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PREDICTION OF THE MAXIMUM SIZE IN WICKSELL'S CORPUSCLE PROBLEM, II

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Abstract. This is a continuing paper of the authors (1998, *Ann. Inst. Statist. Math.*, **50**, 361–377). In the Wicksell corpuscle problem, the maximum size of random spheres in a volume part is to be predicted from the sectional circular distribution of spheres cut by a plane. The size of the spheres is assumed to follow the three-parameter generalized gamma distribution. Prediction methods based on the moment estimation are proposed and their performances are evaluated by simulation. For a practically probable case, one of these prediction methods is as good as a method previously proposed by the authors where the two shape parameters are assumed to be known.

Key words and phrases: Extreme value theory, generalized gamma distribution, Gumbel distribution, metal fatigue, stereology.

1. Introduction

This is a continuing paper of Takahashi and Sibuya (1998). In that paper, the authors studied the prediction of the maximum size of the random spheres in a given volume in the Wicksell's corpuscle problem (see Wicksell (1925) and Sibuya (1999)). Assuming the size distribution of random spheres to be generalized gamma with known shape parameters, Takahashi and Sibuya (1998) proposed a prediction method based on the r largest sizes and total number of the sectional circles on the sectional plane. Simulation results show that the performance of this method is satisfactory.

In this paper, we consider the case that the shape parameters are unknown. First, all three parameters are assumed to be unknown, and next, one shape parameter is assumed to be known. Prediction methods based on the moment estimation are proposed. Estimation methods based on three types of moments are considered, and are evaluated by simulation and their asymptotic variance-covariance matrices.

In Section 2, the Wicksell transform is summarized, and the prediction method in Takahashi and Sibuya (1996) is reviewed. In Section 3, we consider moment estimation methods, and in Section 4 we evaluate their performance by simulation. In Section 5, the asymptotic variance-covariance matrices of the estimates in Section 3 are evaluated to support the simulation in Section 4. For a practically probable case, the moment estimation method using the first two moments of the square root of the data is satisfactory. In the final Section 6, we show a property of the Wicksell transform.

2. A parametric model of Wicksell's corpuscle problem

2.1 *Wicksell's corpuscle problem*

In this paper, we use the framework and notation in Takahashi and Sibuya (1998). λ_V and λ_A are intensities of the spheres in a space and the circles in a sectional plane, respectively. S_V and S_C are the areas of the great circles of sphere in a space and sphere crossing the sectional plane, and S_A is the area of sectional circle. The p.d.f.'s (probability density functions) of S_ω are denoted by $f_\omega(s)$, $\omega = V, C$ and A, respectively.

It is known that

$$\lambda_V = \sqrt{\pi}\lambda_A/(2\mu_0), \qquad \mu_0 := E(\sqrt{S_V}), \tag{2.1}$$

$$f_C(s) = \sqrt{s} f_V(s)/\mu_0, \qquad 0 < s < \infty, \qquad S_A = S_C(1 - U^2), \tag{2.2}$$

$$f_A(s) = \int_0^1 f_C(s/u)\frac{du}{2u\sqrt{1-u}}, \tag{2.3}$$

and

$$\bar{F}_A(s) = \frac{1}{2E(\sqrt{S_V})}\int_0^\infty \frac{1}{\sqrt{w}}\bar{F}_V(s+w)dw, \tag{2.4}$$

where U is the uniform random variable on $(0,1)$ and which is independent of S_C.

2.2 *Generalized gamma model and prediction problem*

Let the generalized gamma distribution with the p.d.f.

$$\frac{1}{\Gamma(\alpha)}\cdot\frac{\gamma}{\xi^{\alpha\gamma}}x^{\alpha\gamma-1}e^{-(x/\xi)^\gamma}1[0 < x < \infty], \qquad \alpha, \gamma, \xi > 0, \tag{2.5}$$

be denoted by $\mathsf{Ga}(\alpha, \gamma, \xi)$. Suppose the area S_V of the great circle of the sphere to follow $\mathsf{Ga}(\alpha, \gamma, \xi)$. The parameters α, γ, ξ and the intensity of the sphere λ_V are unknown. We observe the areas S_A of the circle in k parts of identical area A of the sectional plane. The distribution of S_A is denoted by $\mathsf{WGa}(\alpha, \gamma, \xi)$, whose p.d.f. is the expression (5.6). The number N_A of the circles in a part of area A is the Poisson variable with mean $\lambda_A A$.

We will consider the following prediction problem:

(V) Predict the square root of the maximum area, $\sqrt{W_V}$, of the great circles of the spheres in a part of volume V. The expected number of spheres in the part is $\lambda_V V$.

2.3 *Basis of prediction method*

Our prediction methods are based on the following facts.

PROPOSITION 2.1. *Assume S_V to follow* $\mathsf{Ga}(\alpha, \gamma, \xi)$.

(1) *The area S_C follows* $\mathsf{Ga}(\alpha + \frac{1}{2\gamma}, \gamma, \xi)$.

(2)
$$E(S_A^r) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(r+1)}{\Gamma(r+3/2)}\xi^r\frac{\Gamma\left(\alpha + \dfrac{2r+1}{2\gamma}\right)}{\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)}.$$

(3) *If $\lambda_V V \to \infty$, then the distribution of the square root of the maximum area $\sqrt{W_V}$ is approximated by the Gumbel distribution $\Lambda((t - b_V)/a_V)$, where the scale and*

the location parameters are determined as follows:

$$a_V/\sqrt{\xi} = \tau_V^{1/2\gamma-1}/2\gamma, \qquad \tau_V = \log(\lambda_V V),$$
$$b_V/\sqrt{\xi} = \tau_V^{1/2\gamma} + a_V((\alpha-1)\log\tau_V - \log\Gamma(\alpha))/\sqrt{\xi},$$

where $\Lambda(x) = \exp(-\exp(-x))$.

Remark that

$$\tau_V = \tau_A + \log\frac{V}{A} - \delta, \qquad \delta = \log\left(\frac{2\mu_0}{\sqrt{\pi}}\right) = \frac{1}{2}\log\xi + \log\left\{\frac{2}{\sqrt{\pi}}\Gamma\left(\alpha+\frac{1}{2\gamma}\right) \Big/ \Gamma(\alpha)\right\},$$

where $\tau_A = \log(\lambda_A A)$.

Based on these facts, we predict $\sqrt{W_V}$ as follows. First, we estimate unknown parameters in $\mathsf{WGa}(\alpha,\gamma,\xi)$ and τ_A from the data. Further, from these estimates we estimate δ, τ_V, a_V and b_V. Finally, we estimate the mean and quantiles of $\sqrt{W_V}$ by linear expressions

(2.6) $$\widehat{b_V} + c\,\widehat{a_V},$$

where the coefficient is; for the mean $c = 0.5772\ldots$, Euler's constant, and for the p-quantile $c = -\log(-\log p)$.

The case where the two shape parameters α and γ are known was considered by Takahashi and Sibuya (1998). In that case, the data in k parts of area A are combined, and the prediction method PM3 based on the r largest areas and the total number N of the sectional circles within the part of area kA is satisfactory. The following values of r were recommended

$$r_1 = [\sqrt{N}+0.5], \qquad r_2 = [4\times\log(N)+0.5] \quad \text{and} \quad r_3 = [0.5\times(\log N)^2+0.5],$$

where $[x]$ denotes the integer part of x. The PM3 using these r's have approximately the same mean square errors. The prediction method PM3 with r_1 is denoted by PM31.

3. Estimation methods

In this section, we investigate some methods for estimating the parameters α, γ, ξ and τ_A. The area S_V follows $\mathsf{Ga}(\alpha,\gamma,\xi)$, and the distribution of S_A in k parts of common area A of the sectional plane follows $\mathsf{WGa}(\alpha,\gamma,\xi)$. The parameter τ_A is estimated by $\widehat{\tau_A} = \log\overline{N_A}$, where $\overline{N_A} = \sum_{j=1}^{k} N_{Aj}/k$, N_{Aj} is the number of the circles in a part of area A. So we consider only the estimation of the parameters α, γ and ξ.

The mle's are asymptotically efficient. To obtain the maximum likelihood equations, however, we have to evaluate numerically the integration (5.6), which is troublesome, and the Newton-Raphson method does not work well in the present case. So we consider instead the moment estimation method.

3.1 *The case where α, γ, ξ are unknown*

We consider an estimation method based on the moments of $\log S_A$ (see Stacy and Mihram (1965)). The moment-generating function of $T = \log(S_A/\xi)$ is

$$E(e^{Tt}) = E[(S_A/\xi)^t] = \frac{\sqrt{\pi}}{2}\frac{\Gamma(t+1)}{\Gamma(t+3/2)}\frac{\Gamma\left(\alpha+\dfrac{1}{2\gamma}+\dfrac{t}{\gamma}\right)}{\Gamma\left(\alpha+\dfrac{1}{2\gamma}\right)}.$$

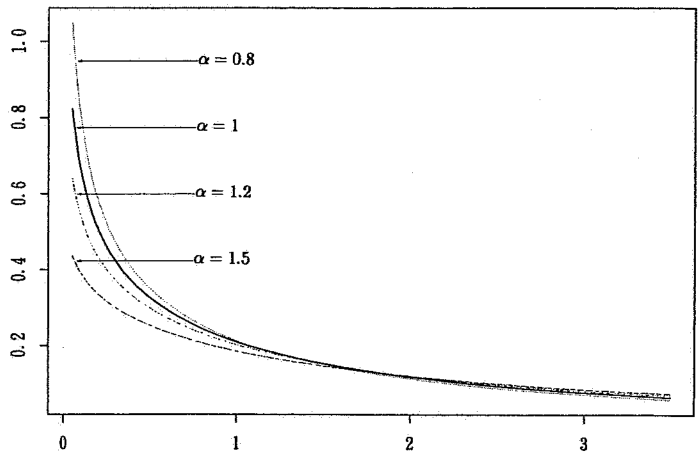

Fig. 1. p.d.f.'s of $\mathsf{WGa}(\alpha, 1/2, 1)$ for $\alpha = 0.8, 1, 1.2, 1.5$.

Hence the cumulant generating function and the cumulant of T are

$$\log\Gamma(\alpha^* + t/\gamma) - \log\Gamma(\alpha^*) + \log\Gamma(t+1) - \log\Gamma(t+3/2) + \log(\sqrt{\pi}/2),$$
$$\kappa_r(T) = \gamma^{-r}\psi^{(r-1)}(\alpha^*) + c_r, \quad r = 1, 2, \dots$$

where $\alpha^* = \alpha + 1/(2\gamma)$, $c_r = \psi^{(r-1)}(1) - \psi^{(r-1)}(3/2)$, $\psi(y) = (d/dy)\log\Gamma(y)$ and $\psi^{(r)}(y) = (d/dy)^r\psi(y)$ are poly-gamma functions. So we have the estimating equation

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\mu_3 - c_3}{(\mu_2 - c_2)^{3/2}} = \dfrac{\psi''(\alpha^*)}{[\psi'(\alpha^*)]^{3/2}}, \\ \gamma = \dfrac{(\mu_2 - c_2)\psi''(\alpha^*)}{(\mu_3 - c_3)\psi'(\alpha^*)}, \\ \xi = \exp[\mu_1' - c_1 - \gamma^{-1}\psi(\alpha^*)], \end{cases}$$

where $\mu_1' = E(\log S_A)$, $\mu_2 = \mathrm{Var}(\log S_A)$, $\mu_3 = E(\log S_A - \mu_1')^3$. Replace population moments in the above equations by sample moments, and obtain successively α^*, γ, ξ and α. The first nonlinear equation of (3.1) has a unique solution, because its right hand side decreasing in α^*.

The prediction methods based on this estimate is denoted by PLM. Unfortunately, its error is rather large as will be shown in Section 4.

3.2 *The case where γ, ξ are unknown*

The p.d.f. of S_A following $\mathsf{WGa}(\alpha, 1/2, 1)$ with $\alpha = 0.8$, 1, 1.2 and 1.5 are numerically computed and shown in Fig. 1. If S_A follows $\mathsf{WGa}(1, 1/2, 1)$ then the diameter of the sphere, $\sqrt{S_V}$, follows the exponential distribution and the parameter value $(\alpha, \gamma) = (1, 1/2)$ is practically probable. The p.d.f.'s for these values of α are so close and this fact makes the estimation of α difficult. Attraction to the exponential distribution of the Wicksell transformed distribution occurs (see Section 6 and Takahashi *et al.*, (1996)). Hence, we consider the case where α is known.

If the moments of $S_A^{1/2}$ and S_A are used, then we have the equations

$$\text{(3.2)}\qquad \begin{cases} \dfrac{E(S_A)}{E^2(S_A^{1/2})} = \dfrac{32}{3\pi^2} \dfrac{\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)\Gamma\left(\alpha + \dfrac{3}{2\gamma}\right)}{\Gamma^2\left(\alpha + \dfrac{2}{2\gamma}\right)}, \\ \xi = \dfrac{3\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)E(S_A)}{2\Gamma\left(\alpha + \dfrac{3}{2\gamma}\right)}. \end{cases}$$

If the two moments of S_A and S_A^2 are used, then we have the equations

$$\text{(3.3)}\qquad \begin{cases} \dfrac{E(S_A^2)}{E^2(S_A)} = \dfrac{6}{5} \dfrac{\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)\Gamma\left(\alpha + \dfrac{5}{2\gamma}\right)}{\Gamma^2\left(\alpha + \dfrac{3}{2\gamma}\right)}, \\ \xi = \dfrac{3\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)E(S_A)}{2\Gamma\left(\alpha + \dfrac{3}{2\gamma}\right)}. \end{cases}$$

Replacing population moments in these equations by sample moments, we have the estimates. The first nonlinear equation of (3.2) has a unique solution, because its right hand side decreasing in γ. The same thing holds for the first nonlinear equation of (3.3). The prediction methods based on (3.2) and (3.3) are denoted by PAKM and PAKM$'$, respectively.

The estimation method based on the first two cumulants of $\log S_A$ is less accurate than PAKM (see Subsection 5.3).

4. Simulation results

The methods described in Section 3 were examined by simulation using S-Plus. The parameter were set to the following practically probable values (see Murakami (1993)). The area S_V of the great circle of the sphere follows $\mathsf{Ga}(1, 1/2, 1)$, that is the diameter of the sphere follows the exponential distribution, and $V/A = 200{,}000$, $\lambda_A A = 10$. Fig. 2 shows the p.d.f.'s of f_V, f_C and f_A in this case. Since $\alpha\gamma \le 1$ and $\gamma \le 1$, F_V is a decreasing failure rate distribution (see Sibuya (1984)). Thus, by Proposition 6.1, F_A is stochastically larger than F_V. The prediction was repeated 1,000 times.

Simulation results for prediction of the expectation $E(\sqrt{W_V})$ are summarized in Table 1 and Fig. 3. PAKM$'$ is less accurate than PAKM, so we omit its simulation result. PLM($k = 700$), PAKM($k = 150$) and PM31($k = 40$) have the similar mean square errors of prediction of $E(\sqrt{W_V})$. It is found that PLM is more accurate than those prediction methods based on the moments of S_A^r with $r = 1/2$, 1, 3/2, or $r = 1$, 2, 3. But PLM needs larger k and it is not practical. In Subsection 5.3 we confirm the above fact by calculating the asymptotic variance-covariance matrices (a.v.c.m.'s) of the related moment estimators.

The p.d.f.'s of $\mathsf{WGa}(\alpha, 1/2, 1)$ are similar for close α values as remarked in Subsection 3.2, so we examine the robustness of the PAKM. Table 2 shows simulation results of the

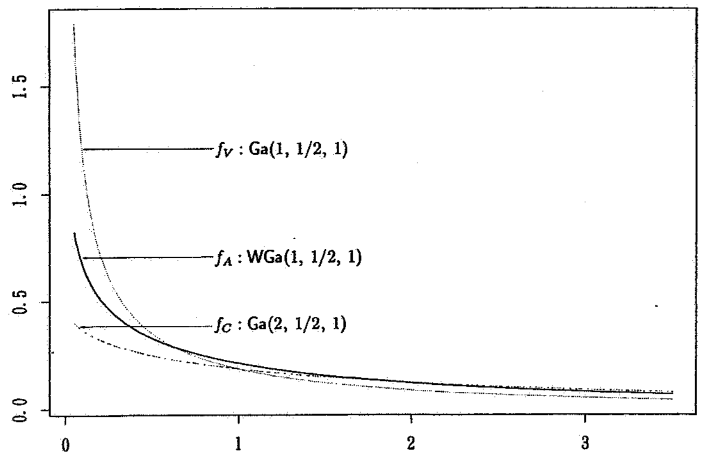

Fig. 2. p.d.f.'s f_V, f_C and f_A for $(\alpha, \gamma, \xi) = (1, 1/2, 1)$.

Table 1. Errors of PM31, PAKM and PLM in predicting $E(\sqrt{W_V})$. $\lambda_A A = 10$.

Parameters		α	γ	ξ	τ_V	$E(\sqrt{W_V})$
Methods	True	1.000	0.500	1.000	14.388	14.965
PM31 ($k = 40$)	Bias			−0.018	0.020	−0.256
α, γ known	S. D.			0.057	0.076	0.796
	M.S.E.			0.004	0.006	0.700
PM31 ($k = 80$)	Bias			−0.022	0.026	−0.311
α, γ known	S. D.			0.045	0.059	0.634
	M.S.E.			0.003	0.004	0.498
PAKM ($k = 100$)	Bias		0.001	0.006	0.001	−0.001
α known	S. D.		0.021	0.131	0.058	0.936
	M.S.E.		0.000	0.017	0.003	0.877
PAKM ($k = 150$)	Bias		0.001	0.004	0.000	−0.012
α known	S. D.		0.017	0.106	0.050	0.756
	M.S.E.		0.000	0.011	0.002	0.572
PAKM ($k = 200$)	Bias		0.001	0.003	−0.000	−0.017
α known	S. D.		0.016	0.096	0.041	0.669
	M.S.E.		0.000	0.009	0.002	0.448
PLM ($k = 700$)	Bias	0.038	−0.000	0.058	0.002	0.031
	S. D.	0.251	0.048	0.511	0.033	0.862
	M.S.E.	0.064	0.002	0.265	0.001	0.743

prediction of $E(\sqrt{W_V})$ by PM31 misspecifying (α, γ) as $(1, 1/2)$ (see Takahashi and Sibuya (1998), Table 3), and PAKM misspecifying α as 1. It shows that PM31 and PAKM are robust if (α, γ) is close to $(1, 1/2)$. PAKM needs larger k than PM31, but it is more accurate than PM31 if γ departs from 1/2.

We did the simulation work for (α, γ), $\alpha = 0.5, 1.5, 2, \gamma = 0.3, 0.5, 1, 1.5, 2$. In these

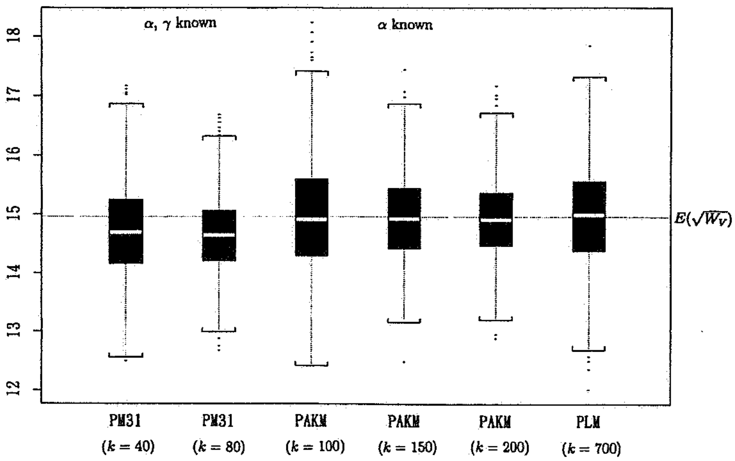

Fig. 3. Boxplots of PM31, PAKM and PLM in predicting $E(\sqrt{W_V})$. $\lambda_A A = 10$.

cases, PAKM is relatively good except $\gamma = 0.3$.

It is known that the log-normal distribution is a limit of the generalized gamma distribution (Lawless (1980)). That is, as $\alpha \to \infty$, $\mathsf{Ga}(\alpha, 1/\sqrt{\alpha}, 1/\alpha^{\sqrt{\alpha}})$ approaches the standard log-normal distribution. We had done the simulation work for the case $\sqrt{S_V} \sim \mathsf{Ga}(\alpha, 1/\sqrt{\alpha}, 1/\alpha^{\sqrt{\alpha}})$. In this case, $\gamma = 1/\sqrt{\alpha}$ is too small for large α and the performance of PAKM is worse.

5. Asymptotic variances

In this section, the size of a random sample is always n.

5.1 *Fisher Information of* $\mathsf{Ga}(\alpha, \gamma, \xi)$

The Fisher Information matrix for (α, γ, ξ) of the generalized gamma distribution $\mathsf{Ga}(\alpha, \gamma, \xi)$ is

$$\begin{pmatrix} \psi'(\alpha) & -\psi(\alpha)/\gamma & \gamma/\xi \\ -\psi(\alpha)/\gamma & D_0/\gamma^2 & -[\alpha\psi(\alpha)+1]/\xi \\ \gamma/\xi & -[\alpha\psi(\alpha)+1]/\xi & \alpha\gamma^2/\xi^2 \end{pmatrix}, \tag{5.1}$$

where $D_0 = \alpha\psi'(\alpha+1) + \alpha[\psi(\alpha+1)]^2 + 1 = \alpha\psi'(\alpha) + \alpha[\psi(\alpha)]^2 + 2\psi(\alpha) + 1$. And the a.v.c.m.'s of mle $(\hat{\alpha}, \hat{\gamma}, \hat{\xi})$ and $(\hat{\gamma}, \hat{\xi})$, α known, are

$$\frac{1}{nD_1}\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{pmatrix}, \tag{5.2}$$

and

$$\frac{1}{nD_2}\begin{pmatrix} \alpha\gamma^2 & \xi[\alpha\psi(\alpha)+1] \\ \xi[\alpha\psi(\alpha)+1] & \xi^2 D_0/\gamma^2 \end{pmatrix}, \tag{5.3}$$

where $D_1 = \alpha^2[\psi'(\alpha)]^2 - \psi'(\alpha) - 1$, $D_2 = \alpha^2\psi'(\alpha) + \alpha - 1$, $D_{11} = \alpha^2\psi'(\alpha) + \alpha - 1$, $D_{12} = -\gamma$, $D_{13} = -\xi[\alpha\psi'(\alpha)+\psi(\alpha)+1]/\gamma$, $D_{22} = \gamma^2[\alpha\psi'(\alpha)-1]$, $D_{23} = \xi\{\psi'(\alpha)[\alpha\psi(\alpha)+1] - \psi(\alpha)\}$, $D_{33} = \xi^2\{\psi'(\alpha)D_0 - [\psi(\alpha)]^2\}/\gamma^2$, respectively. See Hager and Bain (1970).

Table 2. Robustness of PM31 and PAKM in predicting $E(\sqrt{W_V})$. Three numbers in each entry show (1) the true values of $E(\sqrt{W_V})$, (2) m.s.e. of PM31 ($k = 40$) misspecifying (α, γ) as $(1, 0.5)$, (3) m.s.e. of PAKM ($k = 150$) misspecifing α as 1. $\lambda_A A = 10$.

$\gamma \setminus \alpha$	0.7	0.8	0.9	1.0	1.1	1.2	1.3
				46.626			
0.35				190.023			
				11.330			
			28.606	29.120	29.614		
0.4			39.024	35.041	31.817		
			3.674	3.407	3.457		
		19.463	19.806	20.135	20.453	20.760	
0.45		8.040	6.797	5.504	4.529	3.785	
		1.916	1.426	1.330	1.435	1.485	
	17.654	17.977	18.287	18.586	18.874	19.153	19.422
0.4625	6.273	5.121	4.194	3.396	2.651	2.054	1.601
	1.778	1.383	1.241	1.153	1.106	1.155	1.364
	16.378	16.672	16.954	17.227	17.489	17.743	17.988
0.475	3.999	3.255	2.530	1.875	1.446	1.100	0.979
	1.530	1.194	0.994	0.872	0.934	1.030	1.058
	15.252	15.521	15.779	16.028	16.268	16.500	16.725
0.4875	2.435	1.806	1.461	1.080	0.875	0.783	0.839
	1.279	0.999	0.876	0.782	0.764	0.862	0.977
	14.254	14.500	14.737	14.965	15.186	15.399	15.605
0.5	1.493	1.135	0.871	0.700	0.697	0.773	1.003
	1.165	0.862	0.738	0.572	0.635	0.741	0.811
	12.567	12.777	12.978	13.173	13.361	13.542	13.718
0.525	0.563	0.476	0.529	0.661	0.913	1.215	1.698
	0.872	0.648	0.494	0.466	0.480	0.493	0.630
		11.387	11.560	11.728	11.890	12.047	
0.55		0.572	0.819	1.102	1.552	2.007	
		0.468	0.365	0.335	0.343	0.388	
			9.438	9.567	9.691		
0.6			1.972	2.494	3.074		
			0.226	0.186	0.204		
				8.049			
0.65				3.761			
				0.114			

The $n\times$a.v.c.m.'s of mle $(\hat{\alpha}, \hat{\gamma}, \hat{\xi})$ and $(\hat{\gamma}, \hat{\xi})$, α is known, calculated for Ga$(1, 1/2, 1)$, are given by

$$\begin{pmatrix} 27.02 & -8.21 & -67.93 \\ -8.21 & 2.65 & 20.91 \\ -67.93 & 20.91 & 175.22 \end{pmatrix}, \tag{5.4}$$

and

$$\begin{pmatrix} 0.15 & 0.26 \\ 0.26 & 4.43 \end{pmatrix}, \tag{5.5}$$

respectively. This is to be compared with the numerical values of the next section.

5.2 *Fisher Information of* $\mathsf{WGa}(\alpha, \gamma, \xi)$

By (2.3) and Proposition 2.1 (1), the p.d.f. f_A of $\mathsf{WGa}(\alpha, \gamma, \xi)$ is

$$f_A(s) = \frac{\gamma s^{\alpha\gamma - 1/2}}{2\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)\xi^{\alpha\gamma+1/2}} \int_0^1 \frac{\exp\left\{-\left(\dfrac{s}{\xi u}\right)^\gamma\right\}}{u^{\alpha\gamma+1/2}\sqrt{1-u}}du, \quad s > 0. \tag{5.6}$$

Hence,

$$\begin{aligned} \log f_A(s) &= \log\gamma + (\alpha\gamma - 1/2)\log s - \log 2 - \log\Gamma\left(\alpha + \frac{1}{2\gamma}\right) \\ &\quad -(\alpha\gamma + 1/2)\log\xi + \log I, \end{aligned} \tag{5.7}$$

where

$$I = I(\alpha, \gamma, \xi) = \int_0^1 \frac{\exp\left\{-\left(\dfrac{s}{\xi u}\right)^\gamma\right\}}{u^{\alpha\gamma+1/2}\sqrt{1-u}}du.$$

The Fisher Information matrix for (α, γ, ξ) of $\mathsf{WGa}(\alpha, \gamma, \xi)$ is

$$\begin{pmatrix} w_{\alpha\alpha} & w_{\alpha\gamma} & w_{\alpha\xi} \\ w_{\alpha\gamma} & w_{\gamma\gamma} & w_{\gamma\xi} \\ w_{\alpha\xi} & w_{\gamma\xi} & w_{\xi\xi} \end{pmatrix}, \tag{5.8}$$

where

$$\begin{aligned} &w_{\alpha\alpha} = \psi'(\alpha^*) + e_{\alpha\alpha}, \quad w_{\alpha\gamma} = -\{\psi'(\alpha^*)/(2\gamma^2) + \psi(\alpha^*)/\gamma + \psi(1) - \psi(3/2)\} + e_{\alpha\gamma}, \\ &w_{\alpha\xi} = \gamma/\xi + e_{\alpha\xi}, \quad w_{\gamma\gamma} = \{\psi'(\alpha^*)/(4\gamma^2) + \psi(\alpha^*)/\gamma + 1\}/\gamma^2 + e_{\gamma\gamma}, \\ &w_{\gamma\xi} = \alpha/\xi + e_{\gamma\xi}, \quad w_{\xi\xi} = -(\alpha\gamma + 1/2)/\xi^2 + e_{\xi\xi}, \quad \alpha^* = \alpha + \frac{1}{2\gamma}, \end{aligned}$$

and

$$e_{\delta\varepsilon} = E\left[\frac{I_\delta I_\varepsilon - I_{\delta\varepsilon}I}{I^2}\right], \quad I_\delta = \frac{\partial}{\partial\delta}I, \quad I_{\delta\varepsilon} = \frac{\partial^2}{\partial\delta\partial\varepsilon}I, \quad \delta, \varepsilon \in \{\alpha, \gamma, \xi\}.$$

The $n\times$a.v.c.m.'s of mle $(\hat\alpha_W, \hat\gamma_W, \hat\xi_W)$ and $(\hat\gamma_W, \hat\xi_W)$, α is known, calculated for WGa $(1, 1/2, 1)$, are given by

$$\begin{pmatrix} 170.7 & -32.2 & -354.0 \\ -32.2 & 6.5 & 69.0 \\ -354.0 & 69.0 & 749.1 \end{pmatrix}, \tag{5.9}$$

and

$$\begin{pmatrix} 0.39 & 2.19 \\ 2.19 & 14.82 \end{pmatrix} \tag{5.10}$$

respectively. For numerical integration of $e_{\delta\varepsilon}$, $\delta, \varepsilon \in \{\alpha, \gamma, \xi\}$, we use the Double Exponential Formula (Takahasi and Mori (1974)).

The a.v.c.m. of mle of (α,γ,ξ) of $\mathsf{WGa}(1,1/2,1)$ are very large compared with those of the generalized gamma distribution $\mathsf{Ga}(1,1/2,1)$. If (α,γ) is close to $(1,1/2)$, then the same thing holds.

It is observed by numerical experience that the asymptotic variances of the mle $(\hat{\alpha},\hat{\gamma})$ depend on (α,γ) but not on ξ.

5.3 *Asymptotic variance-covariance matrix of the moment estimation method*

The following result is easily obtained (see, for example, Rao (1973), Chapter 6).

PROPOSITION 5.1. *Let Θ be an open subset of $\mathcal{R}^k$, and let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with the p.d.f. $f(x \mid \boldsymbol{\theta})$, $\boldsymbol{\theta} \in \Theta$. If the p.d.f. $f(x \mid \boldsymbol{\theta})$ has the $2k$-th moment, then the moment estimator $\hat{\boldsymbol{\theta}}_{Mn}$ is an asymptotically normal estimator for $\boldsymbol{\theta}$:*

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_{Mn} - \boldsymbol{\theta}) \xrightarrow{\mathrm{w}} \mathsf{N}(0, \Sigma_M(\boldsymbol{\theta})), \qquad as \quad n \to \infty, \tag{5.11}$$

where $\Sigma_M(\boldsymbol{\theta}) = M(\boldsymbol{\theta})D(\boldsymbol{\theta})M(\boldsymbol{\theta})'$,

$$M(\boldsymbol{\theta}) = \left(\frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{\mu}(\boldsymbol{\theta})\right)^{-1}, \qquad \boldsymbol{\mu}(\boldsymbol{\theta}) = (\mu_1(\boldsymbol{\theta}), \ldots, \mu_k(\boldsymbol{\theta}))',$$

$\mu_j(\boldsymbol{\theta})$ is the j-th moment, and $D(\boldsymbol{\theta})$ is the $k \times k$ variance-covariance matrix of $f(x \mid \boldsymbol{\theta})$.

Suppose that S_A follows $\mathsf{WGa}(\alpha,\gamma,\xi)$ and put

$$\nu_r = \nu_r(\alpha,\gamma,\xi) = E(S_A^r) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(r+1)}{\Gamma(r+3/2)}\xi^r \frac{\Gamma\left(\alpha + \dfrac{1}{2\gamma} + \dfrac{r}{\gamma}\right)}{\Gamma\left(\alpha + \dfrac{1}{2\gamma}\right)}, \qquad r > 0.$$

If the moments of $S_A^{1/2}$, S_A and $S_A^{3/2}$ are used, then $\boldsymbol{\theta} = (\alpha,\gamma,\xi)'$, $\boldsymbol{\mu}_1(\boldsymbol{\theta}) = (\nu_{1/2}, \nu_1, \nu_{3/2})$ and the $n\times$a.v.c.m., $\Sigma_{\{1/2,1,3/2\}} = \Sigma_{\{1/2,1,3/2\}}(\boldsymbol{\theta})$, is obtained as follows:

$$\Sigma_{\{1/2,1,3/2\}} = M_1(\boldsymbol{\theta})D_1(\boldsymbol{\theta})M_1(\boldsymbol{\theta})',$$

where $D_1(\boldsymbol{\theta}) = (\nu_{(i+j)/2} - \nu_{i/2}\nu_{j/2})_{i,j=1,2,3}$ and

$$M_1(\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{\partial \nu_{1/2}}{\partial \alpha} & \dfrac{\partial \nu_{1/2}}{\partial \gamma} & \dfrac{\partial \nu_{1/2}}{\partial \xi} \\ \dfrac{\partial \nu_1}{\partial \alpha} & \dfrac{\partial \nu_1}{\partial \gamma} & \dfrac{\partial \nu_1}{\partial \xi} \\ \dfrac{\partial \nu_{3/2}}{\partial \alpha} & \dfrac{\partial \nu_{3/2}}{\partial \gamma} & \dfrac{\partial \nu_{3/2}}{\partial \xi} \end{pmatrix}^{-1}$$

$$
= \begin{pmatrix}
\nu_{1/2}\left[\psi\left(\alpha+\frac{1}{\gamma}\right)-\psi\left(\alpha+\frac{1}{2\gamma}\right)\right] & \frac{\nu_{1/2}\left[\psi\left(\alpha+\frac{1}{2\gamma}\right)-2\psi\left(\alpha+\frac{1}{\gamma}\right)\right]}{2\gamma^2} & \frac{\nu_{1/2}}{2\xi} \\
\nu_{1}\left[\psi\left(\alpha+\frac{3}{2\gamma}\right)-\psi\left(\alpha+\frac{1}{2\gamma}\right)\right] & \frac{\nu_{1}\left[\psi\left(\alpha+\frac{1}{2\gamma}\right)-3\psi\left(\alpha+\frac{3}{2\gamma}\right)\right]}{2\gamma^2} & \frac{\nu_{1}}{\xi} \\
\nu_{3/2}\left[\psi\left(\alpha+\frac{2}{\gamma}\right)-\psi\left(\alpha+\frac{1}{2\gamma}\right)\right] & \frac{\nu_{3/2}\left[\psi\left(\alpha+\frac{1}{2\gamma}\right)-4\psi\left(\alpha+\frac{2}{\gamma}\right)\right]}{2\gamma^2} & \frac{3\nu_{3/2}}{2\xi}
\end{pmatrix}^{-1}.
$$

Suppose α to be known, then $\boldsymbol{\theta} = (\gamma, \xi)'$. If the moments of $S_A^{1/2}$ and S_A are used, then $\boldsymbol{\mu}_2(\boldsymbol{\theta}) = (\nu_{1/2}, \nu_1)$ and the $n\times$a.v.c.m., $\Sigma_{\{1/2,1\}} = \Sigma_{\{1/2,1\}}(\boldsymbol{\theta})$, is obtained as follows:

$$
\Sigma_{\{1/2,1\}} = M_2(\boldsymbol{\theta})D_2(\boldsymbol{\theta})M_2(\boldsymbol{\theta})',
$$

where $D_2(\boldsymbol{\theta}) = (\nu_{(i+j)/2} - \nu_{i/2}\nu_{j/2})_{i,j=1,2}$, and

$$
M_2(\boldsymbol{\theta}) = \begin{pmatrix}
\frac{\nu_{1/2}\left[\psi\left(\alpha+\frac{1}{2\gamma}\right)-2\psi\left(\alpha+\frac{1}{\gamma}\right)\right]}{2\gamma^2} & \frac{\nu_{1/2}}{2\xi} \\
\frac{\nu_{1}\left[\psi\left(\alpha+\frac{1}{2\gamma}\right)-3\psi\left(\alpha+\frac{3}{2\gamma}\right)\right]}{2\gamma^2} & \frac{\nu_{1}}{\xi}
\end{pmatrix}^{-1}.
$$

If the moments of S_A, S_A^2 and S_A^3 are used, then $\boldsymbol{\theta} = (\alpha, \gamma, \xi)'$, $\boldsymbol{\mu}_3(\boldsymbol{\theta}) = (\nu_1, \nu_2, \nu_3)$ and the $n\times$a.v.c.m., $\Sigma_{\{1,2,3\}} = \Sigma_{\{1,2,3\}}(\boldsymbol{\theta})$, is obtained as follows:

$$
\Sigma_{\{1,2,3\}} = M_3(\boldsymbol{\theta})D_3(\boldsymbol{\theta})M_3(\boldsymbol{\theta})',
$$

where $D_3(\boldsymbol{\theta}) = (\nu_{i+j} - \nu_i\nu_j)_{i,j=1,2,3}$, and

$$
M_3(\boldsymbol{\theta}) = \begin{pmatrix}
\nu_{1}\left[\psi\left(\alpha+\frac{3}{2\gamma}\right)-\psi\left(\alpha+\frac{1}{2\gamma}\right)\right] & \frac{\nu_{1}\left[\psi\left(\alpha+\frac{1}{2\gamma}\right)-3\psi\left(\alpha+\frac{3}{2\gamma}\right)\right]}{2\gamma^2} & \frac{\nu_{1}}{\xi} \\
\nu_{2}\left[\psi\left(\alpha+\frac{5}{2\gamma}\right)-\psi\left(\alpha+\frac{1}{2\gamma}\right)\right] & \frac{\nu_{2}\left[\psi\left(\alpha+\frac{1}{2\gamma}\right)-5\psi\left(\alpha+\frac{5}{2\gamma}\right)\right]}{2\gamma^2} & \frac{2\nu_{2}}{\xi} \\
\nu_{3}\left[\psi\left(\alpha+\frac{7}{2\gamma}\right)-\psi\left(\alpha+\frac{1}{2\gamma}\right)\right] & \frac{\nu_{3}\left[\psi\left(\alpha+\frac{1}{2\gamma}\right)-7\psi\left(\alpha+\frac{7}{2\gamma}\right)\right]}{2\gamma^2} & \frac{3\nu_{3}}{\xi}
\end{pmatrix}^{-1}.
$$

Suppose α to be known, then $\boldsymbol{\theta} = (\gamma, \xi)'$. If the moments of S_A and S_A^2 are used, then $\boldsymbol{\mu}_4(\boldsymbol{\theta}) = (\nu_1, \nu_2)$ and the $n\times$a.v.c.m., $\Sigma_{\{1,2\}} = \Sigma_{\{1,2\}}(\boldsymbol{\theta})$ is obtained as follows:

$$
\Sigma_{\{1,2\}} = M_4(\boldsymbol{\theta})D_4(\boldsymbol{\theta})M_4(\boldsymbol{\theta})',
$$

where $D_4(\boldsymbol{\theta}) = (\nu_{i+j} - \nu_i\nu_j)_{i,j=1,2}$, and

$$M_4(\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{\nu_1\left[\psi\left(\alpha+\dfrac{1}{2\gamma}\right) - 3\psi\left(\alpha+\dfrac{3}{2\gamma}\right)\right]}{2\gamma^2} & \dfrac{\nu_1}{\xi} \\ \dfrac{\nu_2\left[\psi\left(\alpha+\dfrac{1}{2\gamma}\right) - 5\psi\left(\alpha+\dfrac{5}{2\gamma}\right)\right]}{2\gamma^2} & \dfrac{2\nu_2}{\xi} \end{pmatrix}^{-1}.$$

For estimating $\boldsymbol{\theta} = (\alpha, \gamma, \xi)'$ we use the first three cumulants of $\log S_A$ (see Subsection 3.1), the $n\times$a.v.c.m., $\Sigma^*_{\{1,2,3\}} = \Sigma^*_{\{1,2,3\}}(\boldsymbol{\theta})$, of this estimation method is obtained by the δ method :

$$\Sigma^*_{\{1,2,3\}} = (H_5^*(\boldsymbol{\theta})M_5(\boldsymbol{\theta})H_5(\boldsymbol{\theta}))D_5(\boldsymbol{\theta})(H_5^*(\boldsymbol{\theta})M_5(\boldsymbol{\theta})H_5(\boldsymbol{\theta}))',$$

where $D_5(\boldsymbol{\theta}) = (\mu'_{i+j} - \mu'_i\mu'_j)_{i,j=1,2,3}$,

$$H_5(\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{\partial\kappa_1}{\partial\mu'_1} & \dfrac{\partial\kappa_1}{\partial\mu'_2} & \dfrac{\partial\kappa_1}{\partial\mu'_3} \\ \dfrac{\partial\kappa_2}{\partial\mu'_1} & \dfrac{\partial\kappa_2}{\partial\mu'_2} & \dfrac{\partial\kappa_2}{\partial\mu'_3} \\ \dfrac{\partial\kappa_3}{\partial\mu'_1} & \dfrac{\partial\kappa_3}{\partial\mu'_2} & \dfrac{\partial\kappa_3}{\partial\mu'_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2\mu'_1 & 1 & 0 \\ -3\mu'_2 + 6\mu'^2_1 & -3\mu'_1 & 1 \end{pmatrix},$$

$$M_5(\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{\partial\kappa_1}{\partial\alpha^*} & \dfrac{\partial\kappa_1}{\partial\gamma} & \dfrac{\partial\kappa_1}{\partial\xi} \\ \dfrac{\partial\kappa_2}{\partial\alpha^*} & \dfrac{\partial\kappa_2}{\partial\gamma} & \dfrac{\partial\kappa_2}{\partial\xi} \\ \dfrac{\partial\kappa_3}{\partial\alpha^*} & \dfrac{\partial\kappa_3}{\partial\gamma} & \dfrac{\partial\kappa_3}{\partial\xi} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{\psi'(\alpha^*)}{\gamma} & -\dfrac{\psi(\alpha^*)}{\gamma^2} & \dfrac{1}{\xi} \\ \dfrac{\psi''(\alpha^*)}{\gamma^2} & -\dfrac{2\psi'(\alpha^*)}{\gamma^3} & 0 \\ \dfrac{\psi'''(\alpha^*)}{\gamma^3} & -\dfrac{3\psi''(\alpha^*)}{\gamma^4} & 0 \end{pmatrix}^{-1},$$

$$H_5^*(\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{\partial\alpha}{\partial\alpha^*} & \dfrac{\partial\alpha}{\partial\gamma} & \dfrac{\partial\alpha}{\partial\xi} \\ \dfrac{\partial\gamma}{\partial\alpha^*} & \dfrac{\partial\gamma}{\partial\gamma} & \dfrac{\partial\gamma}{\partial\xi} \\ \dfrac{\partial\xi}{\partial\alpha^*} & \dfrac{\partial\xi}{\partial\gamma} & \dfrac{\partial\xi}{\partial\xi} \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{1}{2\gamma^2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mu'_j = E(\log S_A)^j$ and κ_j is the j-th cumulant of $\log S_A$. The relations between moments and cumulants are given in Stuart and Ord (1987).

Assume α to be known. For estimating $\boldsymbol{\theta} = (\gamma, \xi)'$ we use the first two cumulants of $\log S_A$, then the $n\times$a.v.c.m., $\Sigma^*_{\{1,2\}} = \Sigma^*_{\{1,2\}}(\boldsymbol{\theta})$, of this estimation method is obtained:

$$\Sigma^*_{\{1,2\}} = (M_6(\boldsymbol{\theta})H_6(\boldsymbol{\theta}))D_6(\boldsymbol{\theta})(M_6(\boldsymbol{\theta})H_6(\boldsymbol{\theta}))',$$

where $D_6(\boldsymbol{\theta}) = (\mu'_{i+j} - \mu'_i\mu'_j)_{i,j=1,2}$,

$$H_6(\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{\partial\kappa_1}{\partial\mu'_1} & \dfrac{\partial\kappa_1}{\partial\mu'_2} \\ \dfrac{\partial\kappa_2}{\partial\mu'_1} & \dfrac{\partial\kappa_2}{\partial\mu'_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2\mu'_1 & 1 \end{pmatrix},$$

and

$$M_6(\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{\partial\kappa_1}{\partial\gamma} & \dfrac{\partial\kappa_1}{\partial\xi} \\ \dfrac{\partial\kappa_2}{\partial\gamma} & \dfrac{\partial\kappa_2}{\partial\xi} \end{pmatrix}^{-1} = \begin{pmatrix} -\dfrac{2\gamma\psi\left(\alpha+\frac{1}{2\gamma}\right)+\psi'\left(\alpha+\frac{1}{2\gamma}\right)}{2\gamma^3} & \dfrac{1}{\xi} \\ -\dfrac{4\gamma\psi'\left(\alpha+\frac{1}{2\gamma}\right)+\psi''\left(\alpha+\frac{1}{2\gamma}\right)}{2\gamma^4} & 0 \end{pmatrix}^{-1}.$$

Suppose S_A follows WGa(1, 1/2, 1) then we have the following $n\times$a.v.c.m.'s:

$$(5.12)\quad \Sigma_{\{1/2,1,3/2\}} = \begin{pmatrix} 609.7 & -107.8 & -1223.0 \\ -107.8 & 19.5 & 218.6 \\ -1223.0 & 218.6 & 2469.5 \end{pmatrix}, \quad \Sigma_{\{1/2,1\}} = \begin{pmatrix} 0.48 & 2.74 \\ 2.74 & 18.00 \end{pmatrix},$$

$$(5.13)\quad \Sigma_{\{1,2,3\}} = \begin{pmatrix} 17117.6 & -2422.7 & -30464.6 \\ -2422.7 & 344.1 & 4319.1 \\ -30464.6 & 4319.1 & 54266.8 \end{pmatrix}, \quad \Sigma_{\{1,2\}} = \begin{pmatrix} 1.87 & 11.80 \\ 11.80 & 77.21 \end{pmatrix},$$

and

$$(5.14)\quad \Sigma^*_{\{1,2,3\}} = \begin{pmatrix} 380.3 & -77.4 & -817.1 \\ -77.4 & 16.2 & 168.7 \\ -817.1 & 168.7 & 1772.0 \end{pmatrix}, \quad \Sigma^*_{\{1,2\}} = \begin{pmatrix} 0.69 & 3.62 \\ 3.62 & 21.67 \end{pmatrix}.$$

If we have to estimate (α, γ, ξ), then the estimation method using the first three cumulants is the best among the above three methods. If α is known, then the moment estimation method based on the moments $S_A^{1/2}$ and S_A is the best among the above three methods. By (5.9) and (5.12), the asymptotic relative efficiencies of this method are

$$\frac{0.39}{0.48} \doteq 0.81 \quad \text{and} \quad \frac{14.82}{18.00} \doteq 0.82,$$

respectively. So we recommend to use moments $S_A^{1/2}$ and S_A for the estimation of (γ, ξ).

If (α, γ) is close to $(1, 1/2)$ then the same thing holds.

6. Supplement

In this section, we show a property of the Wicksell transform:

PROPOSITION 6.1. *If the function* $-\log \bar{F}_V(s)$ *is super-additive (or sub-additive) the Wicksell transform* F_A *of* F_V *is stochastically smaller (or larger) than* F_V.

PROOF. From (2.4),

$$\bar{F}_A(s) = \int_0^\infty \frac{1}{\sqrt{w}}\bar{F}_V(s+w)dw \Big/ \int_0^\infty \frac{1}{\sqrt{w}}\bar{F}_V(w)dw,$$

so, $\bar{F}_A(s) \leq \bar{F}_V(s)$ is equivalent to

$$\int_0^\infty \frac{1}{\sqrt{w}}(\bar{F}_V(s+w) - \bar{F}_V(s)\bar{F}_V(w))dw \leq 0.$$

The condition of the proposition is sufficient for the inequality. □

REFERENCES

Hager, H. W. and Bain, L. J. (1970). Inferential procedures for the generalized gamma distribution, *J. Amer. Statist. Assoc.*, **65**, 1601–1609.

Lawless, J. F. (1980). Inference in the generalized gamma and log gamma distribution, *Technometrics*, **22**, 409–419.

Murakami, Y. (1993). *Metal Fatigue : Effects of Small Defects and Nonmetallic Inclusions*, Yokendo, Tokyo (in Japanese).

Murakami, Y. (1994). Inclusion rating by statistics of extreme values and its application to fatigue strength prediction and quality control of materials, *Journal of Research of the National Institute of Standards and Technology*, **99**, 345–351.

Rao, C. R. (1973). *Linear Statistical Inference and Its Applications*, 2nd ed. Wiley, New York.

Sibuya, M. (1984). Unimodal hazard functions, *Japanese Journal of Applied Statistics*, **13**, 115–124 (in Japanese).

Sibuya, M. (1999). Wicksell's corpuscle problem, *Encyclopedia of Statistical Sciences*: Update Vol. 3 (eds. S. Kotz, C. Read and D. L. Bankd), 809, Wiley, New York.

Stacy, E. W. and Mihram, G. A. (1965). Parameter estimation for a generalized gamma distribution, *Technometrics*, **7**, 349–358.

Stuart, A. and Ord, J. K. (1987). *Kendall's Advanced Theory of Statistics*, Vol. 1, 5th ed., Griffin, London.

Takahashi, R. and Sibuya, M. (1996). The maximum size of the planar sections of random spheres and its application to metallurgy, *Ann. Inst. Statist. Math.*, **48**, 127–144.

Takahashi, R. and Sibuya, M. (1998). Prediction of the maximum size in Wicksell's corpuscle problem, *Ann. Inst. Statist. Math.*, **50**, 361–377.

Takahashi, R., Nawata, S. and Sibuya, M. (1996). Wicksell inverse transformation: An ill-posed problem, *Japanese Journal of Applied Statistics*, **25**, 121–142 (in Japanese).

Takahasi, H. and Mori, M. (1974). Double exponential formulas for numerical integration, *Pub. Res. Inst. Math. Sci.*, **9**, 721–741.

Wicksell, S. D. (1925). The corpuscle problem. A mathematical study of a biometric problem, *Biometrika*, **17**, 84–99.