Ann. Inst. Statist. Math.
Vol. 53, No. 4, 746–759 (2001)

ESTIMATION OF AN EXPONENTIAL QUANTILE UNDER A GENERAL LOSS AND AN ALTERNATIVE ESTIMATOR UNDER QUADRATIC LOSS

CONSTANTINOS PETROPOULOS AND STAVROS KOUROUKLIS

Department of Mathematics, University of Patras, 26 500 Rio, Patras, Greece,
e-mail: stavros@math.upatras.gr

(Received November 22, 1999; revised April 5, 2000)

Abstract. Estimation of the quantile $\mu + \kappa\sigma$ of an exponential distribution with parameters (μ, σ) is considered under an arbitrary strictly convex loss function. For κ obeying a certain condition, the inadmissibility of the best affine equivariant procedure is established by exhibiting a better estimator. The LINEX loss is studied in detail. For quadratic loss, sufficient conditions are given for a scale equivariant estimator to dominate the best affine equivariant one and, when κ exceeds a lower bound specified below, a new minimax estimator is identified.

Key words and phrases: Decision theory, Stein technique, Brewster and Zidek technique, equivariant estimation.

1. Introduction

Let $X_1, X_2, \ldots, X_n$, $n \geq 2$, be a random sample from a two–parameter exponential distribution $E(\mu, \sigma)$ with density given by $\sigma^{-1}\exp\{-\sigma^{-1}(x - \mu)\}$ or 0 according as $x \geq \mu$ or $x < \mu$. The location parameter μ and the scale parameter σ are unknown with $-\infty < \mu < \infty$ and $\sigma > 0$. The problem of interest is to estimate the linear function $\theta = \mu + \kappa\sigma$, where $\kappa \geq 0$ is a given constant, from a decision theoretic point of view. When $\kappa = -\ln p$, $0 < p \leq 1$, θ is the $100(1-p)$-th quantile of $E(\mu, \sigma)$. Quantile estimation, particularly for the exponential distribution, is of importance in reliability and life testing studies.

The best affine equivariant (b.a.e.) estimator of θ under the quadratic loss $(d - \theta)^2/\sigma^2$ is $\delta_0 = X + (\kappa - 1/n)S/n$, where $X = X_{(1)}$ (the minimum of X_i's) and $S = \sum_{i=1}^{n}(X_i - X_{(1)})$. Rukhin and Strawderman (1982) and Rukhin and Zidek (1985) established the inadmissibility of δ_0 when $\kappa > 1 + 1/n$ or $\kappa < 1/n$ by deriving better estimators. Subsequently, Rukhin (1986) showed that, for $1/n \leq \kappa \leq 1 + 1/n$, δ_0 is admissible. Elfessi (1997) considered estimation of θ based on a doubly censored sample.

The quadratic loss is a symmetric loss penalizing evenly overestimation and underestimation in the sense that $\lim_{d\to\infty}(d-\theta)^2/\sigma^2 = \lim_{d\to-\infty}(d-\theta)^2/\sigma^2 = \infty$ at the same rate. There are, however, many situations where overestimation is a more serious error than underestimation and therefore it should be penalized more, or vice versa, cf. Varian (1975), Zellner (1986), and Kuo and Dey (1990). In those situations, use of the quadratic loss may be inappropriate.

In Section 2, estimation of θ is considered under a loss of the form $L((d - \theta)/\sigma)$, where $L(t)$ is an arbitrary strictly convex function on $(-\infty, \infty)$. In Theorem 2.1, under a sufficient condition on κ, an estimator which has smaller risk than the b.a.e. estimator of θ is presented. This condition corresponds to "large" values of κ and generalizes the condition $\kappa > 1 + 1/n$ for the quadratic loss. For this loss, the theorem reproduces one of Rukhin's and Zidek's ((1985), Corollary 1) estimators. The construction of the improved estimator is based on Stein's (1964) idea which was later formalized by Brewster and Zidek (1974). The improved estimator admits a plausible interpretation: gains in estimating $\theta = \mu + \kappa\sigma$ by this estimator are due to better estimating the component $\kappa\sigma$ under the loss $L((d - \kappa\sigma)/\sigma)$, see Remark 2.1. Rukhin and Strawderman (1982) and Rukhin and Zidek (1985) gave analogous interpretations for their estimators when $\kappa > 1 + 1/n$.

In particular, Theorem 2.1 is applied to the LINEX loss $L(t) = e^{at} - at - 1, a \neq 0$, and the corresponding range of "large" values of κ is found as a function of a. This asymmetric loss was introduced by Varian (1975) and has ever since attracted the interest of many researchers, e.g., Zellner (1986), Kuo and Dey (1990), Parsian (1990), Sadooghi-Alvandi (1990), Basu and Ebrahimi (1991), and Huang (1995). Note that overestimation is more or less penalized than underestimation according to the sign of a. A consequence of Theorem 2.1 is that if $\kappa > 1+1/n$ and either $a < 0$ or $0 < a < a_1$ for some a_1 depending on κ then the b.a.e. estimator of θ is inadmissible. For this loss "small" values of κ are also treated and estimators dominating the b.a.e. estimator of θ are given in Theorems 2.2 and 2.3. The way of proof of these theorems suggests that convexity alone is not enough and additional conditions are needed to construct improved estimators for θ when κ is "small" for a general loss (see Remark 2.5).

Theorem 2.1 can be extended to $m \geq 2$ independent populations $E(\mu_1, \sigma), \ldots, E(\mu_m, \sigma)$ for the problem of estimation of a linear function of the parameters. For the same problem Madi and Leonard (1996) constructed generalized Bayes estimators under the quadratic loss. Estimation of this function was also addressed by Madi and Tsui (1990) and of the location parameters by Elfessi and Pal (1991).

In Section 3, we use the quadratic loss $(d-\theta)^2/\sigma^2$. For $\kappa \geq 1$, adopting Kubokawa's (1994) unified approach, we give sufficient conditions on the function ϕ for a scale equivariant estimator of the form $\delta = X + \phi(W)S$, $W = X/S$, to dominate δ_0. Since δ_0 is known to be minimax, so is δ. Then when $\kappa > 1 + 1/n$ we identify two such ϕ's: one produces the Stein-type estimator of Theorem 2.1, while the other yields a new minimax estimator which does not belong to Rukhin's (1986) class of minimax estimators. This new estimator has a very simple form and is different from δ_0 on the set $W > 0$ (thus different from δ_0 with probability one when $\mu \geq 0$). It is demonstrated that it can also be constructed using the techniques of Brown (1968) and Brewster and Zidek (1974). In addition, when $W > 0$ this estimator coincides with the generalized Bayes estimator of θ with respect to the prior $\pi(\mu, \sigma) = 1/\sigma$, $\mu \geq 0$, $\sigma > 0$.

Section 4 is an appendix containing some technical results needed in Section 3.

2. Estimation under a general loss

In this section we consider the problem of estimating θ under the loss $L((d - \mu - \kappa\sigma)/\sigma)$, where $L(t)$ is a strictly convex function on $(-\infty, \infty)$ with minimum attained at $t = 0$. We also assume that $L(t)$ is differentiable, integrals involving $L(t)$ are finite and interchange of integral and derivative is permissible.

The affine equivariant estimators of θ have the form $X + aS$. The risk of such an estimator, $R(a)$, does not depend on (μ, σ) and is strictly convex function of a, since

$L(t)$ is strictly convex. To avoid uninteresting situations, we require that $R(a)$ is not monotone. (The same property will be implicitly assumed later on for some other strictly convex functions representing risk functions.) Then $R(a)$ is uniquely minimized at $a = a_0$ satisfying

$$E_0\{L'(X + a_0 S - \kappa)S\} = 0 \tag{2.1}$$

and

$$\delta_1 = X + a_0 S$$

is the best affine equivariant procedure. Here and in the sequel, "E_0" will denote expectation under $\mu = 0$ and $\sigma = 1$.

Following Stein (1964), for improving upon δ_1 we study scale equivariant estimators of the form

$$\delta = \psi(W)S, \tag{2.2}$$

where $W = X/S$ and ψ is a measurable function. For κ satisfying (2.5), Theorem 2.1 establishes the inadmissibility of δ_1 by deriving a Stein (1964)-type estimator of the form (2.2), which has smaller risk than δ_1.

THEOREM 2.1. *Set $Y = \sum_{i=1}^{n} X_i$ and let b_0 be the unique solution to the equation*

$$E_0\{L'(b_0 Y - \kappa)Y\} = 0. \tag{2.3}$$

Then the estimator

$$\delta = \begin{cases} X + \min\{a_0, b_0(1 + nW)\}S, & W > 0 \\ X + a_0 S, & W \leq 0 \end{cases} \tag{2.4}$$

is better than δ_1 provided

$$E_0\{L'(X + b_0 S - \kappa)S\} < 0. \tag{2.5}$$

PROOF. The risk of δ in (2.2) depends on (μ, σ) through μ/σ so that one can take $\sigma = 1$ and write

$$R(\delta; \mu) = E_\mu[E_\mu\{L(\psi(w)S - \mu - \kappa) \mid W = w\}]. \tag{2.6}$$

For $w > 0$ we study the conditional expectation $g(c; \mu) = E_\mu\{L(cS - \mu - \kappa) \mid W = w\}$. Then $g(c; \mu)$ is a strictly convex function of c minimized at $c = c(\mu)$ satisfying

$$E_\mu\{L'(c(\mu)S - \mu - \kappa)S \mid W = w\} = 0. \tag{2.7}$$

We will establish that

$$c(\mu) \leq c(0) \quad \text{for} \quad \mu \leq 0. \tag{2.8}$$

Observe first that for any μ the conditional density of S given $W = w > 0$ (when $\sigma = 1$) is

$$f(s; \mu) \propto s^{n-1} e^{-(1+nw)s}, \qquad s \geq \max(0, \mu/w). \tag{2.9}$$

Consider $\mu \leq 0$. Since $g(c; 0)$ is strictly convex and minimized at $c(0)$, (2.8) will hold if we show that $g'(c(\mu); 0) \leq 0$ or

$$E_0\{L'(c(\mu)S - \kappa)S \mid W = w\} \leq 0. \tag{2.10}$$

Since $L'(t)$ is increasing and the conditional density of $S \mid W = w$ does not depend on $\mu \leq 0$,

$$E_0\{L'(c(\mu)S - \kappa)S \mid W = w\} \leq E_0\{L'(c(\mu)S - \mu - \kappa)S \mid W = w\}$$
$$= E_\mu\{L'(c(\mu)S - \mu - \kappa)S \mid W = w\} = 0,$$

i.e., (2.10) holds.

We proceed to show that

$$c(\mu) < w + c(0) \quad \text{for} \quad \mu > 0. \tag{2.11}$$

Let $\mu > 0$. Notice first that $c(\mu) \leq w + \kappa w/\mu$, since otherwise with probability one, $c(\mu)S - \mu - \kappa > 0$, by (2.9), and consequently $E_\mu\{L'(c(\mu)S - \mu - \kappa)S \mid W = w\} > 0$ as $L'(t)$ is positive on $(0, \infty)$. The last inequality contradicts (2.7). Analogously to (2.8), (2.11) will hold if

$$E_0\{L'((c(\mu) - w)S - \kappa)S \mid W = w\} < 0. \tag{2.12}$$

The left-hand side of (2.12) is equal to

$$\int_0^{\mu/w} L'((c(\mu) - w)s - \kappa)sf(s;0)ds + \int_{\mu/w}^{\infty} L'((c(\mu) - w)s - \kappa)sf(s;0)ds = I_1 + I_2.$$

For $s < \mu/w$, using $c(\mu) \leq w + \kappa w/\mu$, we have $(c(\mu) - w)s - \kappa < 0$ and thus $I_1 < 0$ as $L'(t)$ is negative on $(-\infty, 0)$. Also, for $s > \mu/w$, $(c(\mu) - w)s - \kappa < c(\mu)s - \mu - \kappa$, so that the monotonicity of $L'(t)$ implies

$$I_2 \leq (\text{positive constant}) \times \int_{\mu/w}^{\infty} L'(c(\mu)s - \mu - \kappa)sf(s;\mu)ds = 0,$$

by (2.7). Thus (2.12) holds.

We will need to express $c(0)$ as a function of w. It follows from (2.7) and (2.9) that

$$\int_0^{\infty} L'(c(0)s - \kappa)s^n e^{-(1+nw)s}ds = 0 \quad \text{or} \quad \int_0^{\infty} L'\left((c(0)/(1 + nw))y - \kappa\right)y^n e^{-y}dy = 0.$$

Comparing with (2.3), we get $c(0) = b_0(1+nw)$. Moreover, (2.5) along with (2.1) ensure that

$$b_0 < a_0 \tag{2.13}$$

Define now

$$\psi(w) = \begin{cases} w + \min\{a_0, b_0(1 + nw)\}, & w > 0 \\ w + a_0, & w \leq 0. \end{cases}$$

Then because of (2.8), (2.11), and (2.13), on a set of positive probability we have $c(\mu) < \psi(w) < w + a_0$ for all μ and hence, by strict convexity,

$$E_\mu\{L(\psi(w)S - \mu - \kappa) \mid W = w\} < E_\mu\{L((w + a_0)S - \mu - \kappa) \mid W = w\}.$$

From (2.6) the conclusion is that $\psi(W)S$, i.e. δ in (2.4), has smaller risk than δ_1.

Remark 2.1. With Y as in Theorem 2.1, the estimator δ in (2.4) can be written as

$$\delta = \begin{cases} \delta_2 = X + b_0 Y, & 0 < W < (a_0/b_0 - 1)/n \\ \delta_1 = X + a_0 S, & \text{otherwise.} \end{cases}$$

By (2.3) $b_0 Y$ is the best scale equivariant estimator of $\kappa\sigma$ under the loss $L((d-\kappa\sigma)/\sigma)$ when $\mu = 0$. We see that δ is a "testimator" choosing between δ_2 or δ_1 depending on whether or not the likelihood ratio test for $H_0 : \mu = 0$ with acceptance region $0 < W < (a_0/b_0 - 1)/n$ accepts H_0. Hence, the gain in estimating $\theta = \mu + \kappa\sigma$ by δ is achieved by better estimating the component $\kappa\sigma$.

As an illustration of Theorem 2.1 we present two examples showing that the condition in (2.5) gives rise to "large" values of κ.

Example 2.1. Let $L(t) = t^2$. Then $a_0 = (\kappa - 1/n)/n$, $b_0 = \kappa/(n+1)$, and (2.5) gives $\kappa > 1 + 1/n$. Here the improved estimator δ coincides with that in Rukhin and Zidek ((1985), Corollary 1, $m = 1$, $a_1 = 1$).

Example 2.2. Let $L(t) = e^{at} - at - 1$, $a \neq 0$. Estimation of μ and/or σ under this loss has been studied by Parsian *et al.* (1993), Parsian and Sanjari Farsipour (1993), and Parsian and Sanjari Farsipour (1997). Estimation of θ, however, has not been considered in the literature. In this case $a_0 = \frac{1}{a}\{1 - \frac{e^{-a\kappa/n}}{(1-a/n)^{1/n}}\}(a < n)$, $b_0 = (1/a)(1 - e^{-a\kappa/(n+1)})$ and (2.5) gives $\kappa > -((n+1)/a)\ln(1 - a/n)$. As a function of $a < 0$, $-\frac{(n+1)}{a}\ln(1 - \frac{a}{n})$ increases from 0 to $1 + 1/n$ while for $0 < a < n$ it increases from $1 + 1/n$ to ∞. Consequently for any given $\kappa > 0$ the b.a.e. estimator of $\theta = \mu + \kappa\sigma$, δ_1, is dominated by δ in (2.4) under $L(t)$ if $a < a_1$, where a_1 is the solution to the equation $-\frac{(n+1)}{a_1}\ln(1 - \frac{a_1}{n}) = \kappa$. For instance in the case of the mean $\theta = \mu + \sigma$ it was found numerically that a_1 is increasing in n ranging from $-2.201(n = 3)$ to $-2.013(n = 50)$. Note, in contrast, that the b.a.e. estimator of $\mu + \sigma$ is admissible under the quadratic loss (Rukhin (1986)). Finally, as $a \downarrow 0$ or $a \uparrow 0$ the range of values of κ approaches $(1 + 1/n, \infty)$ which is the range of "large" κ's corresponding to quadratic loss. This seems reasonable as $L(t) \approx a^2t^2/2$ for $a \approx 0$.

As mentioned in Section 1, under the quadratic loss Rukhin and Strawderman (1982) and Rukhin and Zidek (1985) also derived improved estimators of θ for small values of κ, namely $0 \le \kappa < 1/n$. In the case of the LINEX loss (with given a) we can prove the following results for small κ's. First we treat the case where the parameter a is negative and then the case of positive a.

THEOREM 2.2. *Let $L(t) = e^{at} - at - 1$. Then, for negative a such that $a > -n(n+1)$ and $(1 + \frac{a}{n(n+1)})^n(1 - \frac{a}{n}) - 1 > 0$, if*

$$0 \le \kappa < -\frac{1}{a}\ln\left(1 - \frac{a}{n}\right) - \frac{n}{a}\ln\left(1 + \frac{a}{n(n+1)}\right), \tag{2.14}$$

the estimator

$$\delta = \begin{cases} \max\left\{W + a_0, \dfrac{n+2}{n+1}W\right\}S, & W < -\dfrac{1}{n} \\ X + a_0 S, & W \ge -\dfrac{1}{n} \end{cases} \tag{2.15}$$

is better than δ_1.

PROOF. As in the proof of Theorem 2.1 we need to study the conditional expectation $E_\mu\{L(cS-\mu-\kappa) \mid W=w\}$. For $w<0$ and $\mu<0$ the conditional density of S given $W=w$ (when $\sigma=1$) is $f(s;\mu) \propto s^{n-1}e^{-(1+nw)s}$, $0 \le s \le \mu/w$. For any $\kappa \ge 0$ let $c(\mu,\kappa)$ be the minimizer of the strictly convex function $g(c;\mu,\kappa) = E_\mu\{L(cS-\mu-\kappa) \mid W=w\}$. (The different notation for the same quantities as in Theorem 2.1 is used here to stress the dependency on κ.) The monotonicity of $L'(t)$ and nonnegativity of κ imply

$$\begin{aligned} 0 &= E_\mu\{L'(c(\mu,\kappa)S-\mu-\kappa)S \mid W=w\} \\ &\le E_\mu\{L'(c(\mu,\kappa)S-\mu)S \mid W=w\} = g'(c(\mu,\kappa);\mu,0). \end{aligned}$$

Thus

$$c(\mu,\kappa) \ge c(\mu,0). \tag{2.16}$$

In addition, $g'(0;\mu,0) = L'(-\mu)E_\mu(S \mid W=w) > 0$ and therefore $c(\mu,0)<0$. We will now establish a lower bound for $c(\mu,0)$ (and hence for $c(\mu,\kappa)$). To this end, set $d = c(\mu,0)/w > 0$ and note that $c(\mu,0)$ satisfies $\int_0^{\mu/w} L'(c(\mu,0)s-\mu)s^n e^{-(1+nw)s}ds = 0$ or

$$\int_0^1 L'(\mu d y-\mu)y^n e^{-(1+nw)(\mu/w)y}dy = 0. \tag{2.17}$$

Apart from a normalizing constant, the left-hand side of (2.17) is the expectation of a decreasing function with respect to the density $h(y;\mu) \propto y^n e^{-(1+nw)(\mu/w)y}$, $0 \le y \le 1$. For $w < -1/n$, $h(y;\mu)/h(y;0)$ is increasing. Appealing to Lehmann ((1986), p. 85), we get $\int_0^1 L'(\mu dy-\mu)y^n dy > 0$. So far we have not used the specific form of the loss. Since $L'(t) = a(e^{at}-1) < a^2 t$ we further obtain that $\int_0^1 (dy-1)y^n dy < 0$, so that $d < (n+2)/(n+1)$, or

$$c(\mu,0) > \frac{n+2}{n+1}w. \tag{2.18}$$

Now, the condition about a ensures that κ in (2.14) is well defined. In addition, since here $a_0 = \frac{1}{a}(1-\frac{e^{-a\kappa/n}}{(1-a/n)^{1/n}})$, (2.14) means that $a_0 < -\frac{1}{n(n+1)}$. Consequently, from (2.16) and (2.18), $c(\mu,\kappa) > (n+2)/(n+1)w > w+a_0$ on the set $(n+1)a_0 < w < -1/n$, which by strict convexity implies

$$E_\mu\{L(((n+2)/(n+1))wS-\mu-\kappa) \mid W=w\} < E_\mu\{L((w+a_0)S-\mu-\kappa) \mid W=w\}.$$

It follows that for $\mu<0$, δ in (2.15) has smaller risk than δ_1. For $\mu \ge 0$, $\delta = \delta_1$ with probability one and the proof is complete.

Remark 2.2. The condition on a in Theorem 2.2 is satisfied at least when $-1 \le a < 0$. Indeed, since $(1+x)^n > 1+nx$, $-1<x<0$, we have

$$\left(1+\frac{a}{n(n+1)}\right)^n\left(1-\frac{a}{n}\right)-1 > \left(1+\frac{a}{n+1}\right)\left(1-\frac{a}{n}\right)-1 = -\frac{a(a+1)}{n(n+1)} \ge 0.$$

Further, it can be shown that this condition holds iff $a(n) < a < 0$, where $a(n)$ is the unique root of $(1+\frac{a}{n(n+1)})^n(1-\frac{a}{n})-1$ in the interval $(-n(n+1),0)$. By numerical computation we have found $a(n)$ to be decreasing in n ranging from $-1.57(n=3)$ to $-1.92(n=50)$. Note also that as $a \uparrow 0$ the range of values of κ in (2.14) increases to $0 \le \kappa < \frac{1}{n(n+1)}$ which is the range of "small" κ's given in Rukhin and Zidek ((1985), Corollary 2, $m=1$, $a_1=1$) for the quadratic loss.

THEOREM 2.3. *Let $L(t) = e^{at} - at - 1$. Then, for positive a such that $a < \frac{n+1}{n+2}n$ and $(1 + \frac{a}{n(n+1)-a(n+2)})^n(1 - \frac{a}{n}) - 1 < 0$, if*

$$0 \leq \kappa < -\frac{1}{a}\ln\left(1 - \frac{a}{n}\right) - \frac{n}{a}\ln\left(1 + \frac{a}{n(n+1) - a(n+2)}\right), \tag{2.19}$$

the estimator

$$\delta = \begin{cases} \max\left(W + a_0, \frac{n+2}{n+1}W\right)S, & W < -\frac{1}{n - ((n+2)/(n+1))a} \\ X + a_0 S, & \text{otherwise} \end{cases} \tag{2.20}$$

is better than δ_1.

PROOF. As in the proof of Theorem 2.2, for $w < 0$ and $\mu < 0$, we need to bound $c(\mu, 0)$ from below. We will again show that $c(\mu, 0) > \frac{n+2}{n+1}w$ for some negative w-values to be specified. It suffices to establish $g'(((n+2)/(n+1))w; \mu, 0) < 0$ or

$$\int_0^1 L'(((n+2)/(n+1))\mu y - \mu)\, y^n e^{-(1+nw)(\mu/w)y} dy < 0.$$

Since $L'(t) = a(e^{at} - 1) < a^2 t e^{at}$, this inequality will hold provided that

$$\int_0^1 ((n+2)/(n+1)y - 1) y^n e^{[a(n+2)/(n+1)w - (1+nw)](\mu/w)y} dy > 0. \tag{2.21}$$

For $a < \frac{n+1}{n+2}n$ and $w < -\frac{1}{n-((n+2)/(n+1))a}$ the coefficient of y in the exponential is positive. Therefore, by an application of Lehmann ((1986), p. 85) the left-hand side of (2.21) is larger than a positive constant times $\int_0^1 (((n+2)/(n+1))\, y - 1)\ y^n dy = 0$. Thus (2.21) holds. The rest of the proof as in Theorem 2.2.

Remark 2.3. One can show that the function

$$F(a) = \left(1 + \frac{a}{n(n+1) - a(n+2)}\right)^n \left(1 - \frac{a}{n}\right) - 1$$

is decreasing in $(0, n/(n+2))$ and increasing in $(n/(n+2), (n+1)n/(n+2)$ with $F(0) = 0$ and $F((n+1)n/(n+2)) = \infty$. Thus the condition on a is satisfied for $0 < a \leq n/(n+2) < a(n)$, where $a(n)$ is the root of $F(a)$ in $(0, (n+1)n/(n+2))$. By numerical computation we have found that $a(n)$ is increasing in n ranging from $1.03(n = 3)$ to $1.90(n = 50)$. Analogously to Remark 2.2, as $a \downarrow 0$ the limiting range of values of κ in (2.19) is $0 \leq \kappa < \frac{1}{n(n+1)}$.

Remark 2.4. For $\kappa = 0$, Theorems 2.2 and 2.3 give improved estimators of μ. We note here that an improved estimator for σ under the LINEX loss is given in Parsian and Sanjari Farsipour ((1993), p. 2893). Also, when σ is known, Parsian *et al.* (1993) proved the minimaxity of the best location equivariant estimator of μ for the same loss.

Remark 2.5. The two main steps in the proofs of Theorems 2.2 and 2.3 were establishing the inequality $c(\mu, \kappa) \geq c(\mu, 0)$ in (2.16) and then deriving an appropriate lower bound for $c(\mu, 0)$. It is important to stress that this inequality holds regardless of the specific functional form of the loss $L(t)$. This suggests that in order to construct improved estimators of θ for "small" κ's and a general loss a lower bound for $c(\mu, 0)$ should be sought. On the other hand, the form of $L(t)$ was crucial in bounding $c(\mu, 0)$ from below in Theorems 2.2 and 2.3. In the case of the quadratic loss $L(t) = t^2$, one can follow the proof of Theorem 2.2 up to the point $\int_0^1 L'(\mu dy - \mu)\, y^n dy > 0$ and substitute $L'(t) = 2t$ to get $\int_0^1 (dy-1) y^n dy < 0$, or $c(\mu, 0) > \frac{n+2}{n+1} w$. Then as in Theorem 2.2, one reproduces Rukhin and Zidek's ((1985), Corollary 2, $m = 1$, $a_1 = 1$, $\gamma = 1$) improved estimator.

In the remainder of this section we extend Theorem 2.1 to $m \geq 2$ exponential populations $E(\mu_i, \sigma), i = 1, \ldots, m$. Assume that independent random samples of sizes $n_i, i = 1, \ldots, m$, are available, denote by X_i and S_i the corresponding X and S, and set now $S = \sum_{i=1}^m S_i$, $W_i = X_i/S$, $n = \sum_{i=1}^m n_i$. Further, consider the problem of estimating $\theta = \sum_{i=1}^m a_i \mu_i + \kappa\sigma$, for given constants $a_i \geq 0$ and $\kappa \geq 0$, under a strictly convex loss $L((d-\theta)/\sigma)$. The b.a.e. estimator is

$$\delta_1 = \sum_{i=1}^m a_i X_i + a_0 S,$$

where a_0 satisfies $E_0\{L'(\sum_{i=1}^m a_i X_i + a_0 S - \kappa) S\} = 0$. Then for a_i and κ obeying (2.22), the following result gives an estimator with smaller risk than δ_1.

THEOREM 2.4. *Let Y have Gamma $G(n,1)$ distribution and b_0 be the solution to the equation*

$$E_0\{L'(b_0 Y - \kappa) Y\} = 0.$$

Then the estimator

$$\delta = \begin{cases} \sum_{i=1}^m a_i X_i + \min\left\{a_0, b_0\left(1 + \sum_{i=1}^m n_i W_i\right)\right\} S, & W_i > 0 \quad \forall i = 1, \ldots, m \\ \sum_{i=1}^m a_i X_i + a_0 S, & \textit{otherwise} \end{cases}$$

is better than δ_1 provided that

$$E_0\left\{L'\left(\sum_{i=1}^m a_i X_i + b_0 S - \kappa\right) S\right\} < 0. \tag{2.22}$$

The proof of the theorem is analogous to that of Theorem 2.1 and is omitted. In the case of the quadratic loss, (2.22) yields $\kappa > \frac{n+1}{m} \sum_{i=1}^m \frac{a_i}{n_i}$ and the estimator δ is also given in Rukhin and Zidek ((1985), Corollary 1). For the LINEX loss, (2.22) gives $\kappa > -\frac{n+1}{am} \sum_{i=1}^m \ln(1 - a\frac{a_i}{n_i}$, $a < \min\{\frac{n_i}{a_i} : i = 1, \ldots, m\}$.

3. Estimation under quadratic loss

In this section we deal with the estimation of θ under the quadratic loss $(d - \mu - \kappa\sigma)^2/\sigma^2$ in the one population case. Employing Kubokawa's (1994) approach we study the dominance of a scale equivariant estimator of the form

$$\delta = \begin{cases} X + \phi(W)S, & W > 0 \\ \delta_0, & W \leq 0 \end{cases} \tag{3.1}$$

where ϕ is an absolutely continuous function, over the b.a.e. estimator $\delta_0 = X + (\kappa - 1/n)S/n$. As before, we assume that the risk of δ is finite, interchange of integral and derivative is permissible, and take $\sigma = 1$ when we evaluate risk functions below.

THEOREM 3.1. *For $\kappa \geq 1$ assume that the following conditions hold:*
(a) *$\phi(w)$ is nondecreasing and* $\lim_{w \to \infty} \phi(w) = (\kappa - 1/n)/n$.
(b) $\phi(w) \geq \phi_0(w) \forall w > 0$,

where

$$\phi_0(w) = \left(\kappa - \frac{1}{n}\right)\frac{1}{n} - \left(\kappa - 1 - \frac{1}{n}\right)\frac{w}{(1+nw)^{n+1} - 1}. \tag{3.2}$$

Then the estimator δ in (3.1) *has risk nowhere larger than that of δ_0.*

PROOF. Let $g(v) = \frac{1}{(n-2)!}v^{n-2}e^{-v}$, $v > 0$, $h(u;\mu) = ne^{-n(u-\mu)}$, $H(x;\mu) = \int_0^x h(u;\mu)I(u > \mu)du$, $F(x;\mu) = \int_0^x uh(u;\mu)I(u > \mu)du$, $x > 0$. Then, using Kubokawa's (1994) definite integral approach, by the condition (a), the risk difference $RD = R(\delta_0;\mu) - R(\delta;\mu)$ can be written as

$$RD = 2\int_0^\infty \phi'(w)\left\{\phi(w)\int_0^\infty v^2 g(v)H(wv;\mu)dv + \int_0^\infty vg(v)F(wv;\mu)dv - (\mu+\kappa)\int_0^\infty vg(v)H(wv;\mu)dv\right\}dw. \tag{3.3}$$

Since $\phi(w)$ is nondecreasing, RD will be nonnegative if

$$\phi(w) \geq \frac{(\mu+\kappa)\int_0^\infty vg(v)H(wv;\mu)dv - \int_0^\infty vg(v)F(wv;\mu)dv}{\int_0^\infty v^2 g(v)H(wv;\mu)dv} = \phi(w;\mu)\forall w > 0. \tag{3.4}$$

For $\mu \leq 0$ it is easily seen that $\phi(w;0) \geq \phi(w;\mu)$. For $\mu > 0$ and $\kappa \geq 1$ it follows from Lemma 4.4 that $\phi(w;0) > \phi(w;\mu)$. Finally, $RD \geq 0$ if $\phi(w) \geq \phi(w;0)$. Carrying out the computations it is seen that $\phi(w;0)$ is just $\phi_0(w)$ in (3.2), and thus the proof is complete by the condition (b).

From Theorem 3.1 and for $\kappa > 1 + 1/n$ we obtain the following two improved estimators. The Stein-type estimator in (2.4) for quadratic loss, i.e.,

$$\delta = \begin{cases} X + \min\left\{a_0, \dfrac{\kappa(1+nW)}{n+1}\right\}S, & W > 0 \\ X + a_0 S, & W \leq 0, \end{cases}$$

$a_0 = (\kappa - 1/n)/n$, satisfies the conditions (a) and (b) of Theorem 3.1. Indeed, for $\phi(w) = \min\{a_0, \kappa(1+nw)/(n+1)\}$ the first of them clearly holds. For $\kappa > 1 + 1/n$ and $w \in B = (0, \frac{1}{n^2}(1 - (n+1)/n\kappa))$, we have $\phi(w) = \kappa(1+nw)/(n+1)$. Now, with $\psi_S(w) =$

$\min\{\frac{1}{n}, \frac{1+nw}{n+1}\}$, $\psi_S(W)S$ is the Stein (1964)-type estimator of σ (cf. Zidek (1973)) and $\phi(w) = \kappa\psi_S(w)$ for $w \in B$. Also, with

$$\psi(w) = \frac{\int_0^\infty vg(v)H(wv;0)dv}{\int_0^\infty v^2 g(v)H(wv;0)dv},$$

$\psi(W)S$ is the Brewster and Zidek (1974)-type estimator of σ (cf. Brewster (1974)). It is well known that $\psi_S(w) \geq \psi(w)$ (cf. Kubokawa (1994)). It follows that for $w \in B$, $\phi(w) \geq \kappa\psi(w) > \phi_0(w)$ (see (3.4)). For $w \notin B$, $\phi(w) = a_0 > \phi_0(w)$ as shown in Theorem 3.2. Thus condition (b) is satisfied.

THEOREM 3.2. *For $\kappa > 1 + 1/n$ the estimator*

$$\delta_{BZ} = \begin{cases} X + \phi_0(W)S, & W > 0 \\ \delta_0, & W \leq 0, \end{cases}$$

where $\phi_0(w)$ is given in (3.2), *has risk nowhere larger than that of δ_0.*

PROOF. For $\kappa > 1 + 1/n$ it is clear that $\phi_0(w)$ is increasing and $\lim_{w\to\infty}\phi_0(w) = (\kappa - 1/n)/n$. Hence Theorem 3.1 applies.

The estimator δ_{BZ} is a Brewster and Zidek (1974)-type estimator as the following argument demonstrates. For $w > 0$ and a constant c consider the Brown (1968)-type estimator of θ

$$\delta_c = \begin{cases} X + cS, & 0 < W < w \\ X + a_0 S, & \text{otherwise.} \end{cases}$$

The risk of δ_c is minimized at $c = \phi(w;\mu)$ given in (3.4). Because of the properties of $\phi(w;\mu)$ and $\phi(w;0)$ stated above the choice $c = \phi(w;0) = \phi_0(w)$ produces an improved estimator δ_c which in turn yields δ_{BZ} by the standard limiting argument of Brewster and Zidek (1974). By this method of proof and analogously to Corollary 2.1 in Brewster (1974), the condition (a) can be relaxed to $\phi(w) \leq (\kappa - 1/n)/n$, $w > 0$, in the place of $\lim_{w\to\infty}\phi(w) = (\kappa - 1/n)/n$.

Rukhin (1986) has obtained a class of minimax procedures for θ. Theorem 3.1 describes another one, a member of which is δ_{BZ}. We next show that δ_{BZ} cannot be produced through Rukhin's (1986) sufficient conditions for minimaxity. Note first that when $W > 0$, δ_{BZ} is of the form (2.2) in Rukhin (1986) with $f(z) = \frac{a-1}{2a}\frac{z^{-1}}{(1+z^{-1})^{n+1}-1}$, $z > 0$, where $a = \kappa - 1/n$. The first of the conditions requires that $f(z)(1+z^{-1})^p = \frac{a-1}{2a}\frac{(1+z)^p z^{n-p}}{(1+z)^{n+1}-z^{n+1}}$ be nondecreasing for some $p > 0$. By differentiation the sign of the derivative is the sign of $g(z) = (n - p + nz)\{(1+z)^{n+1} - z^{n+1}\} - (n+1)z(1+z)\{(1+z)^n - z^n\}$. This is a polynomial of degree n with coefficient of z^n $(n+1)(n/2 - p)$. Therefore for $g(z) \geq 0$ $\forall z > 0$ to hold we must have $p \leq n/2$. In general, for any $m = 0, 1, \ldots, n$ the coefficient of z^m is nonnegative if $p \leq n - \frac{m}{n-m+2}$. Consequently, $g(z) \geq 0$ iff $p \leq n/2$. With $\tilde{f} = \frac{a-1}{2a}\frac{1}{n+1}$ the second of the conditions in Rukhin (1986) is that $a\tilde{f} \leq (a-1)\min[1, p(n+2p+1)(n+p)^{-1}(n+p+1)^{-1}]/(n+1)$ which is not satisfied for $0 < p \leq n/2$.

The next theorem reveals an interesting Bayesian property of δ_{BZ}.

THEOREM 3.3. *For $W > 0$, δ_{BZ} coincides with the generalized Bayes estimator of θ with respect to the prior $\pi(\mu, \sigma) = 1/\sigma$, $\mu > 0$, $\sigma > 0$.*

PROOF. The posterior density of (μ, σ) given $S = s$, $X = x > 0$, is

$$\pi(\mu, \sigma \mid s, x) \propto \frac{1}{\sigma^{n+1}} \exp\{-(s + nx)/\sigma + n\mu/\sigma\}, \qquad 0 < \mu < x, \quad \sigma > 0.$$

The generalized Bayes estimator of θ under the quadratic loss $(d - \mu - \kappa\sigma)^2/\sigma^2$ is given by

$$d = \frac{E\frac{\mu}{\sigma^2} + \kappa E\frac{1}{\sigma}}{E\frac{1}{\sigma^2}},$$

where the expectations are taken under $\pi(\mu, \sigma \mid s, x)$. By direct computations, it is easy to verify that $d = \delta_{BZ}$.

Some comments about the magnitude of improvement of δ_{BZ} over δ_0 are now in order. At the origin ($\mu = 0$) δ_{BZ} and δ_0 have the same risk. This is evident from (3.3) and the expression of $\phi_0(w)$ in (3.4) (and it is a common feature of Brewster and Zidek (1974)-type procedures, see, e.g., Kubokawa ((1994), p. 294) for the case of estimating a scale parameter). The percentage risk reduction of δ_{BZ} was computed numerically for selected values of the quantile $p = e^{-\kappa}$ and the sample size n, along with that of δ_{RS} – Rukhin's and Strawderman's (1982) estimator. These numerical results (which are available upon request) indicate that in general for $\mu \geq 0$, δ_{RS} exhibits larger maximal risk reduction, whereas δ_{BZ} offers larger improvement for wide ranges of parameter values. For $\mu < 0$ the two estimators appear to have similar perfomance.

Acknowledgements

Suggestions for improvement by two referees are sincerely appreciated.

Appendix

We first prove some technical results needed to establish that $\phi(w; \mu)$ in (3.4) is decreasing in $\mu > 0$. Then, the proof of this property is given in Lemma A.4.

LEMMA A.1. *For $x = n, \ldots, 2n-1$, $n \geq 1$, and $0 < \alpha < \beta$ we have that*

$$\text{(A.1)}\ D_x = \sum_{k+l=x} \frac{(n-1-l)(k-l)}{(n-k)!(n-l)!} \beta^k \alpha^l - \frac{2^{2n-1-x}}{(2n-1-x)!} \beta^{-1} \alpha^{1+x} > 0 \quad (0 \leq k, l \leq n).$$

PROOF. Denote by E_x the sum in (A.1) and set $x = n + y$. Then

$$E_x = \frac{\alpha^y \beta^y}{(n-y-1)!} \{(n-y-1)\beta(\alpha+\beta)^{n-y-2}(\beta-\alpha)\} + \frac{\alpha^y \beta^y}{(n-y-1)!} \alpha(\alpha+\beta)^{n-y-1}.$$

The term in the brackets is nonnegative while the other term is larger than $\frac{\alpha^y \alpha^{y+1} \beta^{-1}}{(n-y-1)!} \alpha$ $(\alpha+\alpha)^{n-y-1} = \frac{2^{2n-1-x}}{(2n-1-x)!} \beta^{-1} \alpha^{1+x}$.

LEMMA A.2. *For $x = 1, \ldots, n-1$, $n \geq 2$, and $0 < \alpha < \beta$ we have that*

$$F_x = \sum_{k=0}^{x} \frac{n!}{(n-x+k)!(n-k)!}(n-1-k)(x-2k)\beta^{1+x-k}\alpha^k + \left\{ \frac{1+x}{(n-1-x)!} - \sum_{k=0}^{x} \frac{n!}{(n-1-x+k)!(n-k)!} \right\} \alpha^{1+x} > 0. \tag{A.2}$$

PROOF. Denote by G_x the first sum in (A.2) and set $y = (x-1)/2$ or $x/2 - 1$ according as x is odd or even. Combining equidistant terms from the middle we can write G_x as

$$G_x = \sum_{k=0}^{y} e_k \beta^{1+x-k}\alpha^k + \sum_{k=0}^{y} d_k \beta^{1+k}\alpha^k(\beta^{x-2k} - \alpha^{x-2k})$$

where $d_k = \frac{n!(n-1-x+k)(x-2k)}{(n-x+k)!(n-k)!}$ and $e_k = \frac{n!}{(n-x+k)!(n-k)!}(x-2k)^2$. Denote the second sum by H_x and notice that $H_x > 0$. Then F_x in (A.2) becomes

$$F_x > H_x + \alpha^{1+x} \left\{ \frac{1}{2}\sum_{k=0}^{x} e_k - \sum_{k=0}^{x} \frac{n!}{(n-1-x+k)!(n-k)!} + \frac{(1+x)}{(n-1-x)!} \right\} = H_x,$$

since the quantity in the brackets can be shown to be zero. Hence (A.2) holds.

LEMMA A.3. *For $0 < \alpha < \beta$ and $n \geq 2$ let*

$$A = (\alpha^{-1} - \beta^{-1}) \left\{ \sum_{k=0}^{n} (n)_k \beta^k \sum_{k=0}^{n-1} (n-1)_k \alpha^k - \sum_{k=0}^{n} (n)_k \beta^k \sum_{k=0}^{n} (n)_k \alpha^k \right\},$$

$$B = \sum_{k=0}^{n-1} (n-1)_k \beta^k \sum_{k=0}^{n} (n)_k \alpha^k - \alpha\beta^{-1} \sum_{k=0}^{n-1} (n-1)_k \alpha^k \sum_{k=0}^{n} (n)_k \alpha^k,$$

$$C = \sum_{k=0}^{n} (n)_k \beta^k \sum_{k=0}^{n-1} (n-1)_k \alpha^k - \sum_{k=0}^{n-1} (n-1)_k \beta^k \sum_{k=0}^{n} (n)_k \alpha^k.$$

Then, $C > 0$ and $A + B + nC > 0$.

PROOF. We have

$$C = \sum_{k=0}^{n} \sum_{l=0}^{n} \frac{n!(n-1)!}{(n-k)!(n-l)!}(k-l)\beta^k\alpha^l = \sum_{j=1}^{n} \sum_{i=0}^{j-1} \frac{n!(n-1)!}{(n-i)!(n-j)!}(j-i)(\beta^j\alpha^i - \beta^i\alpha^j) > 0$$

since $0 < \alpha < \beta$. Now, collecting the coefficients of $\beta^k\alpha^l$ $k, l = 0, \ldots, n$, $\beta^{-1}\alpha^l$ $l = 1, \ldots, n$, $\beta^{-1}\alpha^l$ $l = n+1, \ldots, 2n$ we obtain

$$D = A + B + nC = \sum_{k=0}^{n} \sum_{l=0}^{n} \frac{n!(n-1)!}{(n-k)!(n-l)!}(n-1-l)(k-l)\beta^k\alpha^l + \sum_{l=1}^{n} \left\{ \frac{l(n-1)!}{(n-l)!} - \sum_{k=0}^{l-1} \frac{n!(n-1)!}{(n-k)!(n-l+k)!} \right\} \beta^{-1}\alpha^l - \sum_{l=n+1}^{2n} \frac{n!(n-1)!}{(2n-l)!} 2^{2n-l}\beta^{-1}\alpha^l.$$

Next, we split D into two parts: one includes the terms $\beta^k\alpha^l$ and $\beta^{-1}\alpha^{1+x}$ with $k+l=x$ and $x=1,\dots,n-1$ and the other one includes the same terms but for $x=n,\dots,2n-1$. Then D is written as $D=\sum_{x=1}^{n-1}(n-1)!\beta F_x+\sum_{x=n}^{2n-1}n!(n-1)!D_x>0$, by Lemmas A.1 and A.2.

LEMMA A.4. *For $\kappa\ge 1$ the function $\phi(w;\mu)$ in (3.4) is decreasing in $\mu>0$ for each $w>0$.*

PROOF. Write

$$\phi(w;\mu)=\frac{(\mu+\kappa)\int_{\mu/w}^{\infty}vg(v)H(wv;\mu)dv-\int_{\mu/w}^{\infty}vg(v)F(wv;\mu)dv}{\int_{\mu/w}^{\infty}v^2g(v)H(wv;\mu)dv},$$

differentiate with respect to μ, and substitute $g(v)$, $H(wv;\mu)$, and $F(wv;\mu)$. After lengthy but direct calculations the derivative will be negative if

$$\begin{aligned}
&n(\mu+\kappa)e^{-n\mu}\int_{\mu/w}^{\infty}v^ne^{-v}dv\int_{\mu/w}^{\infty}v^{n-1}e^{-(1+nw)v}dv\\
&>(n\kappa-1)e^{-n\mu}\int_{\mu/w}^{\infty}v^{n-1}e^{-v}dv\int_{\mu/w}^{\infty}v^ne^{-(1+nw)v}dv\\
&+ne^{-n\mu}\int_{\mu/w}^{\infty}v^ne^{-v}dv\int_{\mu/w}^{\infty}wv^ne^{-(1+nw)v}dv\\
&+\int_{\mu/w}^{\infty}v^{n-1}e^{-(1+nw)v}dv\int_{\mu/w}^{\infty}v^ne^{-(1+nw)v}dv.
\end{aligned}$$

Making the change of variable $v=\frac{\mu}{w}y$, using the identity $\int_1^\infty y^ne^{-y/\gamma}dy=\gamma e^{-\gamma^{-1}}\sum_{k=0}^{n}(n)_k\gamma^k$ and setting $\beta^{-1}=\mu/w$, $\alpha^{-1}=n\mu+\mu/w$ so that $0<\alpha<\beta$ and $n\mu=\alpha^{-1}-\beta^{-1}$, the above inequality equivalently becomes $n\kappa C+A+B>0$, which holds because of Lemma A.3 and the assumption $\kappa\ge 1$.

REFERENCES

Basu, A. P. and Ebrahimi, N. (1991). Bayesian approach to life testing and reliability estimation using asymmetric loss function, *J. Statist. Plann. Inference*, **29**, 21–31.

Brewster, J. F. (1974). Alternative estimators for the scale parameter of the exponential distribution with unknown location, *Ann. Statist.*, **2**, 553–557.

Brewster, J. F. and Zidek, J. V. (1974). Improving on equivariant estimators, *Ann. Statist.*, **2**, 21–38.

Brown, L. D. (1968). Inadmissibility of the usual estimators of scale parameters in problems with unknown location and scale parameters, *Ann. Math. Statist.*, **39**, 29–48.

Elfessi, A. (1997). Estimation of a linear function of the parameters of an exponential distribution from doubly censored samples, *Statist. Probab. Lett.*, **36**, 251–259.

Elfessi, A. and Pal, N. (1991). On location and scale parameters of exponential distributions with censored observations, *Comm. Statist. Theory Methods*, **20**, 1579–1592.

Huang, S. Y. (1995). Empirical Bayes testing procedures in some nonexponential families using asymmetric linex loss function, *J. Statist. Plann. Inference*, **46**, 293–309.

Kubokawa, T. (1994). A unified approach to improving equivariant estimators, *Ann. Statist.*, **22**, 290–299.

Kuo, L. and Dey, D. K. (1990). On the admissibility of the linear estimators of the Poisson mean using LINEX loss function, *Statist. Decisions*, **8**, 201–210.

Lehmann, E. L. (1986). *Testing Statistical Hypotheses*, 2nd ed., Wiley, New York.

Madi, M. T. and Leonard, T. (1996). Bayesian estimation for shifted exponential distributions, *J. Statist. Plann. Inference*, **55**, 345–351.

Madi, M. T. and Tsui, K. W. (1990). Estimation of the common scale of several shifted exponential distributions with unknown locations, *Comm. Statist. Theory Methods*, **19**, 2295–2313.

Parsian, A. (1990). On the admissibility of an estimator of a normal mean vector under a LINEX loss function, *Ann. Inst. Statist. Math.*, **42**, 657–669.

Parsian, A. and Sanjari Farsipour, N. (1993). On the admissibility and inadmissibility of estimators of scale parameters using an asymmetric loss function, *Comm. Statist. Theory Methods*, **22**, 2877–2901.

Parsian, A. and Sanjari Farsipour, N. (1997). Estimation of parameters of exponential distribution in the truncated space using asymmetric loss function, *Statist. Papers*, **38**, 423–443.

Parsian, A., Sanjari Farsipour, N. and Nematollahi, N. (1993). On the minimaxity of Pitman type estimator under a LINEX loss function, *Comm. Statist. Theory Methods*, **22**, 97–113.

Rukhin, A. L. (1986). Admissibility and minimaxity results in the estimation problem of exponential quantiles, *Ann. Statist.*, **14**, 220–237.

Rukhin, A. L. and Strawderman, W. E. (1982). Estimating a quantile of an exponential distribution, *J. Amer. Statist. Assoc.*, **77**, 159–162.

Rukhin, A. L. and Zidek, J. V. (1985). Estimation of linear parametric functions for several exponential samples, *Statist. Decisions*, **3**, 225–238.

Sadooghi-Alvandi, S. M. (1990). Estimation of the parameter of a Poisson distribution using a LINEX loss function, *Austral. J. Statist.*, **32**, 393–398.

Stein, C. (1964). Inadmissibility of the usual estimator for the variance of a normal distribution with unknown mean, *Ann. Inst. Statist. Math.*, **16**, 155–160.

Varian, H. L. (1975). A Bayesian approach to real estate assesment, *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage* (eds. Stephen E. Fienberg and Arnold Zellner), 195–208, North-Holland, Amsterdam.

Zellner, A. (1986). Bayesian estimation and prediction using asymmetric loss functions, *J. Amer. Statist. Assoc.*, **81**, 446–451.

Zidek, J. V. (1973). Estimating the scale parameter of the exponential distribution with unknown location, *Ann. Statist.*, **1**, 264–278.