ON NON-EQUALLY SPACED WAVELET REGRESSION

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Abstract. Wavelet-based regression analysis is widely used mostly for equally-spaced designs. For such designs wavelets are superior to other traditional orthonormal bases because of their versatility and ability to parsimoniously describe irregular functions. If the regression design is random, an automatic solution is not available. For such non equispaced designs we propose an estimator that is a projection onto a multiresolution subspace in an associated multiresolution analysis. For defining scaling empirical coefficients in the proposed wavelet series estimator our method utilizes a probabilistic model on the design of independent variables. The paper deals with theoretical aspects of the estimator, in particular MSE convergence rates.

Key words and phrases: Irregular design, NES regression, nonparametric statistical procedures, projection estimators, wavelets.

1. Introduction

Function estimation is of fundamental importance in statistics and science in general where it applies to a wide range of problems and has a multitude of different objectives. For the majority of data sets encountered in real life the most appropriate procedures are nonparametric. Wavelet-based non-parametric methods, introduced in statistics by the work of Donoho and Johnstone in early 90's, represent a novel, break-through technology in theory and practice of nonparametric function estimation. The benefit of wavelets is their ability to adapt to unknown smoothness (Donoho et al. (1995)), although the adaptivity can be obtained by using traditional kernel estimators (Lepski et al. (1997)). For the equally spaced observations (for example, measurements at equal time increments), Donoho and Johnstone developed a simple and adaptive procedure, called WaveShrink, based on discrete wavelet transformations. WaveShrink is a fast procedure that has very broad asymptotic near-optimal properties.

Generalizations of WaveShrink-type techniques to non-equally spaced (NES) designs impose additional conceptual and calculational burdens. There are several proposals on how to estimate regression function by wavelets when a design is irregular. The simplest proposal is to ignore the design and is to carry out the analysis as if the data were equally spaced. This "method" is known as the coercion to equal spacing.

Another class of wavelet-based methods applicable on non-equally spaced data utilize interpolations and averaging. Based on the available data, the approximations (inter-

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polations) of the function are done at equally spaced dyadic points. On such approximate values the standard wavelet methods are applied. References on that method include Antoniadis et al. (1997), Deslauriers and Dubuc (1989), Foster (1996), Hall and Turlach (1997), and Härdele et al. (1998), among others. Sveldens (1995) proposes the "lifting scheme" technique to define wavelet transform of data sampled over a variety of topological objects. Sardy et al. (1999) consider the Haar basis and propose four approaches that extend the Haar wavelet transform to a NES data. Each approach is formulated in terms of continuous wavelet functions applied to a piecewise constant interpolation of the observed data, and each approach leads to wavelet coefficients that can be computed via a matrix transform of the original data. Some related approaches can be found in Antoniadis et al. (1994), Antoniadis and Pham (1998), Cai and Brown (1998), Delyon and Juditsky (1995), and Hall et al. (1998).

In this paper we propose a linear wavelet-based regression estimator and explore some of its large sample properties. The proposed estimator can be re-expressed in a form which reminds the wavelet modification of Nadaraja-Watson estimator (Antoniadis et al. (1994)) but conditions on the "pre-estimator" of the design distribution are relaxed. The practical implementations are subject of ongoing simulational study.

2. The estimator and its large sample properties

Let \( \phi \) be a compactly supported scaling function generated by an \( s \)-regular multiresolution analysis (MRA) of \( L_2(\mathbb{R}) \). Assume that \( s \geq 2 \), and that the support of \( \phi \) is contained in the interval \([ -A, A ]\).

Let

\[
(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n),
\]

be a sample of size \( n \). Denote by \( f \) the marginal density of \( X \) and by \( m(x) = E(Y \mid X = x) \) the regression function to be estimated. Assume that \( m(x) \in L^2(\mathbb{R}) \).

Instead of estimating \( m(x) \) directly we estimate its projection on a multiresolution subspace \( V_j \),

\[
m_j(x) = \text{Proj}_{V_j} m(x) = \sum_{k \in \mathbb{Z}} c_{j,k} \phi_{j,k}(x), \text{ where } \phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k).
\]

By the properties of MRA, \( U_j V_j \) is dense in \( L_2(\mathbb{R}) \) and the linear approximation \( m_j(x) \) converges to \( m(x) \) uniformly on \( J \), compact, when \( J \to \infty \).

The coefficients are

\[
c_{j,k} = 2^{j/2} \int_{-\infty}^{\infty} \phi(2^j x - k) m(x) dx
\]

\[
= \int_{-\infty}^{\infty} 2^{j/2} \phi(2^j x - k) m(x) f(x) dx
\]

\[
= \mathbb{E} \left[ \frac{\phi_{j,k}(X)}{f(x)} Y \right].
\]

Consider an estimator \( \hat{f}_n(x) \) of \( f(x) \) constructed on basis of \( n - 1 \) observations,

\[
\hat{f}_n(x) = \hat{f}_n(x \mid Z_1, \ldots, Z_{n-1})
\]

which is symmetric with respect to \( Z_1, \ldots, Z_{n-1} \), that is,

\[
\hat{f}_n(x \mid Z_1, \ldots, Z_{n-1}) = \hat{f}_n(x \mid Z_{i_1}, \ldots, Z_{i_{n-1}}).
\]
Let $\hat{m}_n(x)$ be an estimator of $m(x)$, more precisely, of $\text{Proj}_{V_j} m(x)$,

$$
(2.3) \quad \hat{m}_n(x) = \sum_k \hat{c}_{J,k} \phi_{j,k}(x),
$$

with $\hat{c}_{J,k}$ motivated by (2.2),

$$
(2.4) \quad \hat{c}_{J,k} = \frac{1}{n} \sum_{i=1}^n \left[ \frac{\phi_{J,k}(X_i)Y_i \cdot 1(\hat{f}_n(X_i) \mid X_{-i}) \geq \delta_n}{\hat{f}_n(X_i) \mid X_{-i}} \right],
$$

where $X_{-i}$ is the sample with $i$-th observation excluded, i.e., $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$. We will call estimator in (2.3) the NES linear wavelet estimator. Estimator similar to (2.3) was recently proposed by Pinheiro (1997). He proved that under mild regularity conditions on $\hat{f}_n$, $\hat{c}_{J,k} \to c_{J,k}$, a.s.

To derive quantitative results, we now discuss the MSE convergence rates of the NES estimator. Denote $v(x) = \mathbb{E}(Y^2 \mid X)$ and notice that $v(x) - m^2(x) \geq 0$. Let us consider the class of linear estimators of the density $f(x)$, that is, estimators of the form

$$
(2.5) \quad \hat{f}_n(x \mid X_{-i}) = \frac{1}{n-1} \sum_{j=1 \atop j \neq i}^n \mathbb{K}_n(x, x_j),
$$

where $\mathbb{K}_n(x, y)$ is a bounded kernel, symmetric in its arguments. Assume the following conditions hold:

A1. $\|f\|_c = \sup_x f(x) < \infty$;

A2. $\sup_x (\frac{\nu(x)}{f(x)}) < V_1 < \infty$, $\sup_x (\frac{m^2(x)}{f^3(x)}) < V_2 < \infty$;

A3. the kernel $\mathbb{K}_n(x, y)$ satisfy the conditions

(a) $\sup_{x, y} |\mathbb{K}_n(x, y)| \leq C_1 n^{-1/(2\gamma+1)}$,  
(b) $\sup_x \mathbb{E}(\mathbb{K}_n^2(x, X_1)) \leq C_2 n^{1/(2\gamma+1)}$;

A4. the estimator (2.5) achieves the following convergence rate

$$
\sup_x \mathbb{E}(\hat{f}_n(x) - f(x))^2 \leq C_3 n^{-2\gamma/(2\gamma+1)}, \quad \sup_x \mathbb{E}(\hat{f}_n(x) - f(x))^4 \leq C_4 n^{-(4\gamma-1)/(2\gamma+1)},
$$

with $\gamma \geq 1$;

A5. $m(x)$ is $s$ times continuously differentiable with $|m^{(s)}(x)| \leq m_s < \infty$.

Denote $\|\phi\|_c = \sup_x \phi(x)$, $\|\phi\|_L = \int |\phi(x)| dx$. The main result of this paper is expressed in the following theorem.

**Theorem 2.1.** Let the conditions A1–A5 hold, and $\delta_n$ be such that

$$
(2.6) \quad \delta_n = \delta_0 n^{-\gamma/(2\gamma+1)} \ln n,
$$

with $\delta_0 \geq 4C_1$. Then

$$
\sup_x \mathbb{E}(\hat{m}_n(x) - m(x))^2 \leq \begin{cases} 
(C_5 + C_7)n^{-4\gamma/(2\gamma+1)(2s+1)}(\ln n)^4s/(2s+1)(1 + o(1)), & \text{if } 2\gamma \geq 2s + 1 \\
(C_6 + C_7)n^{-(2\gamma-1)/(2\gamma+1)}(\ln n)^{-2}(1 + o(1)), & \text{if } 2\gamma < 2s + 1,
\end{cases}
$$
provided
\begin{equation}
2^J \sim \begin{cases}
\frac{n^{2\gamma/(2\gamma+1)(2s+1)}(\ln n)^{-2/(2s+1)}}{n^{(2\gamma-1)/(2s(\gamma+1))}(\ln n)^{1/s}}, & \text{if } 2\gamma \geq 2s + 1 \\
\frac{n^{2\gamma-1}/(2s(\gamma+1))}{(\ln n)^{1/s}}, & \text{if } 2\gamma < 2s + 1.
\end{cases}
\end{equation}

Here \( C_5 = 24(2A+1)^2\|\phi\|_2^2(C_4 + C_4')V_2\delta_0^{-2}\|\phi\|_2^2, \ C_6 = 27/2 \cdot (2A+1)^2\|\phi\|_2^2 V_2^2 \delta_0^2, \) and \( C_7 = (2A+1)^2\|\phi\|_2^2\|\phi\|_2^2 (s')^{-2}m_2^22^{2s}A^{2s}. \)

The proof of Theorem 2.1 is given in Section 3.

It is easy to see that conditions A3 and A4 are satisfied if \( \hat{f}_n(x) \) is a kernel density estimator or a linear wavelet estimator provided \( f(x) \) is \( \gamma \) times continuously differentiable with \( \sup_x |f^{(\gamma)}(x)| < \infty. \) The choice of the parameter \( \delta_n \) is determined by \( \gamma \) and \( C_1 \) where \( C_1 \) depends on \( k_n(x,y) \) which is known.

From Theorem 2.1 it follows that the rate of convergence of the estimator \( \hat{m}_n(x) \) is determined significantly by \( \gamma. \) If \( \gamma \) is fixed, then the rate of convergence of \( \hat{m}_n(x) \) is \( O(n^{-(2\gamma-1)/(2\gamma+1)}(\ln n)^{-2}) \) for any \( s > \gamma - 0.5. \) Hence, even infinite grows of \( s \) does not improve convergence rate. If \( s \) is fixed, the rate of convergence grows slowly with the increase of \( \gamma \) and reaches \( O(n^{-(4\gamma)/(2\gamma+1)}(\ln n)^{4s/(2s+1)}) \) as \( \gamma \to \infty. \) Therefore, the estimator (2.3) should be applied only if we expect \( f(x) \) to be reasonably smooth, that is, when \( \gamma \) is sufficiently large.

3. Proof of Theorem 2.1

We give now a proof of Theorem 2.1. This proof is based on a series of auxiliary lemmas. To simplify the notations, in what follows we will denote \( \hat{f}_n(x_i) \equiv \hat{f}_n(x_i | X_{-i}), \ \hat{f}_n(x) \equiv \hat{f}_n(x | X_{-n}). \) Also, the index \( n \) in \( J_n \) will be suppressed and we will simply write \( J. \) However, the level \( J \) is a function of a sample size as conditioned in (2.6).

**Lemma 1.** Under the conditions A1–A4

\begin{equation}
\sup_{X_1,X_2} \mathbb{E}[|f_n(X_1)| - f(X_1)|^2 | X_1, X_2] \leq 2(C_3 + C_4^2)n^{-2\gamma/(2\gamma+1)}(1 + o(1)),
\end{equation}

\begin{equation}
\sup_{X_1,X_2} \mathbb{E}[|\hat{f}_n(X_1) - f(X_1)|^4 | X_1, X_2] \leq 8(C_4 + C_2^2)n^{-(4\gamma)/(2\gamma+1)}(1 + o(1)).
\end{equation}

**Proof of Lemma 1.** Let us prove the first assertion. Partitioning \( \hat{f}_n(X_1) \) into the part containing and not containing \( X_2 \) and applying the inequalities \( (a+b)^2 \leq 2a^2 + 2b^2 \) and \( (n-2)/(n-1) < 1, \) we obtain

\[ \mathbb{E}[|\hat{f}_n(X_1) - f(X_1)|^2 | X_1, X_2| \leq 2 \int \cdots \int \left( \sum_{k=3}^{n} \frac{IK_n(X_1, x_k)}{n-2} - f(X_1) \right)^2 \prod_{k=3}^{n} f(x_k) dx_k \]

\[ + \frac{2}{(n-1)^2} \int \cdots \int (IK_n(X_1, X_2) - f(X_1))^2 \prod_{k=3}^{n} f(x_k) dx_k. \]

Therefore, \( \mathbb{E}[|\hat{f}_n(X_1) - f(X_1)|^2 | X_1, X_2] \leq 2C_3(n-1)^{-2\gamma/(2\gamma+1)} + 2(n-1)^{-2}(C_4^2 + \|f\|_2^2), \) which implies (3.1). The second inequality in Lemma 1 can be derived in a similar manner.
LEMMA 2. If assumptions A1–A4 and (2.6) are valid, then

\[ \sup_x \mathbb{P}(|\hat{f}_n(x) - f(x)| > \delta_n) \leq 2n^{-1}. \]

PROOF OF LEMMA 2. According to condition A4, \( \sup_x |\mathbb{E}\hat{f}_n(x) - f(x)| \leq \sqrt{3n} \gamma/(2\gamma + 1) \), which implies that for every \( x \)

\[ \mathbb{P}(|\hat{f}_n(x) - f(x)| > \delta_n) \leq \mathbb{P}(|\hat{f}_n(x) - \mathbb{E}\hat{f}_n(x)| > (\delta_0 \ln n - \sqrt{C_3})n^{-\gamma/(2\gamma + 1)}). \]

To majorize the right-hand side of the last inequality, recall Bernstein’s inequality (see Pollard (1984)). If \( Z_1, Z_2, \ldots, Z_M \) are i.i.d. bounded random variables such that \( \mathbb{E}Z_i = 0 \), \( \mathbb{E}Z_i^2 = \sigma^2 \), and \( |Z_i| \leq \|Z\|_{\infty} < \infty \), then for any \( \lambda > 0 \)

\[ (3.2) \quad \mathbb{P}
\left( \left| M^{-1} \sum_{i=1}^{M} Z_i \right| > \lambda \right) \leq 2 \exp \left( -\frac{M \lambda^2}{2 \sigma^2 + (2/3) \lambda \|Z\|_{\infty}} \right). \]

Apply (3.2) with \( Z_i = \mathbb{E}k_n(x, X_i) - \mathbb{E}k_n(x, X_i), M = n - 1, \sigma^2 \leq C_2n^{-\gamma/(2\gamma + 1)}, \|Z\|_{\infty} \leq 2C_1n^{-\gamma/(2\gamma + 1)} \), and \( \lambda = (\delta_0 \ln n - \sqrt{C_3})n^{-\gamma/(2\gamma + 1)} \). Taking into account that for large \( n \) the following inequalities hold: \( C_2 + (2/3)C_1 (\delta_0 \ln n - \sqrt{C_3})n^{-(\gamma-1)/(2\gamma + 1)} \leq C_1 \delta_0 \ln n \) and \( n^{-1}(n-1)(\delta_0 \ln n - \sqrt{C_3})^2 \geq 0.5\delta_0^2 \ln^2 n \), we obtain

\[ \mathbb{P}(|\hat{f}_n(x) - f(x)| > \delta_n) \leq 2 \exp \{-\delta_0 \ln n/(4C_1)\}, \]

which completes the proof.

LEMMA 3. Under the assumptions A1–A4 and (2.6)

\[ (3.3) \quad \sup_k \mathbb{E}(\hat{c}_{J,k} - c_{J,k})^2 \leq n^{-2\gamma/(2\gamma + 1)}[C_8n^{1/(2\gamma + 1)}2^{-J}(\ln n)^{-2} + C_9(\ln n)^2](1 + o(1)), \]

where \( C_8 = 24(C_4 + C_4^2)V_2\delta_0^2\|\phi\|_{L^2}^2 \) and \( C_9 = 27/2V_2^2\delta_0^2 \).

PROOF OF LEMMA 3. Denote by \( \Delta = \mathbb{E}(\hat{c}_{J,k} - c_{J,k})^2 \) and observe that \( \Delta \leq 3(\Delta_1 + \Delta_2 + \Delta_3) \) where

\[ \Delta_1 = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_{J,k}(X_i)X_i \left( \frac{1}{\hat{f}_n(X_i)} - \frac{1}{f(X_i)} \right) 1(\hat{f}_n(X_i) \geq \delta_n) \right]^2, \]

\[ \Delta_2 = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_{J,k}(X_i)Y_i \left( \hat{f}_n(X_i) \geq \delta_n \right) \right]^2, \]

\[ \Delta_3 = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_{J,k}(X_i)Y_i \left( \frac{1}{f(X_i)} - c_{J,k} \right) \right]^2. \]

In the proof of the theorem we will bound from above \( \Delta_1, \Delta_2, \) and \( \Delta_3, \) separately.
An upper bound for $\Delta_1$. Observe that

$$
\Delta_1 = \mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \phi_{J, k}(X_i) Y_i \left( \frac{f_n(X_i) - f(X_i)}{f_n^2(X_i)} \right)^2 \mathbb{1}(f_n(X_i) - f(X_i)) \right] 
$$

$$
+ \mathbb{E} \left[ \frac{1}{n^2} \sum_{i,j=1, i \neq j}^{n} \phi_{J, k}(X_i) \phi_{J, k}(X_j) \frac{(f_n(X_i) - f(X_i))(f_n(X_j) - f(X_j))}{f_n(X_i) f_n(X_j) f(X_i) f(X_j)} \right] 
$$

$$
\cdot \mathbb{1}(f_n(X_i) \geq \delta_n) \mathbb{1}(f_n(X_j) \geq \delta_n) 
$$

$$
= \Delta_{11} + \Delta_{12}. 
$$

Since $(X_i, Y_i), i = 1, \ldots, n$, are identically distributed, it is easy to see that

$$
\Delta_{11} = \frac{1}{n} \int_{-\infty}^{\infty} \frac{\phi_{J, k}(x) v(x)}{f^2(x)} \mathbb{E} \left[ \frac{(f_n(x) - f(x))^2}{f_n^2(x)} \mathbb{1}(f_n(X_i) \geq \delta_n) \right] f(x) dx 
$$

$$
\leq V_1 C_3 (n \delta_n^{-2})^{-1} n^{-2\gamma/(2\gamma+1)} = o(n^{-2\gamma/(2\gamma+1)}). 
$$

Similar considerations lead to the following upper bound for $\Delta_{12}$:

$$
\Delta_{12} \leq \frac{n - 1}{n} \mathbb{E} \left\{ \left| \phi_{J, k}(X_1) \right| \left| \phi_{J, k}(X_2) \right| \mathbb{E} \left[ \frac{\mathbb{1}(f_n(X_1) \geq \delta_n) \mathbb{1}(f_n(X_2) \geq \delta_n) |X_1, X_2|}{f(X_1)f(X_2)} \right] \right\}. 
$$

Representing $f_n^{-1}$ as $f^{-1} + (f - f_n)(f_n f)^{-1}$ we majorize $\Delta_{12}$ by the sum of two terms: $\Delta_{12} \leq \Delta_{121} + \Delta_{122}$. Here

$$
\Delta_{121} = \mathbb{E} \left\{ \prod_{i=1}^{2} \left[ \frac{\left| \phi_{J, k}(X_i) \right| m(X_i)}{f^2(X_i)} \right] \right\}, 
$$

$$
\cdot \mathbb{E} \left[ |f_n(X_1) - f(X_1)| |f_n(X_2) - f(X_2)| |X_1, X_2| \right], 
$$

$$
\Delta_{122} = \mathbb{E} \left\{ \prod_{i=1}^{2} \left[ \frac{\left| \phi_{J, k}(X_i) \right| m(X_i)}{f^2(X_i) f_n(X_i)} \right] \right\}, 
$$

$$
\cdot \mathbb{E} \left[ (f_n(X_1) - f(X_1))^2 (f_n(X_2) - f(X_2))^2 |X_1, X_2| \right]. 
$$

According to Lemma 1, as $n \to \infty$

$$
\Delta_{121} \leq \mathbb{E} \left\{ \prod_{i=1}^{2} \left[ \frac{\left| \phi_{J, k}(X_i) \right| m(X_i)}{f^2(X_i)} \right] \cdot \sqrt{\mathbb{E} \left[ (f_n(X_i) - f(X_i))^2 |X_1, X_2| \right]} \right\}. 
$$
\[
\leq 2(C_3 + C_1^2) \left[ \int_{-\infty}^{\infty} \frac{\phi_{J,k}(x) ||m(x)||}{f^2(x)} f(x) \, dx \right]^2 n^{-2\gamma/(2\gamma + 1)} (1 + o(1)) \\
\leq 2(C_3 + C_1^2) V_2 \|\phi\|^2 n^{-2\gamma/(2\gamma + 1)} \cdot 2^{-J} (1 + o(1)).
\]

For \(\Delta_{122}\) we can construct a similar upper bound
\[
\Delta_{122} \leq \delta_n^{-2} \prod_{i=1}^{2} \left\{ \mathbb{E} \left[ \frac{\phi_{j,k}(X_i) ||m(X_i)||}{f^2(X_i)} \cdot \sqrt{\mathbb{E}(\hat{f}_n(X_i) - f(X_i))^4 \mid X_1, X_2} \right] \right\}
\leq 8(C_4 + C_1^4) V_2 \|\phi\|^2 n^{-(4\gamma-1)/(2\gamma + 1)} \cdot 2^{-J} \delta_n^{-2} (1 + o(1)).
\]

Plugging in the value of \(\delta_n\) and combining \(\Delta_{11}, \Delta_{121}\) and \(\Delta_{122}\), we derive that
\[
\Delta_1 = (1/3) C_8 n^{-(2\gamma - 1)/(2\gamma + 1)} (\ln n)^{-2} 2^{-J}.
\]

An upper bound for \(\Delta_2\). Note that
\[
\Delta_2 = \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\phi_{J,k}(X_i) Y_i}{f(X_i)} 1(\hat{f}_n(X_i) \leq \delta_n) \right)^2 = \Delta_{21} + \Delta_{22},
\]
where the first term \(\Delta_{21}\) has the form
\[
\Delta_{21} = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left( \frac{\phi_{J,k}(X_i) Y_i^2}{f^2(X_i)} 1(\hat{f}_n(X_i) \leq \delta_n) \right) \\
\leq \frac{1}{n} \mathbb{E} \left( \frac{\phi_{J,k}(X_i) Y_i^2}{f^2(X_i)} \right) \\
= \frac{1}{n} \int_{-\infty}^{\infty} \frac{\phi_{J,k}(x) v(x)}{f(x)} \, dx \leq \frac{V_i}{n}.
\]

For the second term the following relations are valid
\[
\Delta_{22} = \frac{1}{n^2} \sum_{i,j=1; i \neq j}^{n} \mathbb{E} \left[ \frac{\phi_{J,k}(X_i) \phi_{J,k}(X_j) Y_i Y_j}{f(X_i) f(X_j)} 1(\hat{f}_n(X_i) \leq \delta_n) 1(\hat{f}_n(X_j) \leq \delta_n) \right] \\
\leq \mathbb{E} \left[ \frac{\phi_{J,k}(X_i) \phi_{J,k}(X_j) m(X_i) m(X_j)}{f(X_1) f(X_2)} 1(\hat{f}_n(X_i) \leq \delta_n) 1(\hat{f}_n(X_j) \leq \delta_n) \right] \\
\leq \left\{ \mathbb{E} \left[ \frac{\phi_{J,k}(X_i) m^2(X_i)}{f^2(X_i)} 1(\hat{f}_n(X_i) \leq \delta_n) \right] \right\}^2.
\]

Observe that \(1(\hat{f}_n(x) \leq \delta_n) \leq 1(f(x) \leq 1.5 \delta_n) + 1(|\hat{f}_n(x) - f(x)| > 0.5 \delta_n)\). Therefore,
\[
\Delta_{22} \leq 2\Delta_{221} + 2\Delta_{222},
\]
where
\[
\Delta_{221} = \left\{ \mathbb{E} \left[ \frac{\phi_{J,k}(X_1) m^2(X_1)}{f^2(X_1)} 1(f(X_1) \leq 1.5 \delta_n) \right] \right\}^2
\]
\[
\begin{align*}
\Delta_{22} &= \text{IE} \left( \frac{\phi_{j,k}^2(X_1)}{f^2(X_1)} \left( |\hat{f}_n(x) - f(x)| > 0.5 \delta_n \right) \right)^2 \\
&\leq \left\{ \text{IE} \left( \frac{\phi_{j,k}^2(X_1) m^2(X_1)}{f^2(X_1)} \right) \right\}^2 \leq \frac{4}{n^2} V_2^2
\end{align*}
\]

according to Lemma 2. Combining the upper bounds for \(\Delta_{221}\) and \(\Delta_{222}\) in (3.5) and taking into account that \(\Delta_{221} = O(n^{-1})\) and \(\Delta_{222} = o(n^{-1})\), we obtain

\[\Delta_{22} \leq (1/3) C_0 n^{-2\gamma/(2\gamma+1)} (\ln n)^2 (1 + o(1)).\]  

An upper bound for \(\Delta_3\).

\[
\Delta_3 = \text{IE} \left( \frac{1}{n} \sum_{i=1}^n \frac{\phi_{j,k}(X_i) Y_i}{f(X_i)} - c_{j,k} \right)^2
\]

\[
= \text{Var} \left( \frac{1}{n} \sum_{i=1}^n \frac{\phi_{j,k}(X_i) Y_i}{f(X_i)} \right) \leq \frac{1}{n} \text{IE} \left( \frac{\phi_{j,k}^2(X) Y^2}{f^2(X)} \right)
\]

\[
\leq \frac{1}{n} \int_{-\infty}^{\infty} \frac{\phi_{j,k}^2(x)}{f^2(x)} m^2(x) f(x) dx \leq \frac{V_2}{n},
\]

so that \(\Delta_3 = o(n^{-2\gamma/(2\gamma+1)})\). Combination of the last result, (3.4) and (3.6) completes the proof.

**Proof of Theorem 1.** Partitioning

\[
\hat{m}_n(x) - m(x) = \sum_k (\hat{c}_{j,k} - c_{j,k}) \phi_{j,k}(x) + \sum_k c_{j,k} \phi_{j,k}(x) - m(x),
\]

we obtain that

\[
\text{IE} (\hat{m}_n(x) - m(x))^2 = \text{IE} \left( \sum_k (\hat{c}_{j,k} - c_{j,k}) \phi_{j,k}(x) \right)^2 + \left( \sum_k c_{j,k} \phi_{j,k}(x) - m(x) \right)^2
\]

\[= R_1 + R_2^2.
\]

Denote \(K_x = \{k \mid k \in \mathbb{Z}, 2^J x - A \leq k \leq 2^J x + A\}\) and observe that \(K_x\) contains at most \((2A+1)\) elements. Then

\[
R_1 = \sum_{k,l} 2^J \text{IE} (\hat{c}_{j,k} - c_{j,k})(\hat{c}_{j,l} - c_{j,l}) \phi(2^J x - k) \phi(2^J x - l)
\]
\[
\leq 2^J \sum_{k,l \in \mathcal{K}_x} \sqrt{\mathbb{E}(\hat{c}_{j,k}-c_{j,k})^2} \sqrt{\mathbb{E}(\hat{c}_{j,l}-c_{j,l})^2} |\phi(2^j x - k)||\phi(2^j x - l)| \\
\leq (2A + 1)^2 \|\phi\|_c^2 2^j \sup_k \mathbb{E}(\hat{c}_{j,k} - c_{j,k})^2,
\]

where \(\|\phi\|_c = \sup_x |\phi(x)|\). Thus, from Lemma 3 it follows that

\[(3.7) R_1 \leq (2A + 1)^2 \|\phi\|_c^2 n^{-2\gamma/(2\gamma+1)} |C_5 n^{1/(2\gamma+1)} (\ln n)^{-2} + C_9 2^J (\ln n)^2 | (1 + o(1)).\]

To construct an upper bound for \(R_2\) recall that \(\text{supp}\ \phi(y) \subset [-A, A]\). Hence, using Taylor's expansion and the fact that \(|k - 2^j x| \leq A\) and \(|y| \leq A\) imply \(|2^{-j}(y + k) - x| \leq 2^{1-j}A\), we derive that

\[
R_2 = \sum_k \phi(2^j x - k) \int_{-\infty}^\infty 2^j \phi(2^j z - k) \ m(z)dz - m(x)
\]

\[
= \sum_{k \in \mathcal{K}_x} \phi(2^j - k) \int_{-A}^A \phi(y) [m(2^{-j}(y + k)) - m(x)] dy
\]

\[
= \sum_{k \in \mathcal{K}_x} \phi(2^j - k) \left\{ \sum_{l=0}^{s-1} \int_{-A}^A \phi(y) \left[ 2^{-j}(y + k) - x \right]^l \frac{m^{(l)}(x)}{l!} dy \right. \\
\left. \int_{-A}^A \phi(y) \left[ 2^{-j}(y + k) - x \right]^s \frac{m^{(s)}(x + \xi(2^{-j}(y + k) - x))}{s!} dy \right\},
\]

where \(0 \leq \xi = \xi(y) \leq 1\). Since \(\int_{-\infty}^\infty y^i \phi(y) = 0, \ i = 1, \ldots, s\), all the integrals \(\int_{-A}^A \phi(y) [2^{-j}(y + k) - x]^i dy = 0\). Therefore, we obtain

\[(3.8) \quad |R_2| \leq (2A + 1) m_s 2^s A^s (s!)^{-1} \|\phi\|_c \|\phi\|_c 2^{-j} = \sqrt{C_7 2^{-j}}.
\]

Combination of (3.7) and (3.8) results in

\[
sup_x \mathbb{E}(\hat{m}_n(x) - m(x))^2 \leq |C_5 n^{1-2\gamma/(2\gamma+1)} (\ln n)^{-2} + C_9 2^J n^{-2\gamma/(2\gamma+1)} (\ln n)^2 + C_7 2^{-2J} | \\
\cdot (1 + o(1)).
\]

To complete the proof choose \(J\) according to (2.7).

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REFERENCES


