attempt to calculate Pitman efficiencies for some statistics with non-normal asymptotic distributions can be found in Rothe (1981). The results of that paper are restricted to asymptotically chi-square distributed statistics with different degrees of freedom. The resulting efficiencies depend both on the given level α and the power β.

Another approach to measuring asymptotic efficiency has been proposed by Bahadur. In this approach, which is known as the exact Bahadur efficiency, an alternative is fixed while the significance level tends to 0 at an exponential rate, which is determined by the large deviation index of the test statistic under the null hypothesis. Since explicit expressions for this index are not available, the next step is to consider the efficiency measure under alternatives converging to the null distribution. This approach, applied to classical nonparametric tests, is presented clearly in Nikitin (1995). A related approach aiming to calculate the limiting (as the significance level tends to 0) Pitman efficiency by limiting (as the alternative approaches the hypothesis) the approximate efficiency has been proposed by Wieand (1976). The approximate Bahadur efficiency, however, is in itself of little value as a measure of the performance of tests, since monotone transformations of a test statistic may lead to entirely different approximate Bahadur slopes. In this sense the exact Bahadur efficiency is superior to the approximate one. Some results, stating when exact and approximate Bahadur efficiencies locally coincide, can be found in Ledwina (1987), Inglot and Ledwina (1990, 1993) among others.

Despite of the technical complexity of the above approaches, there are statistics for which they simply fail or give conclusions, which are in sharp contrast with finite sample results. An example is the data-driven Neyman test we present in Section 2.

The aim of this paper is to present and develop an alternative solution to investigating asymptotic power and measuring the efficiency of nonparametric test procedures.

Our main goal is to show some application orientated advantages of this alternative approach. Namely, we aim to show that this solution yields an explicit expression for the efficiency, is more widely applicable than classical approaches, allows an intuitive interpretation of finite sample results and gives conclusions consistent both with those following from standard solutions and with simulation experiments with moderate sample sizes.

The idea of this approach goes back to Oosterhoff (1969), Oosterhoff and van Zwet (1972) and Kallenberg (1978, 1983) and shall be called the intermediate approach. Further extensions can be found in Inglot and Ledwina (1996), Inglot (1999), Inglot *et al.* (1998*a*, 2000). Its essence lies in calculating the asymptotic power and efficiencies of a test in situations where the significance level tends to 0 more slowly than in the Bahadur approach and simultaneously the alternative tends to the null distribution more slowly than in the Pitman case. To be more specific, we first give some general comments on the setting and then briefly discuss our results. Throughout the paper we resign from generality to the benefit of technical simplicity. We restrict attention to sequences of alternative densities of the form

$$p_n(x) = 1 + n^{-\xi}\rho a(x), \qquad \xi \in (0, 1/2). \tag{1.4}$$

The Pitman approach would correspond to $\xi = 1/2$ while in the Bahadur approach $\xi = 0$. It is easily seen that the distribution of n i.i.d. observations obeying (1.4) is asymptotically orthogonal to the n-fold product of $p_0(x) \equiv 1$ (cf. Appendix). So, perfect discrimination between p_n and p_0 is possible under (1.4), since n is increasing. This seems to be a natural reflection of the fact that the information contained in the sample is increasing in n. In this situation the natural approach of a statistician seems to be to require that the precision of inference is also increasing in n. This is realized in the intermediate approach by requiring that the significance level tends to 0 as $n \to \infty$ while the asymptotic power under p_n is nondegenerate (staying away 0 and 1). This is the basic difference to the Pitman approach, which is designed to protect against the possibility of perfect discrimination between p_n and p_0 and then avails it allowing test comparisons at a fixed asymptotic level and supposing asymptotic power to be nondegenerate. On the other hand, in the Bahadur approach the alternative is fixed and any consistent test has degenerate power. In this case only efficiency calculations are common practice.

In this paper we show the advantages of the intermediate approach, applying it to the comparison of three nonparametric tests for testing uniformity: the CvM test, the Neyman-Pearson test for uniformity against a nonparametric alternative p_n and some data-driven Neyman test. The results yield interesting conclusions about the powers and efficiencies of the three tests. They can be summarized as follows.

First of all, explicit results for the power and efficiency of Cramér-von Mises test in comparison to the Neyman-Pearson test are obtained and shown to be fully consistent with the qualitative results developed using the Pitman approach by Neuhaus (1976) and the Bahadur efficiency calculations of Nikitin (1995). The essence of our approach to intermediate power comparison is examined in Subsection 3.3. The results on the intermediate power and efficiency of the Cramér-von Mises test with respect to the Neyman-Pearson test we obtained (cf. Theorem 3.3.(2), Remark 3.3.(2) and Remark 4.(2)) clearly show that our approach is well suited to classical ideas of measuring the efficiency of tests (cf. e.g. Hajék and Šidák (1967), Chapter VII.2.1., Remarks (c), (d)). We would like to emphasize once more that the Cramér-von Mises goodness of fit statistic is probably the simplest empirical process functional to analyse using the Pitman approach. How complicated this approach can be for other statistics and testing problems is seen from Milbrodt and Strasser (1990) and Stute (1997). Note also, that exploiting the results of Inglot and Ledwina (1993) and Inglot*et al.* (1993), the findings of the present paper can be extended to a large class of statistics, which are weighted or bilinear functionals of the empirical process. A case where some nuisance parameters ought to be estimated can also be treated by the approach. Inglot and Ledwina (2001) can serve as an illustration.

Parallel to results for the Cramér-von Mises test, we present results for a data driven Neyman test. On one hand this statistic serves to illustrate that classicial approaches to test comparison may fail. On the other hand, we show that for such statistics the intermediate approach is a reliable remedy for this drawback of the classical approaches. The intermediate approach to investigating a data driven Neyman test with the original Schwarz rule incorporated has been developed in Inglot and Ledwina (1996) and Inglot *et al.* (1998*a*) under a very general setting. Here we only use the simplified Schwarz rule $S2$ and contamination alternatives (1.4). This allows us to get slightly stronger results and to present the essence of the approach due to avoiding technicalities needed in the general case. The results we get for the data driven test are qualitatively different to those known for classical tests and those derived here, in particular for the Cramér-von Mises test. Contrary to the previously mentioned tests being asymptotically efficient and most powerful under one particular departure from uniformity, the data driven test we consider can be declared to be an adaptive one in a very natural sense. Simply, it is asymptotically as powerful and efficient as if the alternative were known.

The content of the paper is as follows. In Section 2 we present the data- driven Neyman test and show that for this test classical asymptotics fail or say very little about power behaviour for moderate sample sizes. We also give a few additional remarks on some papers, where classical asymptotics have not appeared to be the most natural one. Section 3 is the main part of the paper. We state limit theorems concerning the asymptotic behaviour of the power functions of the three tests under the above setting. Most of the auxiliary technical results we present here are conclusions from results derived by Inglot, Kallenberg and Ledwina in recent papers. The basic results on the shortcoming of the CvM test are new, while Theorem 3.3.(3) on the shortcoming of data driven test is similar to results of Inglot *et al.* (1998*a*).

In Section 4 we present the notion of intermediate efficiency using a formulation of Inglot (1999) and use it to compare the three tests. Again efficiency calculations for the CvM test are new, while the data driven test is considered for comparison. Some proofs are given in the Appendix.

For completeness, note that in Inglot *et al.* (2000) the problem of the relation of vanishing shortcoming and asymptotic relative efficiency is studied in general. General results are illustrated by an application to the CvM test among others.

2. Data-driven test and classical approaches

The data-driven test we discuss here is Neyman's smooth test with the number of components determined by the simplified Schwarz rule. Tests of this kind were introduced and investigated in Inglot *et al.* (1997), Kallenberg and Ledwina (1997*a*, 1997*b*) and Inglot (1999). For simplicity of presentation, we shall consider here a particular member of the class of such tests. To define the test statistic we introduce some notation.

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with values in [0,1]. Let $\Phi_1, \Phi_2, \ldots$ be normalized Legendre polynomials on [0,1]. For some fixed c set $m_n = [cn^{1/8}]$, where $[x]$ denotes the integer part of x. Define

$$S2 = \min\{1 \le k \le m_n : n|\bar{\Phi}|_k^2 - k\log n \ge n|\bar{\Phi}|_j^2 - j\log n,\ j = 1, \ldots, m_n\}, \tag{2.1}$$

where

$$\bar{\Phi} = (\bar{\Phi}_1, \bar{\Phi}_2, \ldots), \quad \bar{\Phi}_j = \frac{1}{n}\sum_{i=1}^{n} \Phi_j(X_i) \quad \text{and} \quad |\bar{\Phi}|_j^2 = \sum_{l=1}^{j} (\bar{\Phi}_l)^2. \tag{2.2}$$

$S2$ is the simplified Schwarz model selection rule. The original Schwarz rule shall be presented and related to $S2$ later in this section.

The data-driven test rejects uniformity for large values of

$$N_{S2} = n|\bar{\Phi}|_{S2}^2 = \sum_{j=1}^{S2} \left\{ n^{-1/2} \sum_{i=1}^{n} \Phi_j(X_i) \right\}^2. \tag{2.3}$$

In Inglot (2000) it was shown that for any $x > 0$

$$-\lim_{n\to\infty} \frac{1}{nx^2} \log P_0(N_{S2} \geq x\sqrt{n}) = 0,$$

where P_0 stands for the uniform distribution on [0,1]. This means that large deviation index degenerates to 0 and the Bahadur approach does not apply to this statistic.

As to the Pitman approach, a similar argument to the proof of Theorem 3.2 in Kallenberg and Ledwina (1995) yields

$$\lim_{n\to\infty} P_0(S2 = 1) = 1. \tag{2.4}$$

An immediate consequence of (2.4) is that under any sequence $\{P_n\}$ of alternatives contiguous to P_0 $\lim_{n\to\infty} P_n(S2 = 1) = 1$, as well. So, on one hand, (2.4) implies that the asymptotic null distribution of N_{S2} is central $\chi^2_{(1)}$. On the other hand, from the above, for the contamination family (1.1) the asymptotic power of N_{S2} takes the form $P_0(\{Z + \rho \int_0^1 \Phi_1(x)a(x)dx\}^2 \geq c_\alpha)$, where $Z \sim N(0,1)$ and c_α is the asymptotic critical value. This shows that the Pitman efficiency can be considered only for alternatives for which $\int_0^1 \Phi_1(x)p(x)dx \neq 0$. Also, only such alternatives can be detected with probability greater than the significance level α. Since CvM test has local power under any alternative (1.1) strictly greater than α and for any alternative sequence (1.1) its local efficiency is well defined, it may appear from the above that the CvM will be more effective than N_{S2} in detecting alternatives that are close to the null. The same opinion could be formulated for the data-driven Neyman's test with the original Schwarz rule, which was introduced in Ledwina (1994) and investigated in Kallenberg and Ledwina (1995), Inglot and Ledwina (1996) and Inglot *et al.* (1998*a*). The original Schwarz rule is defined as follows

$$S = \min\left\{1 \leq k \leq m_n : \mathcal{L}_k - \frac{k}{2}\log n \geq \mathcal{L}_j - \frac{j}{2}\log n, j = 1, \ldots, m_n\right\},$$

where

$$\mathcal{L}_k = \sup_{\theta \in R^k} \log \prod_{i=1}^{n} c_k(\theta) \exp\left\{\sum_{j=1}^{k} \theta_j \Phi_j(x_i)\right\},$$

while the related data-driven test is given by

$$N_S = n|\bar{\Phi}|_S^2 = \sum_{j=1}^{S} \left\{ n^{-1/2} \sum_{i=1}^{n} \Phi_j(X_i) \right\}^2. \tag{2.5}$$

Table 1. Empirical powers of N_S and N_{S2} under the alternative $p_n(x) = 1+n^{-1/2}\rho\Phi_2(x)$ and related empirical distributions of S and $S2$.

ρ	n	$\|p_n - p_0\|_2$	$\|p_n - p_0\|_\infty$	power of CvM	test	power	distribution	1	2	3
.2	50	.028	.063	.05	N_S	.05	S	.93	.05	.01
					N_{S2}	.06	$S2$	.93	.05	.01
	500	.009	.020	.05	N_S	.06	S	.98	.02	.00
					N_{S2}	.06	$S2$	.98	.02	.00
.5	50	.071	.158	.06	N_S	.07	S	.92	.06	.01
					N_{S2}	.08	$S2$	.91	.07	.01
	500	.022	.050	.06	N_S	.06	S	.98	.02	.00
					N_{S2}	.06	$S2$	.97	.03	.00
1	50	.141	.316	.08	N_S	.14	S	.84	.14	.02
					N_{S2}	.16	$S2$	.81	.15	.02
	500	.045	.100	.07	N_S	.11	S	.94	.06	.00
					N_{S2}	.11	$S2$	.93	.07	.00
2	50	.283	.632	.15	N_S	.41	S	.53	.43	.03
					N_{S2}	.44	$S2$	.49	.45	.05
	500	.089	.200	.13	N_S	.33	S	.70	.30	.00
					N_{S2}	.35	$S2$	.68	.31	.01
3	50	.424	.949	.33	N_S	.75	S	.19	.75	.04
					N_{S2}	.77	S_2	.16	.75	.07
	500	.134	.300	.31	N_S	.69	S	.32	.66	.01
					N_{S2}	.71	$S2$	.31	.68	.01

S2 arises as an approximation of the maximal log-likelihood, which is in fact the log-likelihood ratio statistic and which is locally equivalent to $\frac{1}{2}n|\bar{\Phi}|_k^2$. In other words, $n|\bar{\Phi}|_k^2$ is the score statistic for testing $\theta = 0$ in the exponential family defining $\mathcal{L}_k$.

However, in our opinion, to be decisive the above pessimistic "first order" asymptotic results should be compared with the existing theoretical results of Neuhaus (1976) showing how CvM distributes its power in the space of all alternatives on one hand and with the empirical rates of the convergence of S and $S2$ to 1 and the respective powers of N_S and N_{S2} to α on the other hand.

As an illustration of the gap between the asymptotics and typical empirical results we now present some simulation results.

Table 1 presents the empirical powers of N_S and N_{S2} under alternative (1.1) with $a(x) = \Phi_2(x)$ i.e. under $p_n(x) = 1 + n^{-1/2}\rho\Phi_2(x)$ with $n = 50$ and $n = 500$. We took $m_{50} = m_{500} = 10$. In the case of such "low dimensional models" large m_n's are not necessary, since starting from relatively small m_n the power is practically constant. For evidence see Kallenberg and Ledwina (1997a). According to the Pitman approach, the asymptotic powers of N_{S2} and N_S under such $p_n(x)$ are α, while $S2$ and S should concentrate on 1 as n becomes large. We took $\alpha = .05$. The simulated empirical distributions of $S2$ and S are also presented. We restrict attention to events $\{S2 = k\}$, $\{S = k\}$ with $k = 1, 2, 3$. For each n and ρ we also give L_2 and supremum distances

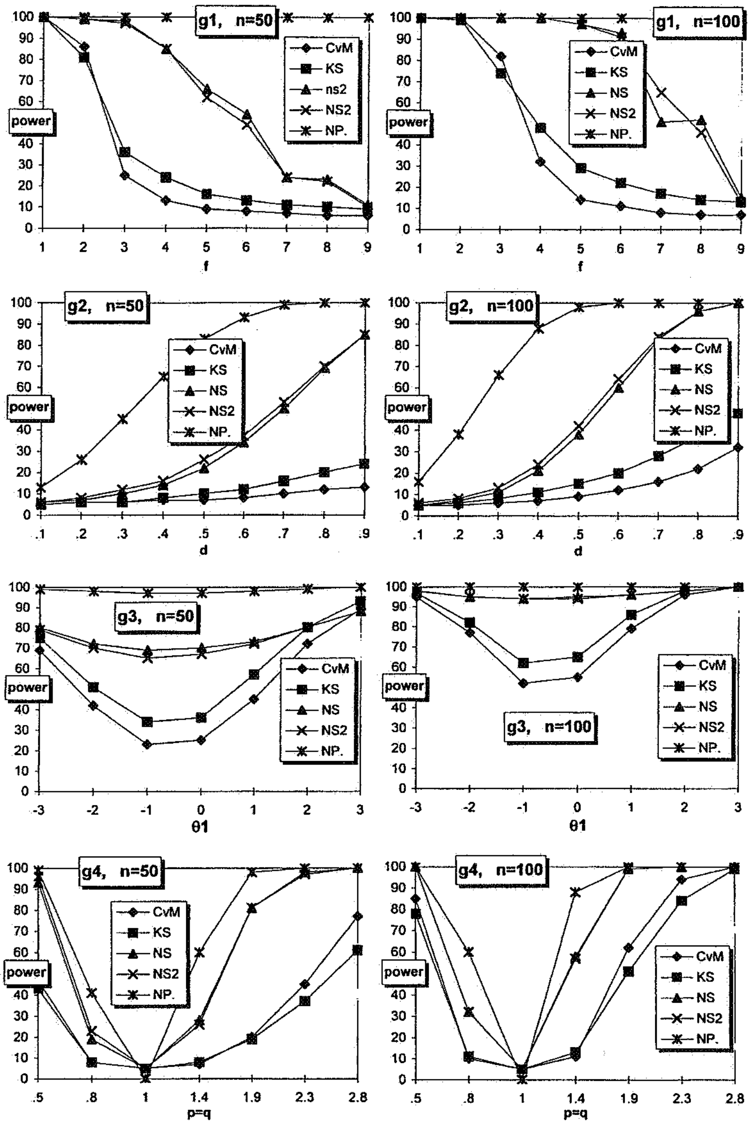

Fig. 1. Empirical powers of five tests under alternatives $g1$-$g4$ and $n = 50$, 100.

between p_n and p_0, which we denote by $\|\cdot\|_2$ and $\|\cdot\|_\infty$, respectively. For comparison empirical powers of CvM test are also reported. The number of Monte Carlo runs is $N = 10000$.

It is seen that only in the cases $\rho = .2$ and $\rho = .5$ one gets results not far from the expected ones. For larger ρ's the rate of convergence of the powers of N_S and N_{S2} to α is so slow that the limiting result gives a very poor approximation of the finite sample situation. It is also evident that, in contrast to the asymptotic results, the CvM test is much worse than data driven tests when the sample size is finite.

One can expect that for more complicated alternatives (i.e. those having several terms in their Fourier expansion in the system $\Phi_1, \Phi_2, \ldots$) the situation is even more transparent. Figure 1 presents the simulated powers of N_{S2} and N_S under the following alternatives:

$$
\begin{aligned}
&g_1(x) = 1 + .9\cos(f\pi x), \qquad f = 1, 2, \ldots, 9,\\
&g_2(x) = 1 + d\cos(4\pi x), \qquad d = .1, .2, \ldots, .9,\\
&g_3(x) = c_4(\theta)\exp\{\theta_1\Phi_1(x) + .2\Phi_2(x) - .3\Phi_3(x) - .4\Phi_4(x)\}, \qquad \theta_1 = -.3, \ldots, .3,\\
&g_4(x) = \beta(p, q) - \text{ the beta distribution density, } p = q \in (.5, 2.8)
\end{aligned}
$$

and $n = 50, 100$. As in Table 1, we took $m_{50} = m_{100} = 10$, $\alpha = .05$ and $N = 10000$. As an illustration consider $g_2(x)$ with $d = .9$. Since $\int \Phi_1(x) g_2(x) dx = 0$, the predicted (local) power under Pitman asymptotics would be .05, while S should concentrate on 1. However, under $n = 50$ the simulated power is .85, $S = 1$ only in 15% of the runs, while $S = 6$ in 57% of cases. The reason is that $\int \Phi_j(x) g_2(x) dx$, $j = 4, 6$ and 2, take Monte Carlo values .50, .35 and .15, respectively. Obviously, for much smaller d's the situation is not so drastic.

For comparison purposes empirical powers of the Cramér-von Mises (CvM) and the Kolmogorov- Smirnov (KS) test as well as the Neyman-Pearson (NP) test for p_0 against g_i, $i = 1, 2, 3, 4$, are also given. For simplicity we write NS and NS2 for N_S and N_{S2}, respectively. It is seen that in most cases NS2 and NS are more sensitive than the CvM and KS tests and adapt well to the data at hand as the sample size increases. It is seen that, as a rule, as n increases the empirical powers of NS, NS2 and NP become closer to each other. We prove in Section 3 that the intermediate approach predicts such a tendency. Moreover, looking at Fig. 1, one has the impression that in most cases the empirical power function of CvM increases more slowly in n than the power functions of NS and NS2. The intermediate approach also predicts this tendency.

We close this informal section with a few remarks on other approaches to test comparison in a nonstandard setting. Obviously we are not the first ones faced with the problem of the necessity of using alternatives not converging at a parametric rate $n^{-1/2}$. Some literature in the area is discussed in Section 2 of Eubank and LaRiccia (1992), for example. A typical and very well known example is the test statistic introduced by Bickel and Rosenblatt (1973). However this case is relatively easy, since asymptotic normality holds under both the null hypothesis and the sequence of alternatives. So, the classical Pitman approach can be extended to cover this case as in Bickel and Rosenblatt (1973) and further discussed in Ghosh and Huang (1991). A similar group of results related to multinomial goodness of fit tests, which is not mentioned in Eubank and LaRiccia (1992), is developed and throughly reviewed in Cressie and Read (1984), Sections 2.3 and 3.2. Further extensions in this direction can be found in Inglot *et al.* (1990a). Parallel results for Neyman test statistics are given in Inglot *et al.* (1990b). A different problem has been solved by Eubank and LaRiccia (1992), where a rather complicated approach is developed to roughly compare CvM statistics with an asymptotically normal one. It is worthy to observe that in all the above mentioned situations the intermediate approach could be applied. As an illustration of the case of asymptotically normal statistics see Corollary 2.2 and Examples 2.1 and 2.2 in Kallenberg (1983). The situation considered by Eubank and LaRiccia (1992) could be treated similarly as N_{S2} and CvM in the present paper (cf. also Theorem 7.23 in Inglot and Ledwina (1996)).

In light of the existing literature we briefly discussed here and in Section 1, it seems that the intermediate approach promises to be the most widely applicable.

3. Asymptotic power functions under a contamination model

3.1 *Neyman-Pearson test*

Throughout this paper we consider the contamination model

$$p_n(x) = 1 + n^{-\xi}\rho a(x), \qquad \xi \in (0, 1/2). \tag{3.1}$$

Define

$$\mathcal{A} = \left\{ a : [0,1] \to R : \sup_x |a(x)| < \infty, \int_0^1 a(x)dx = 0, \int_0^1 a^2(x)dx = 1 \right\}.$$

We shall restrict attention to sequences of alternative densities $\{p_n\}$ obeying (3.1), which belong to the following set

$$\mathcal{P} = \{\{p_n\} : p_n(x) = 1 + n^{-\xi}\rho a(x),\ \xi \in (0,1/2),\ \rho > 0,\ a \in \mathcal{A}\}. \tag{3.2}$$

$X_1, \ldots, X_n$ are independent and identically distributed random variables obeying the law under consideration.

A standardized version of the logarithm of the Neyman-Pearson test statistic for testing $p_0(x) \equiv 1$ against $p_n(x)$ has the form

$$V_{n,p_n}^{(1)} = (\sqrt{n}v_{0n})^{-1} \sum_{i=1}^{n} \{\log p_n(X_i) - e_{on}\}, \tag{3.3}$$

where

$$e_{0n} = \int_0^1 \log p_n(x)dx, \qquad v_{0n}^2 = \int_0^1 \log^2 p_n(x)dx.$$

Note that $v_{0n}^2/\operatorname{Var}\log p_n(X_1) \to 1$ (cf. Lemma 5.4 in Inglot and Ledwina (1996)). For simplicity we shall write

$$V_n^{(1)} = V_{n,p_n}^{(1)}.$$

Additionally define

$$b^{(1)}(p_n) = v_{0n}^{-1/2} \left\{ \int_0^1 p_n(x)\log p_n(x)dx - e_{0n} \right\}. \tag{3.4}$$

For further consideration and to facilitate refering to results contained in other papers we quote below the following chain of relations which, under (3.1) and (3.2), follow from Proposition 2.10 in Inglot (1999),

$$\begin{aligned} \{b^{(1)}(p_n)\}^2 &= 2D(p_n \parallel p_0) + O(H^3(p_n, p_0)) \\ &= \|p_n - p_0\|_2^2 + O(H^3(p_n,p_0)) = 4H^2(p_n,p_0) + O(H^3(p_n,p_0)), \end{aligned} \tag{3.5}$$

where

$$D(p_n \parallel p_0) = \int_0^1 p_n(x)\log p_n(x)dx$$

is the Kullback-Leibler distance,

$$\|p_n - p_0\|_2 = \left\{ \int_0^1 (p_n(x) - 1)^2 dx \right\}^{1/2}$$

is the L_2 distance,

$$H(p_n, p_0) = \left\{ \int_0^1 (\sqrt{p_n(x)} - 1)^2 dx \right\}^{1/2}$$

is the Hellinger distance.

In particular, the above relations imply that for $\{p_n\} \in \mathcal{P}$

$$b^{(1)}(p_n) = n^{-\xi}\rho + O(n^{-2\xi}). \tag{3.6}$$

The asymptotic behaviour of the power function of the Neyman-Pearson test rejecting uniformity for large values of $V_n^{(1)}$ is determined by the following two results.

THEOREM 3.1.(1) *For any* $\{p_n\} \in \mathcal{P}$

$$\lim_{n \to \infty} P_n(V_n^{(1)} - n^{1/2} b^{(1)}(p_n) \le x) = \Phi(x), \qquad x \in R, \tag{3.7}$$

where $\Phi(x)$ *is the standard normal distribution function and* P_n *the distribution with density* p_n.

Theorem 3.1.(1) is a special case of Proposition 6.6 in Inglot and Ledwina (1996).

THEOREM 3.2.(1) *Set* $\sigma_n^2 = \mathrm{Var}_{P_0} V_n^{(1)}$. *For any* $\{p_n\} \in \mathcal{P}$ *and sequence* $\{x_n\}$ *of positive numbers such that* $x_n \to 0$ *and* $nx_n^2 \to \infty$, *it holds that*

$$P_0(\sigma_n^{-1} V_n^{(1)} \ge \sqrt{n} x_n) = \exp\left\{ -\frac{nx_n^2}{2} + O(nx_n^3) + O(\log nx_n^2) \right\}. \tag{3.8}$$

In particular, if $x_n = O(n^{-\xi})$ *and* $\xi \in (1/4, 1/2)$, *then* (3.8) *takes the form*

$$P_0(V_n^{(1)} \ge \sqrt{n} x_n) = \exp\left\{ -\frac{nx_n^2}{2} + o(n^{1/2-\xi}) \right\}. \tag{3.9}$$

Theorem 3.2.(1) follows from Corollary 2.22 of Book (1976) (cf. also Jurečková *et al.* (1988) or Inglot *et al.* (1998*a*)).

Let $\mathbf{1}(B)$ be the indicator of the set B. For given $\{p_n\} \in \mathcal{P}$ and a sequence $\{k_1(n)\}$ of real numbers define the critical region

$$\mathcal{C}_n^{(1)} = \{V_n^{(1)} \ge k_1(n) + \sqrt{n} b^{(1)}(p_n)\} \tag{3.10}$$

with related significance level

$$\alpha_n^{(1)} = P_0(\mathcal{C}_n^{(1)}). \tag{3.11}$$

Moreover, denote

$$\mathcal{V}^{(1)}_{n, p_n, \alpha_n^{(1)}} \triangleq \mathcal{V}^{(1)}_{n, \alpha_n^{(1)}} = \mathbf{1}(\mathcal{C}_n^{(1)}), \tag{3.12}$$

the nonrandomized test based on $V_n^{(1)}$ and

$$\beta^{(1)}(\alpha_n^{(1)}, p_n) = E_{P_n} \mathcal{V}^{(1)}_{n, \alpha_n^{(1)}} = P_n(\mathcal{C}_n^{(1)}) \tag{3.13}$$

its power under p_n.

By Theorem 3.1(1), for $\{p_n\} \in \mathcal{P}$, it holds that

$$\beta^{(1)}(\alpha_n^{(1)}, p_n) - (1 - \Phi(k_1(n))) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.14}$$

while, by Theorem 3.2(1), under the additional assumption that $\limsup_{n\to\infty} n^{-1/2+\xi}|k^{(1)}(n)| < \rho$,

$$\alpha_n^{(1)} = P_0(\mathcal{C}_n^{(1)}) = \exp\{-(k_1(n) + \sqrt{n}b^{(1)}(p_n))^2/2 + O(n^{1-3\xi}) + O(\log n)\}. \tag{3.15}$$

So, if $k_1(n)$ is bounded (3.14) states that the test $\mathcal{V}^{(1)}_{n,\alpha_n^{(1)}}$ has asymptotic power staying away 0 and 1, while (3.15) implies that, contrary to the Pitman case ($\xi = 1/2$), in the intermediate case ($\xi \in (0, 1/2)$) the probability of type I error tends to 0 (at a rate related to the rate of convergence of p_n to p_0).

The next section contains analogous results for the Cramér-von Mises test. Related considerations of the difference in powers of $\mathcal{V}^{(1)}_{n,\alpha_n^{(1)}}$ and the CvM test are given in Subsection 3.3.

3.2 *The Cramér-von Mises test*

Denoting by F_n the empirical distribution function of $X_1, \ldots, X_n$, the CvM test statistic for uniformity is defined by

$$V_n^{(2)} = n\int_0^1 (F_n(x) - x)^2 dx. \tag{3.16}$$

To state some counterparts of Theorems 3.1(1) and 3.2(1) for $V_n^{(2)}$ set

$$A(x) = \int_0^x a(t)dt, \qquad \|A\|_2^2 = \int_0^1 A^2(x)dx \tag{3.17}$$

and

$$\sigma^2 = \int_0^1 \int_0^1 (\min(s,t) - st)A(s)A(t)dsdt. \tag{3.18}$$

THEOREM 3.1.(2) *For any $\{p_n\} \in \mathcal{P}$ it holds that*

$$\lim_{n\to\infty} P_n\left(\frac{V_n^{(2)} - n^{1-2\xi}\rho^2\|A\|_2^2}{2n^{1/2-\xi}\rho\sigma} \le x\right) = \Phi(x), \qquad x \in R. \tag{3.19}$$

Theorem 3.1.(2) can be deduced from Theorem 5.1 in Inglot *et al.* (2000). The lemma is proved by strong approximation techniques in a similar way to Theorem 3.1 in Inglot *et al.* (1993), e.g. Details are given in Inglot *et al.* (1998*b*).

The proof of the next theorem follows from inequalities derived in Inglot and Ledwina (1990) (cf. also Theorem 2.1 and Proposition 2.3 in Inglot *et al.* (1993)). The number π appears due to the tail behaviour of $V_n^{(2)}$ (see (3.20)) and can be deduced from Nikitin (1995), e.g.

THEOREM 3.2.(2) *If $x_n \to 0$ and $nx_n^2 \to \infty$, then for every $2 < \gamma < 3$ it holds that*

$$P_0(V_n^{(2)} \geq nx_n^2) = \exp\left\{-\frac{1}{2}\pi^2 nx_n^2 + O(nx_n^\gamma) + O(\log nx_n^2)\right\}. \tag{3.20}$$

By analogy with (3.10)–(3.13), for a given $k_2 \in R$, set

$$\begin{aligned}
\mathcal{C}_n^{(2)} &= \{V_n^{(2)} \geq n^{1-2\xi}\rho^2\|A\|_2^2 + 2n^{1/2-\xi}\rho\sigma k_2\},\\
\alpha_n^{(2)} &= P_0(\mathcal{C}_n^{(2)}), \mathcal{V}^{(2)}_{n,p_n,\alpha_n^{(2)}} \triangleq \mathcal{V}^{(2)}_{n,\alpha_n^{(2)}} = \mathbf{1}(\mathcal{C}_n^{(2)}),\\
\beta^{(2)}(\alpha_n^{(2)}, p_n) &= E_{P_n}\mathcal{V}^{(2)}_{n,\alpha_n^{(2)}} = P_n(\mathcal{C}_n^{(2)}).
\end{aligned}$$

Then, by Theorem 3.1.(2), for $\{p_n\} \in \mathcal{P}$

$$\beta^{(2)}(\alpha_n^{(2)}, p_n) \to 1 - \Phi(k_2). \tag{3.21}$$

So, (3.21) states that the test $\mathcal{V}^{(2)}_{n,\alpha_n^{(2)}}$ has asymptotic power staying away from 0 and 1. Theorem 3.2.(2) implies that for $\xi \in (0, 1/2)$ and any $\gamma \in (2,3)$

$$\begin{aligned}
\text{(3.22)}\ \alpha_n^{(2)} &= P_0(\mathcal{C}_n^{(2)})\\
&= \exp\left\{-\frac{1}{2}n^{1-2\xi}\pi^2\rho^2\|A\|_2^2 - n^{1/2-\xi}\pi^2\rho\sigma k_2 + O(n^{1-\gamma\xi}) + O(\log n)\right\}.
\end{aligned}$$

3.3 *The shortcoming between powers of the Neyman-Pearson and the Cramér-von Mises tests*

Consider $\alpha_n^{(2)}$ and k_2 as in (3.22), the related test $\mathcal{V}^{(2)}_{n,p_n,\alpha_n^{(2)}}$ and its power $\beta^{(2)}(\alpha_n^{(2)}, p_n)$.

The question is: Under which $\{p_n\}$'s, or equivalently, under which ξ's and a's, does there exist $k_1(n) = k_2 + o(1)$, such that

$$P_0(V_n^{(1)} \geq k_1(n) + \sqrt{n}b^{(1)}(p_n)) = \alpha_n^{(2)}\,? \tag{3.23}$$

Obviously, the existence of such $\{k_1(n)\}$ is equivalent to the existence of $\mathcal{V}^{(1)}_{n,p_n,\alpha_n^{(2)}}$ with power $\beta^{(1)}(\alpha_n^{(2)}, p_n)$.

For the two tests $\mathcal{V}^{(1)}_{n,p_n,\alpha_n^{(2)}}$ and $\mathcal{V}^{(2)}_{n,p_n,\alpha_n^{(2)}}$ set $R_n^{(2)}(\alpha_n^{(2)}, p_n) = \beta^{(1)}(\alpha_n^{(2)}, p_n) - \beta^{(2)}(\alpha_n^{(2)}, p_n)$. The quantity $R_n^{(2)}(\alpha_n^{(2)}, p_n)$ is called **the shortcoming** of $\mathcal{V}^{(2)}_{n,p_n,\alpha_n^{(2)}}$. In view of Theorems 3.1.(2) and 3.2.(2), the question posed above is equivalent to the following one: Under which p_n's (or equivalently, under which ξ's and a's) does

$$\lim_{n\to\infty} R_n^{(2)}(\alpha_n^{(2)}, p_n) = 0\,? \tag{3.24}$$

Property (3.24) is called **vanishing shortcoming**. Vanishing shortcoming means that both tests (the Neyman-Pearson and Cramér-von Mises) have the same asymptotic power and the same probability of type I error.

The conditions for (3.24) are given in Theorem 3.3.(2)(i), below. For a better understanding of the result note that if $\{p_n\} \in \mathcal{P}$, then $\|A\|_2 \leq \pi^{-1}$ and $\sigma \leq \pi^{-2}$

(cf. Lemma 4.(2)). Moreover, in these inequalities equality holds if and only if $a(x) = \sqrt{2}\cos(\pi x)$. Some more comments are given in Remark 3.3.(2).

THEOREM 3.3.(2) (i) *Suppose* $\{p_n\} \in \mathcal{P}$, $\xi \in (1/4, 1/2)$ *and* $\alpha_n^{(2)}$ *is given by* (3.22). *Then*

$$\lim_{n\to\infty} R_n^{(2)}(\alpha_n^{(2)}, p_n) = 0,$$

if and only if $a(x) = \sqrt{2}\cos(\pi x)$. *Moreover, for any other function* a, $\lim_{n\to\infty}\beta^{(1)}(\alpha_n^{(2)}, p_n) = 1$ *and* $R_n^{(2)}(\alpha_n^{(2)}, p_n)$ *does not converge to* 0.

(ii) *Consider the following "shifted" alternatives from* $\mathcal{P}$

$$p_n^{(d)}(x) = 1 + n^{-\xi} d\rho a(x), \qquad d \geq 1, \tag{3.25}$$

and set

$$\begin{aligned} \alpha_n^{(2)}(d) &= P_0(V_n^{(2)} \geq n^{1-2\xi} d^2\rho^2\|A\|_2^2 + 2n^{1/2-\xi} d\rho\sigma k_2), \\ D &= (\pi\|A\|_2)^{-1}. \end{aligned} \tag{3.26}$$

Then it holds that

(ii-a) *If* $d = D$ *then*

$$\lim_{n\to\infty}\{\beta^{(1)}(\alpha_n^{(2)}(d), p_n) - \beta^{(2)}(\alpha_n^{(2)}(d), p_n^{(d)})\} = \Phi(k_2) - \Phi(\kappa k_2),$$

where $\kappa = \pi\sigma\|A\|_2^{-1} = \pi^2 d\sigma$.

(ii-b) *If* $d < D$ *then*

$$\lim_{n\to\infty}\beta^{(1)}(\alpha_n^{(2)}(d), p_n) = 1, \qquad \textit{while} \qquad \lim_{n\to\infty}\beta^{(2)}(\alpha_n^{(2)}(d), p_n^{(d)}) = 1 - \Phi(k_2).$$

(ii-c) *If* $d > D$ *then*

$$\lim_{n\to\infty}\beta^{(1)}(\alpha_n^{(2)}(d), p_n) = 0, \qquad \textit{while} \qquad \lim_{n\to\infty}\beta^{(2)}(\alpha_n^{(2)}(d), p_n^{(d)}) = 1 - \Phi(k_2).$$

(iii) *There exists a shift depending on* n, *say* d_n, *given by* $d_n = D + \epsilon_n$, *where* D *is defined above and* $\epsilon_n = n^{-1/2+\xi} D k_2(1-\kappa)/\rho$, *such that*

$$\lim_{n\to\infty}\{\beta^{(1)}(\alpha_n^{(2)}(d_n), p_n) - \beta^{(2)}(\alpha_n^{(2)}(d_n), p_n^{(d_n)})\} = 0.$$

A proof of this theorem is given in the Appendix.

Remark 3.3.(2) Part (i) of Theorem 3.3.(2) is a counterpart of (2.12) of Theorem 2.2 of Neuhaus (1976). Parts (ii-a)–(ii-c) show how to shift the alternative p_n to get a given difference in the asymptotic powers of Neyman-Pearson test for p_0 against p_n and the CvM test under $p_n^{(d)}$. Point (iii) shows that shifting an alternative by d_n yields asymptotically the same (local) power for CvM and NP. For a simple relation of this result to the intermediate efficiency result see Remark 4.(2).

We illustrate part (ii-a) of Theorem 3.3.(2) by a small simulation study. We consider here the alternatives $p(x) = 1 + \rho\cos(2\pi x)$ and $p^{(d)}(x) = 1 + d\rho\cos(2\pi x)$. By (ii-a),

Table 2. Empirical shortcoming between NP and CvM under alternatives $p(x) = 1+\rho\cos(2\pi x)$ and $p^{(d)} = 1 + d\rho\cos(2\pi x)$.

$n = 100$, $\alpha = .05$, $d = 2$						
NP		CvM			shortcoming	
ρ	$\hat\beta^{(1)}(\alpha,p)$	$d\rho$	$\hat\beta^{(2)}(\alpha,p^{(d)})$	$\hat k_2$	$\hat R_n^{(2)}$	$\hat R_\infty$
.45	.9329	.90	.9992	−3.16	−.0663	−.0560
.40	.8761	.80	.9866	−2.22	−.1105	−.1205
.35	.7732	.70	.9081	−1.33	−.1349	−.1612
.30	.6451	.60	.7467	−.66	−.1016	−.1166
.25	.5064	.50	.5026	−.01	.0038	−.0013
.20	.3635	.40	.2923	.55	.0712	.1000
.15	.2442	.30	.1581	1.00	.0861	.1500

Table 3. Empirical powers of NP and CvM under $p(x) = 1 + \rho\cos(2\pi x)$ and sample sizes n and $n\mathcal{E}^{-1}$, respectively.

$\alpha = .05$			$\alpha = .01$		
	NP	CvM		NP	CvM
	$n = 50$	$n = 200$		$n = 100$	$n = 400$
ρ	$\hat\beta^{(1)}(\alpha,p)$	$\hat\beta^{(2)}(\alpha,p)$	ρ	$\hat\beta^{(1)}(\alpha,p)$	$\hat\beta^{(2)}(\alpha,p)$
.90	.9999	1.0000	.70	.9980	1.0000
.80	.9987	1.0000	.60	.9757	.9999
.70	.9829	.9998	.55	.9415	.9982
.60	.9323	.9888	.50	.8776	.9851
.55	.8860	.9632	.45	.7833	.9280
.50	.8151	.9030	.40	.6621	.7838
.45	.7371	.7971	.35	.5115	.5451
.40	.6387	.6434	.30	.3672	.3183
.35	.5384	.4727	.25	.2475	.1458
.30	.4307	.3228	.20	.1468	.0665
.25	.3305	.2065	.15	.0820	.0334
.20	.2477	.1339	.10	.0404	.0190
.15	.1762	.0898			
.10	.1138	.0648			
.05	.0711	.0506			

(3.26), (4.18) and (4.19) we get $d = D = 2$ and $\kappa = 1/2$. In Table 2 we use the following notation. $\hat\beta^{(1)}(\alpha,p)$ is the simulated power under p of the NP test at the level α. Similarly, $\hat\beta^{(2)}(\alpha,p^{(d)})$ is the simulated power of the CvM test under $p^{(d)}$ with $d = D = 2$.

So, from (ii-a) one could expect $\hat R_n^{(2)} = \hat\beta^{(1)}(\alpha,p) - \hat\beta^{(2)}(\alpha,p^{(d)}) \approx R_\infty$ where $R_\infty = \Phi(k_2) - \Phi(\kappa k_2)$. Obviously R_∞ can also be calculated only approximately. This approximate value of R_∞ we denote $\hat R_\infty$ and calculate inserting $k_2 = \hat k_2$ into the above formula , where $\hat k_2$ is such that $\hat\beta^{(2)}(\alpha,p^{(d)}) = 1 - \Phi(\hat k_2)$. A comparison of $\hat R_n^{(2)}$ with

$\hat{R}_\infty$ shows that we get very similar results, as long as the alternative is not too close to the null hypothesis. For alternatives close to the null hypothesis power functions are rather flat and this causes bigger errors. Table 2 exemplifies the typical power behaviour we observed in a more extensive simulation study we do not present here. Obviously, one can also investigate R_∞ analytically. E.g. in the case of $a(x) = \sqrt{2}\cos(2\pi x)$ we have $\kappa = 1/2$ and R_∞ attains its extremal values at $k_2 = \pm 1.360$ with respective values $R_\infty = \pm 0.1614$.

Point (iii) of Theorem 3.3.(2) supplements (ii-a) by showing that slightly greater than D shift d_n, $d_n \to D$ as $n \to \infty$, guarantees that asymptotically the NP and CvM tests have the same power. This result can be equivalently expressed in terms of intermediate efficiency of CvM to NP. This is detaily described in Remark 4.(2). The remark states that the efficiency, say $\mathcal{E}$, equals to D^{-2}, $D^{-2} \le 1$. It means that we need approximately $\mathcal{E}^{-1}$ times more observations to ensure that the CvM test will achieve the same power as the NP test. Table 3 illustrates how this asymptotic result works for finite samples. For the alternative $p(x) = 1 + \rho\cos(2\pi x)$, we consider, we have $D = 2$ and $\mathcal{E} = 1/4$.

3.4 *Data-driven Neyman test*

To formulate counterparts of Theorems 3.1.(1) and 3.2.(1) for N_{S2}, we shall first introduce some notation and make some comments. Set

$$V_n^{(3)} = N_{S2} \tag{3.27}$$

and

$$\hat{a} = (\hat{a}_1, \hat{a}_2, \ldots) \quad \text{with} \quad \hat{a}_j = \int_0^1 a(x)\Phi_j(x)dx. \tag{3.28}$$

Recall that $m_n = [cn^{1/8}]$ and $|\hat{a}|_{m_n} = \{\sum_{j=1}^{m_n} \hat{a}_j^2\}^{1/2}$. To get the limiting distribution of $V_n^{(3)}$ under p_n, we shall apply Theorem 4.2 of Inglot (1999). To this end, we have to ensure that $\{p_n\}$ from $\mathcal{P}$ belongs to $\mathcal{P}^*_{\Phi m}$ (or equivalently satisfies (3.5), (4.1) and (4.2) in the notation of that paper). Now, (4.1) and (4.2) will be satisfied, if $\xi \in (1/8, 3/8)$, but (3.5) is trivially satisfied as $a \in L_2[0,1]$ (see Remark 3.1 and Example 4.4 in Inglot (1999)). We shall also use Theorem 4.2 of Inglot (1999) in the more restricted case, $\xi \in (5/16, 3/8)$ and $a \in W_2^1$, where W_2^1 is the Sobolev space of absolutely continuous functions, whose derivatives belong to $L_2[0,1]$. Then by (7.4) of Barron and Sheu (1991) we get $1 - |\hat{a}|^2_{m_n} = O(m_n^{-2}) = O(n^{-1/4})$. This allows us to replace the centering constant $n^{1-2\xi}\rho^2|\hat{a}|^2_{m_n}$ in Theorem 3.1.(3)(i) by $n^{1-2\xi}\rho^2$ (see (ii) below).

THEOREM 3.1.(3) (i) *Suppose that* $\{p_n\} \in \mathcal{P}$ *and* $\xi \in (1/8, 3/8)$. *Then*

$$\lim_{n\to\infty} P_n\left(\frac{V_n^{(3)} - n^{1-2\xi}\rho^2|\hat{a}|^2_{m_n}}{2n^{1/2-\xi}\rho|\hat{a}|_{m_n}} \le x\right) = \Phi(x), \qquad x \in R.$$

(ii) *If* $\{p_n\} \in \mathcal{P}$ *is such that* $a \in W_2^1$ *and* $\xi \in (5/16, 3/8)$, *then*

$$\lim_{n\to\infty} P_n\left(\frac{V_n^{(3)} - n^{1-2\xi}\rho^2}{2n^{1/2-\xi}\rho} \le x\right) = \Phi(x), \qquad x \in R.$$

The following theorem can be obtained along the same line of argument as Theorem 4.3 in Inglot *et al.* (1998*a*). As above, we take $m_n = [cn^{1/8}]$ and $\omega = 1/2$ in the proof.

THEOREM 3.2.(3) *Let $\{x_n\}$ satisfy $x_n \to 0$, $nx_n^2 \to \infty$ and $m_n x_n \to 0$. Then*

$$P_0(V_n^{(3)} \geq nx_n^2) \leq \exp\left\{-\frac{1}{2}nx_n^2(1-\eta_n) + \frac{1}{2}m_n \log(enx_n^2/2) + O(1)\right\},$$

where $0 \leq \eta_n \leq Cm_n \max\{x_n, (m_n n^{-1}\log n)^{1/2}\}$ for some constant $C > 0$. In particular, for $\{x_n\}$ such that $n^{1/8}x_n \to 0$ and $n^{3/8}x_n \to \infty$ we have

$$P_0(V_n^{(3)} \geq nx_n^2) \leq \exp\left\{-\frac{1}{2}nx_n^2 + O(n^{9/8}x_n^3) + O(n^{1/8}\log n)\right\}.$$

As in previous cases, for a fixed $k_3 \in R$ set

$$\begin{aligned} C_n^{(3)} &= \{V_n^{(3)} \geq n^{1-2\xi}\rho^2 + 2n^{1/2-\xi}\rho k_3\}, \\ \alpha_n^{(3)} &= P_0(C_n^{(3)}), \\ \mathcal{V}^{(3)}_{n,p_n,\alpha_n^{(3)}} &\triangleq \mathcal{V}^{(3)}_{n,\alpha_n^{(3)}} = \mathbf{1}(C_n^{(3)}) \\ \beta^{(3)}(\alpha_n^{(3)}, p_n) &= E_{P_n}\mathcal{V}^{(3)}_{n,\alpha_n^{(3)}} = P_n(C_n^{(3)}). \end{aligned} \tag{3.29}$$

3.5 *Shortcoming between powers of the Neyman-Pearson and the data- driven Neyman tests*

Consider $\alpha_n^{(3)}$ as in (3.29) and set

$$R_n^{(3)}(\alpha_n^{(3)}, p_n) = \beta^{(1)}(\alpha_n^{(3)}, p_n) - \beta^{(3)}(\alpha_n^{(3)}, p_n).$$

THEOREM 3.3.(3) *Suppose $\{p_n\} \in \mathcal{P}$, $\xi \in (5/16, 3/8)$ and $a \in W_2^1$. Then*

$$\lim_{n\to\infty} R_n^{(3)}(\alpha_n^{(3)}, p_n) = 0.$$

The proof of this theorem is based on Theorem 3.1.(3)(ii) and Theorem 3.2.(3) with $x_n^2 = n^{-2\xi}\rho^2 + 2n^{-1/2-\xi}\rho k_3$ and is given in the Appendix.

Remark 3.3.(3) Theorem 3.3.(3) shows that $V_n^{(3)}$ has qualitatively different properties to $V_n^{(2)}$. Vanishing shortcoming takes place for any smooth ($a \in W_2^1$) departure from p_0.

4. Intermediate efficiency

This section deals with the comparison of $V_n^{(1)}$, $V_n^{(2)}$ and $V_n^{(3)}$ using the notion of intermediate efficiency. We shall start with a definition of this notion and a basic theorem allowing to calculate it. Both the definition and theorem are stated in the framework of Inglot (1999). On the other hand, the statement is adopted to the framework of the present paper. To ensure maximal precision when defining the notion, we shall use new notation. The test statistics shall be denoted by $T_{n,p_k}^{(i)}$, while related tests by $\mathcal{T}_{n,p_k,\alpha}^{(i)}$. Then the theorem is formulated for $T_{n,p_n}^{(i)}$. Subsections 4.2 and 4.3 shall be devoted to the calculation of the efficiencies of $V_n^{(2)}$ and $V_n^{(3)}$ with respect to $V_n^{(1)}$.

4.1 *The notion of efficiency and main theorem*

When defining intermediate efficiency, we shall consider a general sequence $\{p_n\}$ of alternative densities to p_0. Later on we shall restrict attention to some contamination models.

Assume we have a sequence $\{p_n\}$ of densities on [0,1] satisfying

$$\lim_{n\to\infty} H(p_n,p_0)=0 \quad \text{and} \quad \lim_{n\to\infty} nH^2(p_n,p_0)=\infty. \tag{4.1}$$

Let $T^{(i)}_{n,p_k}$, $i=1,2$, be two test statistics for $H_0: p(x)=p_0(x)$ against $H_1: p(x)=p_k(x)$ based on a sample of size n. Moreover, let $\mathcal{T}^{(i)}_{n,p_k,\alpha}$, $i=1,2$, be two tests at significance level α rejecting H_0 for large values of $T^{(i)}_{n,p_k}$. Denote by $\sqrt{n}t^{(i)}_{n,p_k,\alpha}$, $i=1,2$, the respective critical values corresponding to level α. We have

$$P_0(T^{(i)}_{n,p_k}\ge \sqrt{n}t^{(i)}_{n,p_k,\alpha})\le\alpha \quad \text{and} \quad P_0(T^{(i)}_{n,p_k}\ge d)>\alpha \quad \text{for all } d<\sqrt{n}t^{(i)}_{n,p_k,\alpha}. \tag{4.2}$$

Let $\{\alpha_n\}$ be a sequence of significance levels such that

$$\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty} n^{-1}\log\alpha_n=0 \tag{4.3}$$

and

$$0<\liminf_{n\to\infty} P_n(T^{(2)}_{n,p_n}\ge\sqrt{n}t^{(2)}_{n,p_n,\alpha_n})\le\limsup_{n\to\infty} P_n(T^{(2)}_{n,p_n}\ge\sqrt{n}t^{(2)}_{n,p_n,\alpha_n})<1. \tag{4.4}$$

Set

$$\begin{aligned} &N_{T^{(2)}T^{(1)}}(n,\{p_n\})=N_{T^{(2)}T^{(1)}}(n,\{p_n\},\{\alpha_n\})\\ &=\inf\{N: P_n(T^{(1)}_{N+k,p_n}\ge\sqrt{N+k}t^{(1)}_{N+k,p_n,\alpha_n})\ge P_n(T^{(2)}_{n,p_n}\ge\sqrt{n}t^{(2)}_{n,p_n,\alpha_n}) \text{ for all } k\ge 0\}.\end{aligned}$$

DEFINITION. Suppose $\{p_n\}$ satisfies (4.1) and there exists $\{\alpha_n\}$ satisfying (4.2), (4.3) and (4.4) with $p_k=p_n$. Assume

$$\lim_{n\to\infty}\frac{N_{T^{(2)}T^{(1)}}(n,\{p_n\})}{n}=e_{T^{(2)}T^{(1)}}(\{p_n\}) \tag{4.5}$$

and this limit does not depend on a particular choice of $\{\alpha_n\}$. Then we shall say that the asymptotic intermediate efficiency of $\{T^{(2)}_{n,p_n}\}$ with respect to $\{T^{(1)}_{n,p_n}\}$ under $\{p_n\}$ exists and equals $e_{T^{(2)}T^{(1)}}(\{p_n\})$.

As said before, for simplicity of presentation we introduce here the approach for a contamination model and do comparisons of $V_n^{(2)}$ and $V_n^{(3)}$ with respect to $V_n^{(1)}$. So $V_n^{(1)}$ shall play the role of $T^{(1)}_{n,p_n}$ in the above approach. Before stating the theorem providing tools to compare $V_n^{(i)}$, $i=2,3$, with $V_n^{(1)}$ we first introduce the auxiliary contamination family and list some properties of $V_n^{(1)}$, which are needed for the comparison.

Set

$$\mathcal{P}_c=\{\{p_n\}: p_n(x)=1+\theta_n a(x),\ a\in\mathcal{A},\ \theta_n\to 0,\ n\theta_n^2\to\infty\}. \tag{4.6}$$

Obviously, we have $\mathcal{P}\subset\mathcal{P}_c$ and for $\{p_n\}\in\mathcal{P}_c$ the condition (4.1) holds. Moreover, for $\{p_n\}\in\mathcal{P}_c$, (3.5) implies that the function $b^{(1)}(\cdot)$ defined in (3.4) satisfies

$$b^{(1)}(p_n)=2H(p_n,p_0)+O(H^2(p_n,p_0))=\theta_n+O(\theta_n^2). \tag{4.7}$$

By Theorems 5.3 and 5.8 of Inglot and Ledwina (1996) it follows that for any $\{p_n\} \in \mathcal{P}_c$

$$(4.8)\quad \lim_{n\to\infty} P_n\left(\left|\frac{V_n^{(1)}}{\sqrt{n}b^{(1)}(p_n)} - 1\right| \le \epsilon\right) = 1 \quad \text{for each} \quad \epsilon > 0,$$

$$(4.9)\quad -\lim_{n\to\infty} \frac{1}{nx_n^2} \log P_0(V_n^{(1)} \ge x_n\sqrt{n}) = c^{(1)} = \frac{1}{2} \quad \text{for any} \quad x_n \to 0,\ nx_n^2 \to \infty.$$

By Remark 3.3 of Inglot and Ledwina (1996), relations (4.7) and (4.9) imply that the expression

$$(4.10)\qquad c^{(1)}\{b^{(1)}(p_n)\}^2 = \frac{1}{2}\theta_n^2 + O(\theta_n^3)$$

is the intermediate slope of $V_n^{(1)}$.

The following theorem is a special case of Theorem 2.7 in Inglot (1999) (cf. also Lemma 3.2 in Inglot and Ledwina (1996)). We would like to emphasize that the $\{p_n\}$ appearing there are chosen from $\mathcal{P}$ only (cf. also Remark 4.1).

THEOREM 4.1. *Suppose $\{T_{n,p_n}^{(2)}\}$ is a sequence of test statistics for testing p_0 against p_n. Let $\{p_n\} \in \mathcal{P}$ and $\xi \in (0, 1/2)$. Assume that*

(i) *there exists a sequence $\{b^{(2)}(p_n)\}$, such that for each $\epsilon > 0$*

$$\lim_{n\to\infty} P_n\left(\left|\frac{T_{n,p_n}^{(2)}}{\sqrt{n}b^{(2)}(p_n)} - 1\right| \le \epsilon\right) = 1,$$

(ii) *there exists a constant $c^{(2)}$ and a bounded sequence $\{q_n\}$, such that for any $x_n = o(q_n)$, $nx_n^2 \to \infty$*

$$-\lim_{n\to\infty} \frac{1}{nx_n^2} \log P_0(T_{n,p_n}^{(2)} \ge x_n\sqrt{n}) = c^{(2)},$$

(iii) *there exists a sequence $\{\alpha_n\}$ satisfying* (4.2), (4.3) *and* (4.4), *such that* $\log \alpha_n = o(nq_n^2)$,

(iv) *the following limit exists*

$$\lim_{n\to\infty} \frac{c^{(2)}}{c^{(1)}} \left\{\frac{b^{(2)}(p_n)}{b^{(1)}(p_n)}\right\}^2 = \mathcal{E}(\{p_n\}).$$

Then the asymptotic intermediate efficiency $e_{T^{(2)}T^{(1)}}(\{p_n\})$ of $\{T_{n,p_n}^{(2)}\}$ with respect to $\{T_{n,p_n}^{(1)}\}$ under $\{p_n\}$ exists and equals $\mathcal{E}(\{p_n\})$.

Remark 4.1. Note that the intermediate efficiency notion is not symmetric with respect to the two test statistics. In view of this we need stronger assumptions for $\{T_{n,p_n}^{(1)}\}$ than for $\{T_{n,p_n}^{(2)}\}$. In particular, we require (4.8) and (4.9) for $T_{n,p_n}^{(1)}$ for a rich enough family of sequences. Checking (4.8) and (4.9) for an actual sequence $\{p_n\} \in \mathcal{P}$ is far from being sufficient. The minimal assumption we have to impose is so- called renumerability (cf. Definition 2.6 in Inglot (1999)). Here we take a family $\mathcal{P}_c$, which is obviously renumerable.

4.2 *The intermediate efficiency of $V_n^{(2)}$ with respect to $V_n^{(1)}$.*

Let $\{p_n\} \in \mathcal{P}$. We shall check that for $T_{n,p_n}^{(2)} = \{V_n^{(2)}\}^{1/2}$ conditions (i)–(iv) of Theorem 4.1 are fulfilled.

Taking

$$b^{(2)}(p_n) = n^{-\xi}\rho\|A\|_2, \tag{4.11}$$

condition (i) becomes

$$\begin{aligned} P_n & \left(\left| \frac{\{V_n^{(2)}\}^{1/2}}{n^{1/2-\xi}\rho\|A\|_2} - 1 \right| \geq \epsilon \right) \\ &= P_n \left(\frac{V_n^{(2)} - n^{1-2\xi}\rho^2\|A\|_2^2}{2n^{1/2-\xi}\rho\sigma} \geq n^{1/2-\xi}\frac{\epsilon^2+2\epsilon}{2\sigma}\rho\|A\|_2^2 \right) \\ &+ P_n \left(\frac{V_n^{(2)} - n^{1-2\xi}\rho^2\|A\|_2^2}{2n^{1/2-\xi}\rho\sigma} \leq n^{1/2-\xi}\frac{\epsilon^2-2\epsilon}{2\sigma}\rho\|A\|_2^2 \right). \end{aligned} \tag{4.12}$$

By (3.19) of Theorem 3.1(2), this tends to 0 if $\epsilon \in (0,2)$ and condition (i) holds with $b^{(2)}(p_n)$ given by (4.11).

Theorem 3.2(2) implies that (ii) is satisfied with

$$c^{(2)} = \frac{\pi^2}{2}, \tag{4.13}$$

and $q_n \equiv 1$.

Taking $\alpha_n = \alpha_n^{(2)}$ (see Subsection 3.2) and $t_{n,p_n,\alpha_n}^{(2)} = \{n^{-2\xi}\rho^2\|A\|_2^2 + 2n^{-1/2-\xi}\rho\sigma k_2\}^{1/2}$, by (3.22) and (3.21), (iii) is fulfilled with $q_n \equiv 1$.

By (4.6), (4.7), (4.9), (4.11) and (4.13)

$$\lim_{n\to\infty} \frac{c^{(2)}}{c^{(1)}} \left\{ \frac{b^{(2)}(p_n)}{b^{(1)}(p_n)} \right\}^2 = \lim_{n\to\infty} \frac{\frac{1}{2}\pi^2 n^{-2\xi}\rho^2\|A\|_2^2}{\frac{1}{2}n^{-2\xi}\rho^2(1+o(1))} = \{\pi\|A\|_2\}^2.$$

Therefore (iv) holds with

$$\mathcal{E}(\{p_n\}) = \{\pi\|A\|_2\}^2. \tag{4.14}$$

By the above, for each $\{p_n\} \in \mathcal{P}$ the intermediate efficiency of $\{(V_n^{(2)})^{1/2}\}$ with respect to $\{V_n^{(1)}\}$ exists and

$$e_{V^{(2)}V^{(1)}}(\{p_n\}) = \mathcal{E}(\{p_n\}) = \{\pi\|A\|_2\}^2. \tag{4.15}$$

To interpret result (4.15), we shall first state the following auxiliary technical lemma.

LEMMA 4.(2) *Suppose $a(x)$ is such that $\int_0^1 a(x)dx = 0$ and $\int_0^1 a^2(x)dx = 1$. Set*

$$\eta_k(x) = \sqrt{2}\cos(\pi k x) \quad \textit{and} \quad c_k = \int_0^1 a(x)\eta_k(x)dx. \tag{4.16}$$

Then $A(x)$ has the following representation in $L_2[0,1]$

$$A(x) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{c_k}{k}\sqrt{2}\sin(\pi k x). \tag{4.17}$$

Moreover,

$$\|A\|_2^2 = \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{c_k^2}{k^2}, \tag{4.18}$$

and

$$\sigma^2 = \frac{1}{\pi^4}\sum_{k=1}^{\infty}\frac{c_k^2}{k^4}. \tag{4.19}$$

Since $\int_0^1 a^2(x)dx = 1$, then $\|A\|_2^2 \le \pi^{-2}$, $\sigma^2 \le \pi^{-4}$, with the equalities holding if and only if $a(x) = \eta_1(x)$.

The proof of Lemma 4.3 can be found in Inglot *et al.* (1998*b*).

Remark 4.(2) Relations (4.15) and (4.18) imply that the asymptotic intermediate efficiency of $\{(V_n^{(2)})^{1/2}\}$ with respect to $\{V_n^{(1)}\}$ under $\{p_n\} \in \mathcal{P}$ is always smaller or equal to 1. It is 1 if and only if $a(x) = \eta_1(x)$. Moreover, for any other direction η_k, $k > 1$, the efficiency is smaller and decreases monotonically as k increases. This observation provides a counterpart of Corollary 2.6 in Neuhaus (1976). However, contrary to Neuhaus's approach, the intermediate approach provides explicit quantitative results. In particular, the efficiency of $\{(V_n^{(2)})^{1/2}\}$ with respect to $\{V_n^{(1)}\}$ in the direction η_k equals $1/k^2$.

We also see that for $\{p_n\} \in \mathcal{P}$ we have $\mathcal{E}(\{p_n\}) = D^{-2}$ in the notation of Theorem 3.3.(2). Looking at (iii) of this theorem, we see that "shifting" an alternative p_n by $1/\sqrt{\mathcal{E}(\{p_n\})} + \epsilon_n$ the CvM test has the same asymptotic power as the most powerful test. It shows that the notion $\mathcal{E}(\{p_n\})$ posesses an interpretation as ratio of sample sizes and is closely and in a traditional way related to the shift necessary to get comparable powers of both tests.

4.3 *The intermediate efficiency of $V_n^{(3)}$ with repect to $V_n^{(1)}$*

Similar to the case of $V_n^{(2)}$, for $\{p_n\} \in \mathcal{P}$ and $\xi \in (1/8, 3/8)$ we shall check the assumptions of Theorem 4.1 for $T_{n,p_n}^{(2)} = \{V_n^{(3)}\}^{1/2}$. As before $V_n^{(3)}$ is determined by $m_n = [cn^{1/8}]$.

Taking

$$b^{(2)}(p_n) = n^{-\xi}\rho|\hat{a}|_{m_n} \tag{4.20}$$

(cf. (3.27) and what follows), we have

$$\begin{aligned} P_n\left(\left|\frac{\{V_n^{(3)}\}^{1/2}}{n^{1/2-\xi}\rho|\hat{a}|_{m_n}} - 1\right| \ge \epsilon\right) \\ = P_n\left(\frac{V_n^{(3)} - n^{1-2\xi}\rho^2|\hat{a}|_{m_n}^2}{2n^{1/2-\xi}\rho|\hat{a}|_{m_n}} \ge n^{1/2-\xi}\frac{\epsilon^2+2\epsilon}{2}\rho|\hat{a}|_{m_n}\right) \end{aligned} \tag{4.21}$$

$$+P_n\left(\frac{V_n^{(3)}-n^{1-2\xi}\rho^2|\hat{a}|_{m_n}^2}{2n^{1/2-\xi}\rho|\hat{a}|_{m_n}}\le n^{1/2-\xi}\frac{\epsilon^2-2\epsilon}{2}\rho|\hat{a}|_{m_n}\right).$$

Since $|\hat{a}|_{m_n}\to 1$ as $n\to\infty$, by Theorem 3.1(3), this tends to 0 for $\epsilon\in(0,2)$ and condition (i) is satified with $b^{(2)}(p_n)$ given by (4.20).

Since $P_0(V_n^{(3)}\ge nx_n^2)\ge P_0(\frac{1}{n}\sum_{i=1}^n\Phi_1(X_i)\ge\sqrt{n}x_n)$, by Proposition 7.8 in Inglot and Ledwina (1996) and Theorem 3.2(3), if $x_n=o(n^{-1/8})$ and $nx_n^2\to\infty$ then

$$-\lim_{n\to\infty}\frac{1}{nx_n^2}\log P_0(V_n^{(3)}\ge nx_n^2)=\frac{1}{2}. \tag{4.22}$$

Therefore (ii) follows with

$$c^{(2)}=\frac{1}{2} \tag{4.23}$$

and $q_n=n^{-1/8}$.

Taking $\alpha_n=\alpha_n^{(3)}$ (cf. (3.29)) and $t_{n,p_n,\alpha_n}^{(3)}=\{n^{-2\xi}\rho^2|\hat{a}|_{m_n}^2+2n^{-1/2-\xi}\rho|\hat{a}|_{m_n}k_3\}^{1/2}$, by (4.22), $\log\alpha_n^{(3)}=O(n^{1-2\xi})$ and $(\log\alpha_n^{(3)})/n^{1-2\xi}\to 1/2$. Hence (4.3) follows. Since $\xi>1/8$, then $\log\alpha_n^{(3)}=o(nq_n^2)$. Moreover, Theorem 3.1.(3) yields (4.4). Hence (iii) is fulfilled.

By (4.6), (4.7), (4.9), (4.20) and (4.23)

$$\lim_{n\to\infty}\frac{c^{(2)}}{c^{(1)}}\left\{\frac{b^{(2)}(p_n)}{b^{(1)}(p_n)}\right\}^2=\lim_{n\to\infty}\frac{\frac{1}{2}n^{-2\xi}\rho^2|\hat{a}|_{m_n}^2}{\frac{1}{2}n^{-2\xi}\rho^2(1+o(1))}=\lim_{n\to\infty}|\hat{a}|_{m_n}^2=1. \tag{4.24}$$

By the above, for each $\{p_n\}\in\mathcal{P}$ with $\xi\in(1/8,3/8)$, the intermediate efficiency of $\{V_n^{(3)}\}^{1/2}$ with respect to $\{V_n^{(1)}\}$ exists and $e_{V^{(3)}V^{(1)}}(\{p_n\})=\mathcal{E}(\{p_n\})=1$ irrespective of the direction a.

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Appendix

Proof of Theorem 3.3.(2). Let $\alpha_n^{(2)}$ be given by (3.22) and $\xi\in(1/4,1/2)$. Choose γ sufficiently close to 3 that $O(n^{1-\gamma\xi})=o(n^{1/2-\xi})$. Applying Theorem 3.2.(1), with $x_n=\frac{1}{2}\pi\|A\|_2b^{(1)}(p_n)$, we get

$$P_0\left(V_n^{(1)}-\sqrt{n}b^{(1)}(p_n)\ge\left(\frac{1}{2}\pi\|A\|_2-1\right)\sqrt{n}b^{(1)}(p_n)\right)$$
$$=\exp\left\{-\frac{1}{8}n^{1-2\xi}\pi^2\|A\|_2^2\rho^2(1+o(1))\right\},$$

which is greater than $\alpha_n^{(2)}$ for large n. Hence it follows that there exists a unique $k_1(n)$ such that

$$k_1(n) > \left(\frac{1}{2}\pi\|A\|_2 - 1\right)\sqrt{n}b^{(1)}(p_n) \tag{A.1}$$

and $\alpha_n^{(1)} = \alpha_n^{(2)} = P_0(V_n^{(1)} - \sqrt{n}b^{(1)}(p_n) \geq k_1(n))$. On the other hand, by Theorem 3.1.(1) $\beta^{(1)}(\alpha_n^{(2)}, p_n) - (1 - \Phi(k_1(n))) \to 0$, which together with (3.21) gives

$$\{\beta^{(1)}(\alpha_n^{(2)}, p_n) - \beta^{(2)}(\alpha_n^{(2)}, p_n) - (\Phi(k_2) - \Phi(k_1(n)))\} \to 0. \tag{A.2}$$

As $\mathcal{V}^{(1)}_{n,p_n,\alpha_n^{(2)}}$ is the most powerful test, (A.2) implies $k_1(n) \leq k_2 + o(1)$. So, we can again apply (3.9) of Theorem 3.2.(1), taking $x_n = n^{-1/2}k_1(n) + b^{(1)}(p_n)$ and obtaining from (3.6)

$$\begin{aligned} \alpha_n^{(2)} &= \exp\left\{-\frac{1}{2}(k_1(n) + n^{1/2}b^{(1)}(p_n))^2 + o(n^{1/2-\xi})\right\} \\ &= \exp\left\{-\frac{1}{2}n^{1-2\xi}\rho^2 - n^{1/2-\xi}\rho k_1(n) - \frac{1}{2}k_1^2(n) + o(n^{1/2-\xi})\right\}. \end{aligned} \tag{A.3}$$

Comparing this equality to (3.22), we see that if $\pi\|A\|_2 = 1$, i.e. if $a(x) = \sqrt{2}\cos(\pi x)$ (cf. (4.18)), then

$$n^{1/2-\xi}\pi^2\rho\sigma k_2 = n^{1/2-\xi}\rho k_1(n) + \frac{1}{2}k_1^2(n) + o(n^{1/2-\xi}).$$

Consequently $k_1(n) = \pi^2\sigma k_2 + o(1)$ as the second root of this equation does not satisfy (A.1). Now, by (4.18) and (4.19) we see that $\pi^2\sigma = 1$ and $k_1(n) = k_2 + o(1)$. Hence by (A.2) $R_n^{(2)}(\alpha_n^{(2)}, p_n) \to 0$.

If $\pi\|A\|_2 < 1$, i.e. $a(x) \neq \sqrt{2}\cos(\pi x)$, then comparing (A.3) to (3.22), we have $k_1(n) = (\pi\|A\|_2 - 1)n^{1/2-\xi}\rho(1 + o(1)) \to -\infty$, i.e. $\beta^{(1)}(\alpha_n^{(2)}, p_n) \to 1$ as $n \to \infty$. This proves part (i) of the theorem.

To prove (ii) we use an argument analogous to the one in the proof of (i) obtaining

$$\alpha_n^{(2)}(d) = \exp\left\{-\frac{1}{2}n^{1-2\xi}\pi^2 d^2\rho^2\|A\|_2^2 - n^{1/2-\xi}\pi^2 d\rho\sigma k_2 + o(n^{1/2-\xi})\right\}$$

and

$$\alpha_n^{(2)}(d) = \exp\left\{-\frac{1}{2}n^{1-2\xi}\rho^2 - n^{1/2-\xi}\rho k_1(n,d) - \frac{1}{2}k_1^2(n,d) + o(n^{1/2-\xi})\right\}.$$

If $d = D$, then comparing the above expressions we get

$$\frac{1}{2}n^{-1/2+\xi}k_1^2(n,d) + \rho k_1(n,d) - \kappa\rho k_2 + o(1) = 0 \tag{A.4}$$

and consequently $k_1(n,d) = \kappa k_2 + o(1)$. By Theorem 3.1.(1) and (3.19) we see that $\lim_{n\to\infty}\{\beta^{(1)}(\alpha_n^{(2)}(d), p_n) - \beta^{(2)}(\alpha_n^{(2)}(d), p_n^{(d)})\} = \Phi(k_2) - \Phi(\kappa k_2)$, which proves (ii-a).

Similar analysis in the case $d < D$ ($d > D$) gives a related $k_1(n,d)$ tending to $-\infty$ ($+\infty$). This yields $\beta^{(1)}(\alpha_n^{(2)}(d), p_n) \to 1(0)$, respectively. This ends the proof of (ii).

To prove (iii), a slight modification of the above argument is needed. Therefore we again give some hints. First of all a counterpart of (3.26) follows from Theorem 5.1(a) in Inglot *et al.* (2000). Theorem 3.2.(2), with γ as before, gives us an expression for $\alpha_n^{(2)}(d_n)$. Applying Theorem 3.2.(1) with $x_n = D^{-1}b^{(1)}(p_n)$, there exists $k_1(n, d_n) > (D^{-1} - 1)\sqrt{n}b^{(1)}(p_n)$, such that

$$\alpha_n^{(2)}(d_n) = P_0(V_n^{(1)} - \sqrt{n}b^{(1)}(p_n) \geq k_1(n, d_n)). \tag{A.5}$$

Then application of Theorem 3.2.(1), with $x_n = 2b^{(1)}(p_n)$, allows us to infer that $k_1(n, d_n) < \sqrt{n}b^{(1)}(p_n)$ for n sufficiently large. This implies that we can get an expression for (A.5) applying Theorem 3.2.(1) again with $x_n = n^{-1/2}k_1(n, d_n) + b^{(2)}(p_n)$. The rest of the proof relies on comparing the two expressions for $\alpha_n^{(2)}(d_n)$ and exploiting the form of ϵ_n. This yields the equation for $k_1(n, d_n)$ with solution $k_1(n, d_n) = k_2 + o(1)$. Using Lemma 5.3 and Proposition 6.6 of Inglot and Ledwina (1996), we see that these more general CLTs conclude the proof.

Proof of Theorem 3.3.(3) Recall that $\{p_n\} \in \mathcal{P}$ with $\xi \in (5/16, 3/8)$ and

$$\alpha_n^{(3)} = P_0\left(\frac{V_n^{(3)} - n^{1-2\xi}\rho^2}{2n^{1/2-\xi}\rho} \geq k_3\right).$$

By Theorem 3.1.(3)(ii)

$$\lim_{n\to\infty} \beta_n^{(3)}(\alpha_n^{(3)}, p_n) = 1 - \Phi(k_3).$$

Set $x_n^2 = n^{-2\xi}\rho^2 + 2n^{-1/2-\xi}\rho k_3 = n^{-2\xi}\rho^2(1 + o(1))$. Since $\xi > 5/16$ and $\xi < 3/8$, we have $n^{9/8}x_n^3 = o(n^{1/2-\xi})$ and $n^{1/8}\log n = o(n^{1/2-\xi})$. By Theorem 3.2.(3) we get

$$\alpha_n^{(3)} \leq \exp\left\{-\frac{1}{2}nx_n^2 + o(n^{1/2-\xi})\right\} = \exp\left\{-\frac{1}{2}n^{1-2\xi}\rho^2 - n^{1/2-\xi}\rho k_3 + o(n^{1/2-\xi})\right\}. \tag{A.6}$$

Observe now that if we consider $V_n^{(1)}$ and $x_n = \frac{1}{2}b^{(1)}(p_n) = \frac{1}{2}n^{-\xi}\rho + O(n^{-2\xi})$ (cf. (3.6)), then by (3.9) of Theorem 3.2.(1), since $\xi > 1/4$

$$P_0\left(V_n^{(1)} \geq \frac{1}{2}\sqrt{n}b^{(1)}(p_n)\right) = \exp\left\{-\frac{1}{8}n^{1-2\xi}\rho^2 + o(n^{1/2-\xi})\right\}. \tag{A.7}$$

Comparing (5.6) and (5.7), we see that there exists $k_1(n) > -\frac{1}{2}\sqrt{n}b^{(1)}(p_n)$ such that $\alpha_n^{(3)} = P_0(V_n^{(1)} - \sqrt{n}b^{(1)}(p_n) \geq k_1(n))$. By (3.14) we get $\beta^{(1)}(\alpha_n^{(3)}, p_n) - (1 - \Phi(k_1(n))) \to 0$. Hence, for sufficiently large n, $k_1(n) \leq k_3 + o(1)$, since otherwise one would have $\beta^{(1)}(\alpha_n^{(3)}, p_n) < \beta^{(3)}(\alpha_n^{(3)}, p_n)$, which is obviously impossible. The above implies we can again apply Theorem 3.2.(1) for a new x_n given by $x_n = n^{-1/2}k_1(n) + b^{(1)}(p_n)$. This together with the assumption $\xi \in (5/16, 3/8)$ yields

$$\alpha_n^{(3)} = \exp\left\{-\frac{1}{2}n^{1-2\xi}\rho^2 - n^{1/2-\xi}\rho k_1(n) - \frac{1}{2}k_1^2(n) + o(n^{1/2-\xi})\right\}. \tag{A.8}$$

Comparing (A.8) and (A.6), we get $n^{1/2-\xi}\rho k_3 \leq n^{1/2-\xi}\rho k_1(n) + \frac{1}{2}k_1^2(n) + o(n^{1/2-\xi})$. Since, for sufficiently large n, $k_1(n) \geq -\frac{1}{2}n^{1/2-\xi}\rho$, this inequality yields $k_1(n) \geq k_3 + o(1)$. In this way we have proved that $k_1(n) = k_3 + o(1)$. Hence $\lim_{n\to\infty} R_n^{(3)}(\alpha_n^{(3)}, p_n) = 0$.

Orthogonality of product measures. Two probability measures P, Q with densities p, q with respect to some σ-finite measure are orthogonal if and only if $H^2(P,Q) = 2$, where $H^2(P,Q) = 2(1 - \int \sqrt{pq})$ (cf. Rao and Varadarajan (1963)). Let $P^{(n)}$ and $Q^{(n)}$ be n-fold products of P_{ni} and Q_{ni}, respectively. By (1.4) in Oosterhoff and van Zwet (1979)

$$H^2(P^{(n)}, Q^{(n)}) = 2 - 2\prod_{i=1}^{n}\left\{1 - \frac{1}{2}H^2(P_{ni}, Q_{ni})\right\}.$$

Hence, for the contamination model given by $p_n(x) = 1 + n^{-\xi}\rho a(x)$ with $\xi \in (0,1)$, from (3.5) and (3.6)

$$H^2(P_n^{(n)}, P_0^{(n)}) = 2 - 2\left\{1 - \frac{1}{8}n^{-2\xi}\rho^2(1+o(1))\right\}^n \to \begin{cases} 2 & \text{if } 2\xi < 1, \\ 2 - 2e^{-\rho^2/8} & \text{if } 2\xi = 1, \\ 0 & \text{if } 2\xi > 1. \end{cases}$$

This, in particular, proves the asymptotical orthogonality of the products in the contamination model (3.1).

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