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DEPENDENCE PROPERTIES OF MULTIVARIATE MIXTURE DISTRIBUTIONS AND THEIR APPLICATIONS

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Abstract. Consider a multivariate mixture model where the random variables $X_1, \dots, X_n$ given $(\Theta_1, \dots, \Theta_n)$, are conditionally independent. Conditions are obtained under which different kinds of positive dependence hold among X_i's. The results obtained are applied to a variety of problems including the concomitants of order statistics and of record values; and to frailty models.

Key words and phrases: Dependence by total positivity (DTP), dependence by reverse regular (DRR), MTP_2 dependence, associated random variables, concomitants of order statistics and record values, frailty models.

1. Introduction

Let $X_1, \dots, X_n$ be n random variables such that they are conditionally independent given some random vector $\boldsymbol{\Theta} = (\Theta_1, \dots, \Theta_m)$. It is of interest to know which kind of dependence arises among X_i's when $\boldsymbol{\Theta}$ is unknown. If $F_i(\cdot \mid \theta_1, \dots, \theta_m)$ denotes the conditional distribution function of X_i given $\boldsymbol{\Theta} = (\theta_1, \dots, \theta_m)$ and $G(\theta_1, \dots, \theta_m)$ denotes the joint distribution function of $\boldsymbol{\Theta}$, then the joint distribution function of $\boldsymbol{X}$ is given by

$$F(\boldsymbol{x}) = \int_{R^m} \prod_{i=1}^{n} F_i(x_i \mid \theta_1, \dots, \theta_m) dG(\theta_1, \dots, \theta_m).$$

If $F_i(\cdot \mid \theta_1, \dots, \theta_m)$ is absolutely continuous with respect to the Lebesgue measure on R for each $(\theta_1, \dots, \theta_m)$ in the support of $\boldsymbol{\Theta}$ with a density function $f_i(\cdot \mid \theta_1, \dots, \theta_m)$, then the joint distribution of $X_1, \dots, X_n$ is absolutely continuous with respect to the Lebesgue measure on R^n and is given by

$$f(\boldsymbol{x}) = \int_{R^m} \prod_{i=1}^{n} f_i(x_i \mid \theta_1, \dots, \theta_m) dG(\theta_1, \dots, \theta_m). \tag{1.1}$$

Such a model is known as a *mixture model.* An interesting special case of this model is,

$$f(\boldsymbol{x}) = \int_{R^n} \prod_{i=1}^{n} f_i(x_i \mid \theta_i) dG(\theta_1, \dots, \theta_n). \tag{1.2}$$

In this case $m = n$ and the conditional distribution of X_i given $\boldsymbol{\Theta}$ depends on $\boldsymbol{\Theta}$ only through Θ_i for $i = 1, \dots, n$. In Section 3, we give several examples of such models.

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Variants of the multivariate mixture model as given by (1.1) and (1.2) have been studied in the literature by many researchers including Shaked (1977), Jogdeo (1978), Lee (1985), Marshall and Olkin (1988,1991) and Shaked and Spizzichino (1998), beside others.

There are several notions of positive and negative dependence among random variables with varying degree of strength and these have been discussed in detail in Yanagimoto (1972), Barlow and Proschan (1975), Shaked (1977), Karlin and Rinott (1980*a*,1980*b*), Lee (1985) and Kimeldorf and Sampson (1989), among others. In this paper we identify conditions on F_i's and G under which the X_i's possess positive dependence properties of various types. First we introduce some notations and review some of the concepts that will be used later in this paper.

DEFINITION 1.1. (Karlin (1968)) We say that a function $h(x,y)$ is Sign-Regular of order 2 (SR_2) if $\varepsilon_1 h(x,y) \geq 0$ and

$$\varepsilon_2 \begin{vmatrix} h(x_1,y_1) & h(x_1,y_2) \\ h(x_2,y_1) & h(x_2,y_2) \end{vmatrix} \geq 0, \tag{1.3}$$

whenever $x_1 < x_2$, $y_1 < y_2$ for ε_1 and ε_2 equal to $+1$ or -1.

If the above relations hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = +1$, then h is said to be *Totally Positive of order* 2 (TP_2); and if they hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = -1$ then h is said to be *Reverse Regular of order* 2 (RR_2).

Let $X_1, \ldots, X_n$ be random variables with joint distribution function F and density f. For $k > 0$, let $\gamma^{(k)}(t)$ be defined as follows:

$$\gamma^{(k)}(t) = \begin{cases} (-t)^{k-1}/\Gamma(k) & \text{if} \quad t \leq 0, \\ 0 & \text{if} \quad t > 0. \end{cases}$$

Define n fold integral $\psi_{k_1,\ldots,k_n}(x_1, \ldots, x_n)$ by

$$\psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \gamma^{(k_i)}(x_i - t_i) dF(t_1, \ldots, t_n)$$

and define $\psi_{0,\ldots,0}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$. Also define $\psi_{0,\ldots,0,k_{i+1},\ldots,k_n}(x_1, \ldots, x_n)$ to be $(n-i)$ fold integral

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=i+1}^{n} \gamma^{(k_j)}(x_j - t_j) g_i(x_1, \ldots, x_i) dF(t_{i+1}, \ldots, t_n \mid x_1, \ldots, x_i)$$

where g_i is joint density of $(X_1, \ldots, X_i)$ and $F(t_{i+1}, \ldots, t_n \mid x_1, \ldots, x_i)$ is the conditional distribution function of $(X_{i+1}, \ldots, X_n)$ given $X_1 = x_1, \ldots, X_i = x_i$, for $k_{i+1} > 0, \ldots, k_n > 0$. Similarly we can define $\psi_{k_1,\ldots,k_n}(x_1, \ldots, x_n)$ with any subset of $k_1, \ldots, k_n$ consisting of zeros.

DEFINITION 1.2. (Shaked (1977), Lee (1985)) A random vector $(X_1, \ldots, X_n)$ is said to be dependent by total positivity with degree $(k_1, \ldots, k_n)$, denoted by $DTP(k_1, \ldots, k_n)$, if $\psi_{k_1,\ldots,k_n}(x_1, \ldots, x_n)$ is TP_2 in pairs of $\{x_1, \ldots, x_n\}$.

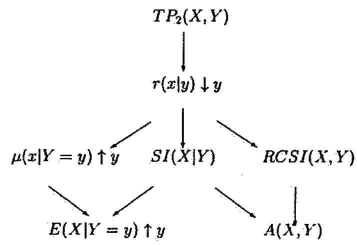

Fig. 1. Implications among notions of positive dependence.

We explain the concept of $DTP(k_1, \ldots, k_n)$ for the bivariate case. As observed in Shaked (1977), two random variables X and Y are likelihood ratio (or TP_2) dependent if and only if X and Y are $DTP(0,0)$ dependent. (X,Y) are $DTP(1,0)$ dependent if the function $\bar{F}(x \mid y) = P[X > x \mid Y = y]$ is TP_2. In this case the conditional hazard rate of X given $Y = y$, $r(x \mid y)$, is decreasing in y. The random variables X and Y are also said to be *right corner set increasing* ($RCSI$) if they are $DTP(1,1)$ dependent. The random variables X and Y are $DTP(2,0)$ dependent if the conditional mean residual life function of X given $Y = y$, $\mu(x \mid Y = y) = E[X - x \mid X > x, Y = y]$, is increasing in y. We say that X is stochastically increasing in Y (denoted by $SI(X \mid Y)$) if $P[X > x \mid Y = y]$ is increasing in y for all x. Two random variables X and Y are said to be associated (denoted by $A(X,Y)$) if $\mathrm{Cov}(u(X,Y), v(X,Y)) \geq 0$ for all increasing functions u and v. Figure 1 shows the chain of implications that hold among the above notions of positive dependence. See Lee (1985) for interpretation of $DTP(k_1, \ldots, k_n)$ distributions for other values of n.

One of the notions of positive dependence in multivariate setting is that of *multivariate total positivity of order* 2 (denoted by MTP_2). A function $\psi : R^n \to [0, \infty)$ is said to be MTP_2 if $\psi(\boldsymbol{x})\psi(\boldsymbol{y}) \leq \psi(\boldsymbol{x} \wedge \boldsymbol{y})\psi(\boldsymbol{x} \vee \boldsymbol{y})$ for every $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathcal{R}^n$, where $\boldsymbol{x} \wedge \boldsymbol{y} = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$ and $\boldsymbol{x} \vee \boldsymbol{y} = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$. Random variables $(X_1, \ldots, X_n)$ are said to be MTP_2 dependent if their joint density function is MTP_2. If a set of random variables is MTP_2 dependent, then they are TP_2 in pairs (i.e. $DTP(0, \ldots, 0)$) and the converse is true if $\boldsymbol{X}$ has a lattice support. See Karlin and Rinott (1980*a*) for this observation and for other properties of MTP_2 functions.

The corresponding concept of negative dependence for the case $n = 2$ is given in Shaked (1977).

DEFINITION 1.3. We say that (X,Y) is dependent by reverse regular rule of degree k_1 and k_2, denoted by $DRR(k_1, k_2)$, if $\psi_{k_1,k_2}(x,y)$ is RR_2.

Lee (1985) considered the model (1.2) when $\boldsymbol{\Theta}$ is a continuous univariate random variable. She proved that if (X_i, Θ) is $DTP(k_i, 0)$ for $i = 1, \ldots, n$, then $(X_1, \ldots, X_n)$ is $DTP(k_1, \ldots, k_n)$. In this paper we extend this problem to the case when $\boldsymbol{\Theta}$ is a random vector with joint density $g(\theta_1, \ldots, \theta_n)$. We also consider the case when X_i and Θ_i are negatively dependent for $i = 1, \ldots, n$. In particular, we prove that if g is TP_2 in pairs and if either (X_i, Θ_i) are all $DTP(k_i, 0)$ or are all $DRR(k_i, 0)$, then $(X_1, \ldots, X_n)$ are $DTP(k_1, \ldots, k_n)$.

Jogdeo (1978) and Shaked and Spizzichino (1998) studied the dependence properties of a random vector $\boldsymbol{X}$ satisfying the general mixture model (1.1). Jogdeo (1978) proved that if X_i is stochastically increasing (decreasing) in $\boldsymbol{\Theta}$ for $i = 1, \ldots, n$; and G is associated, then the random variables $X_1, \ldots, X_n$ are associated. Shaked and Spizzichino (1998) proved that if $f_i(x \mid \theta_1, \ldots, \theta_m)$ is MTP_2 in $(x, \theta_1, \ldots, \theta_m)$ for $i = 1, \ldots, n$ and if $\boldsymbol{\Theta}$ is MTP_2, then $\boldsymbol{X}$ is MTP_2. We prove in the next section that if $f_i(x \mid \theta_1, \ldots, \theta_m)$ is $RR_2(TP_2)$ in (x, θ_j) and is TP_2 in (θ_j, θ_k) for $j, k \in \{1, \ldots, m\}$; and if g is TP_2 in pairs then the joint density function $f(\boldsymbol{x})$ of $\boldsymbol{X}$ is TP_2 in pairs.

In Section 3 we give several applications of the results obtained in Section 2. In particular, we apply these results to study dependence among concomitants of order statistics and of record values for continuous bivariate distributions. An example concerning frailty models is also given. Throughout this paper we assume that expectations, whenever they are defined, exist and we can interchange the order of integration in multiple integrals.

2. Main Results

We shall need the following four lemmas to prove Theorem 2.1 which gives a general composition result for SR_2 functions. Lemmas 2.1 and 2.2 are due to Karlin (1968) and Lemma 2.3 has been proved recently in Khaledi and Kochar (2000). We state Lemmas 2.1–2.3 and prove Lemma 2.4 which may also be of independent interest. In the following μ represents a σ-finite measure.

LEMMA 2.1. *Let A, B and C be subsets of the real line and let $L(x, z)$ be SR_2 for $x \in A$, $z \in B$ and $M(z, y)$ be SR_2 for $z \in B$, $y \in C$. Then $K(x, y) = \int L(x, z) M(z, y)\, d\mu(z)$ is SR_2 for $x \in A$, $y \in C$ and $\varepsilon_i(K) = \varepsilon_i(L) \times \varepsilon_i(M)$ $\forall$ $i = 1, 2$.*

LEMMA 2.2. *Suppose λ, x, ζ traverse the ordered sets Λ, X and Z, respectively and consider the functions $f(\lambda, x, \zeta)$ and $g(\lambda, \zeta)$ satisfying the following conditions,*

(a) *$f(\lambda, x, \zeta) > 0$ and f is TP_2 in each pairs of variables when the third variable is held fixed; and*

(b) *$g(\lambda, \zeta)$ is TP_2. Then the function*

$$h(\lambda, x) = \int_Z f(\lambda, x, \zeta) g(\lambda, \zeta) d\mu(\zeta),$$

defined on $\Lambda \times X$ is TP_2 in (λ, x).

LEMMA 2.3. *Suppose λ, x, ζ traverse the ordered sets Λ, X and Z, respectively and consider the function $f(\lambda, x, \zeta)$ satisfying the following conditions,*

(a) *$f(\lambda, x, \zeta) > 0$ and f is TP_2 in (λ, x),*

(b) *$f(\lambda, x, \zeta)$ is RR_2 in (λ, ζ) as well as in (x, ζ).*

Then the function

$$h(\lambda, x) = \int_Z f(\lambda, x, \zeta) d\mu(\zeta),$$

defined on $\Lambda \times X$ is TP_2 in (λ, x).

LEMMA 2.4. *Suppose λ, x, ζ traverse the ordered sets Λ, X and Z, respectively and consider the functions $f(\lambda, x, \zeta)$ and $g(\lambda, \zeta)$ satisfying the following conditions,*

(a) $f(\lambda, x, \zeta) > 0$, f *and* g *are* TP_2 *in* (λ, ζ).
(b) $f(\lambda, x, \zeta)$ *is* RR_2 *in* (λ, x) *and* (x, ζ).

Then the function

$$h(\lambda, x) = \int_Z f(\lambda, x, \zeta) g(\lambda, \zeta) d\mu(\zeta),$$

defined on $\Lambda \times X$ *is* RR_2 *in* (λ, x).

PROOF. We have to prove that for $\lambda_1 < \lambda_2$ and $x_1 < x_2$,

$$h(\lambda_2, x_2) h(\lambda_1, x_1) - h(\lambda_2, x_1) h(\lambda_1, x_2) \leq 0. \tag{2.1}$$

Karlin ((1968), p. 123) showed that the L. H. S. in (2.1) is equal to,

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_{-\infty}^{\zeta} \left\{ \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} - \frac{f(\lambda_1, x_2, u)}{f(\lambda_1, x_1, u)} \right\} \{ f(\lambda_1, x_1, u) g(\lambda_1, u) f(\lambda_2, x_1, \zeta) g(\lambda_2, \zeta) \\ &\qquad - f(\lambda_2, x_1, u) g(\lambda_2, u) f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta) \} d\mu(u) d\mu(\zeta) \\ &+ \int_{-\infty}^{+\infty} \int_{-\infty}^{\zeta} \left\{ \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} - \frac{f(\lambda_1, x_2, \zeta)}{f(\lambda_1, x_1, \zeta)} + \frac{f(\lambda_2, x_2, u)}{f(\lambda_2, x_1, u)} - \frac{f(\lambda_1, x_2, u)}{f(\lambda_1, x_1, u)} \right\} \\ &\times f(\lambda_2, x_1, u) g(\lambda_2, u) f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta) d\mu(u) d\mu(\zeta). \end{aligned} \tag{2.2}$$

The first expression in (2.2) is non-positive since f is RR_2 in (λ, x) and in (x, ζ). The second expression in the first integral is non-negative since g and f both are TP_2 in (λ, ζ). Hence the first double integral in (2.2) is non-positive. The second integral is non-positive since f is RR_2 in (λ, x). This proves the required result. □

With the help of the above lemmas we prove Theorem 2.1 which is a mathematical tool used to prove dependence properties of mixtures of the type (1.1) and (1.2) in Theorems 2.2 and 2.3.

THEOREM 2.1. *Consider*

$$\phi(\boldsymbol{x}) = \int_{R^m} \left\{ \prod_{i=1}^{n} h_i(x_i, \theta_1, \ldots, \theta_m) \right\} g(\theta_1, \ldots, \theta_m) \prod_{i=1}^{m} d\theta_i. \tag{2.3}$$

Suppose that $g(\theta_1, \ldots, \theta_m)$ *is* TP_2 *in pairs and for each* $i \in \{1, \ldots, n\}$, $h_i(x_i, \theta_1, \ldots, \theta_m)$ *is* TP_2 *in* (θ_j, θ_k) *for* $j, k \in \{1, \ldots, m\}$. *If* $h_i(x_i, \theta_1, \ldots, \theta_m)$ *is either*
(a) RR_2 *in* (x_i, θ_j) *for all* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, m\}$; *or*
(b) TP_2 *in* (x_i, θ_j) *for all* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, m\}$,

then $\phi(\boldsymbol{x})$ *is* TP_2 *in pairs.*

PROOF. (a) Let $I = \{1, \ldots, n\}$ and $J_j = \{j, \ldots, m\}$ for $j = 1, \ldots, m$. Define

$$g_1(x_1, \ldots, x_n, \theta_2, \ldots, \theta_m) = \int_R \left\{ \prod_{i=1}^{n} h_i(x_i, \theta_1, \ldots, \theta_m) \right\} g(\theta_1, \ldots, \theta_m) d\theta_1.$$

By Lemma 2.1, g_1 is TP_2 in (x_i, x_j), $i, j \in I$ and it is TP_2 in (θ_j, θ_k), $j, k \in J_2$ by Lemma 2.2. For fixed $i \in I$ and $j \in J_2$ let

$$z(x_i, \theta_j, \theta_1) = h_i(x_i, \theta_1, \ldots, \theta_j, \ldots, \theta_m)$$

and

$$w(\theta_1,\theta_j) = \left\{ \prod_{\substack{j=1 \\ j\neq i}}^{n} h_j(x_j,\theta_1,\ldots,\theta_j,\ldots,\theta_m) \right\} g(\theta_1,\ldots,\theta_j,\ldots,\theta_m).$$

The function z defined above is RR_2 in (x_i,θ_j), RR_2 in (x_i,θ_1) and TP_2 in (θ_1,θ_j). The function w is TP_2 in (θ_1,θ_j). Hence by Lemma 2.4 the function

$$g_1(x_1,\ldots,x_n,\theta_2,\ldots,\theta_m) = \int_R z(x_i,\theta_j,\theta_1)w(\theta_1,\theta_j)d\theta_1$$

is RR_2 in (x_i,θ_j). Define

$$g_i(x_1,\ldots,x_n,\theta_{i+1},\ldots,\theta_m) = \int_R g_{i-1}(x_1,\ldots,x_n,\theta_i,\ldots,\theta_m)d\theta_i, \qquad \text{for} \quad i=2,\ldots,m-1.$$

We prove the required result by induction. Suppose g_{i-1} is TP_2 in (x_k,x_j) , TP_2 in (θ_p,θ_l) and RR_2 in (x_k,θ_p) for $k,j \in I$ and $p,l \in J_i$. By Lemma 2.3, g_i is TP_2 in (x_k,x_j), and by Lemma 2.2, it is TP_2 in (θ_p,θ_l) for $p,l \in J_{i+1}$. From Lemma 2.4, g_i is RR_2 in (x_k,θ_p), $k \in I$ and $p \in J_{i+1}$. Thus $g_{m-1}(x_1,\ldots,x_n,\theta_m)$ is TP_2 in (x_k,x_j) and RR_2 in (x_j,θ_m) for $k,j \in I$. Using Lemma 2.3 we find that,

$$\phi(x_1,\ldots,x_n) = \int_R g_{m-1}(x_1,\ldots,x_n,\theta_m)d\theta_m$$

is TP_2 in pairs of its arguments.

(b) The proof follows from Lemmas 2.1 and 2.2 and using arguments similar to those used for proving part (a). □

In the next theorem we prove that for the mixture model (1.2), under appropriate conditions, both positive as well as negative monotone dependence between X_i and Θ_i, $i=1,\ldots,n$, imply positive dependence among $X_1,\ldots,X_n$.

THEOREM 2.2. *Consider the mixture model* (1.2) *and suppose that the density function g is TP_2 in pairs. If either*

(a) *(X_i,Θ_i) is $DRR(k_i,0)$ for all $i=1,\ldots,n$; or*

(b) *(X_i,Θ_i) is $DTP(k_i,0)$ for all $i=1,\ldots,n$,*

then $X_1,\ldots,X_n$ is $DTP(k_1,\ldots,k_n)$.

PROOF. (a) By definition, for $k_i>0$ we have

$$\begin{aligned}
\psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n) &= \int_{R^n} \prod_{i=1}^{n} \gamma^{(k_i)}(x_i-t_i) f(t_1,\ldots,t_n) \prod_{i=1}^{n} dt_i \\
&= \int_{R^n}\int_{R^n} \left\{ \prod_{i=1}^{n} \gamma^{(k_i)}(x_i-t_i) f_i(t_i \mid \theta_i) \right\} \\
&\quad \times g(\theta_1,\ldots,\theta_n) \prod_{i=1}^{n} d\theta_i \prod_{i=1}^{n} dt_i \\
&= \int_{R^n} \left\{ \prod_{i=1}^{n} \int_R \gamma^{(k_i)}(x_i-t_i) f_i(t_i \mid \theta_i) dt_i \right\} g(\theta_1,\ldots,\theta_n) \prod_{i=1}^{n} d\theta_i.
\end{aligned}$$

Since (X_i, Θ_i) being $DRR(k_i, 0)$ is equivalent to $\int_R \gamma^{(k_i)}(x_i - t_i) f_i(t_i \mid \theta_i) dt_i$ being RR_2 in (x_i, θ_i), taking $m = n$ and replacing the function $h_i(x_i, \theta_1, \ldots, \theta_m)$ by this function in Theorem 2.1 (a), it follows that $\psi_{k_1,\ldots,k_n}(x_1, \ldots, x_n)$ is TP_2 in pairs.

Let us now consider the case when $k_i = 0$ for each i. In this case the function

$$\begin{aligned}\psi_{0,\ldots,0}(x_1, \ldots, x_n) &= f(x_1, \ldots, x_n) \\ &= \int_{R^n} \left\{ \prod_{i=1}^{n} f_i(x_i \mid \theta_i) \right\} g(\theta_1, \ldots, \theta_n) \prod_{i=1}^{n} d\theta_i\end{aligned}$$

is clearly seen to be $DTP(0, \ldots, 0)$. The proof follows from Theorem 2.1 (a) on replacing $h_i(x_i, \theta_1, \ldots, \theta_m)$ with $f_i(x_i \mid \theta_i)$ and taking $m = n$.

(b) In this case, by assumption, the function $\int_R \gamma^{(k_i)}(x_i - t_i) f_i(t_i \mid \theta_i) dt_i$ is TP_2 in (x_i, θ_i) for each i. The required result follows on the same lines using part (b) of Theorem 2.1. □

The following results are immediate consequences of the above theorem.

COROLLARY 2.1. *Suppose that the density function g in the mixture model* (1.2) *is TP_2 in pairs. Then if*

(i) *X_i and Θ_i are either all TP_2 or all RR_2 dependent, then the joint density of $(X_1, \ldots, X_n)$ is TP_2 in pairs;*

(ii) *the conditional hazard rate of X_i given $\Theta_i = \theta_i$ is nondecreasing (nonincreasing) in θ_i, for $i = 1, \ldots, n$; then the random variables $(X_1, \ldots, X_n)$ are $DTP(1, \ldots, 1)$. In particular X_i and X_j are $RCSI$ for $i \neq j \in \{1, \ldots, n\}$ in this case.*

Remarks. 1. Shaked and Spizzichino (1998) established a different type of dependence result among $(X_1, \ldots, X_n)$ when $\theta_1 = \theta_2 = \cdots = \theta_m$. They proved that if either all (X_i, Θ) are $DTP(1, 0)$ or all are $DRR(1, 0)$, then $(X_1, \ldots, X_n)$ is WBF (weakened by failure) dependent. It is not clear whether there is any relation between $DTP(1, \ldots, 1)$ and WBF dependence.

2. Marshall and Olkin (1991) proved a related result that if each $f_i(x_i \mid \theta_i)$ in (1.2) is TP_2 in (x_i, θ_i) and g is MTP_2, then the function $f(x_1, \ldots, x_n)$ is MTP_2.

For the general model (1.1), Shaked and Spizzichino (1998) proved that if $\mathbf{\Theta}$ is MTP_2 and for each $i \in \{1, \ldots, n\}$, $f_i(x \mid \theta_1, \ldots, \theta_m)$ is MTP_2 in $(x, \theta_1, \ldots, \theta_m)$, then the random vector $\boldsymbol{X}$ is MTP_2. In Theorem 2.3 below we extend this result to prove positive dependence among $X_1, \ldots, X_n$ when X_i and Θ_j are even RR_2 dependent for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$.

THEOREM 2.3. *Let $X_1, \ldots, X_n$ follow the mixture model* (1.1). *Suppose g, the joint density function of $\mathbf{\Theta}$ is TP_2 in pairs. Then for $i \in \{1, \ldots, n\}$ and $j, k \in \{1, \ldots m\}$,*

(a) *$f_i(x \mid \theta_1, \ldots, \theta_m)$ being $RR_2(TP_2)$ in (x, θ_j) and TP_2 in (θ_j, θ_k) implies that $f(x_1, \ldots, x_n)$ is TP_2 in pairs,*

(b) *$\bar{F}_i(x \mid \theta_1, \ldots, \theta_m)$ being $RR_2(TP_2)$ in (x, θ_j) and TP_2 in (θ_j, θ_k) implies that $\bar{F}(x_1, \ldots, x_n)$ is TP_2 in pairs.*

PROOF. The results follow immediately from Theorem 2.1.

As mentioned earlier model (1.2) is a special case of model (1.1) when the conditional distribution of X_i given $\mathbf{\Theta}$ depends on $\mathbf{\Theta}$ only through Θ_i for $i = 1, \ldots, n$. Theorem 2.3

establishes DTP type results for the general mixture model (1.1) only for $k = 0$ and 1 whereas Theorem 2.2 gives more general results for the restricted model (1.2). □

3. Examples and applications

Application 3.1. *Dependence among concomitants of record values:* Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independent and identically distributed random variables from a continuous bivariate distribution. X_n is called a (upper) record value of the sequence $\{X_i, i \geq 1\}$ if $X_n > X_i$ for $i = 1, \ldots, n-1$. By convention X_1 is a record value. The serial numbers at which record values occur are given by the $\{T_n, n \geq 1\}$, defined recursively by $T_1 = 1$, $T_n = \min\{k; k > T_{n-1}, X_k > X_{T_{n-1}}\}$, $n \geq 2$. the sequence $\{T_n, n \geq 1\}$ is called the sequence of (upper) record times and $\{X_{T_n}, n \geq 1\}$ the sequence of (upper) record values corresponding to $\{X_n, n \geq 1\}$. For convenience of notation we shall denote X_{T_n} by R_n so that $\{R_n, n \geq 1\}$ is the sequence of record values. The Y-variate associated with R_n is denoted by $Y_{[n]}$ and is called the concomitant of the n-th record value. That is, the sequence $\{Y_{[i]}, i \geq 1\}$ is the sequence of concomitants of $\{R_n, n \geq 1\}$.

As discussed in Ahsanullah (1994) the joint pdf of the n record values $(R_1, \ldots, R_n)$, is

$$f_{1,\ldots,n}(x_1, \ldots, x_n) = \prod_{i=1}^{n-1} \{f(x_i)/\bar{F}(x_i)\} f(x_n), \qquad \text{for } x_1 < \cdots < x_n.$$

From this we obtain the joint pdf of the concomitants $(Y_{[1]}, \ldots, Y_{[n]})$ as

$$f_{Y_{[1]},\ldots,Y_{[n]}}(y_1, \ldots, y_n) = \int_{-\infty}^{+\infty} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \left\{\prod_{i=1}^{n} f(y_i \mid x_i)\right\} f_{1,\ldots,n}(x_1, \ldots, x_n) \prod_{i=1}^{n} dx_i\ ,$$

which can also be written as

$$\begin{aligned} f_{Y_{[1]},\ldots,Y_{[n]}}(y_1, \ldots, y_n) &= \int_{R^n} \left\{\prod_{i=1}^{n} f(y_i \mid x_i)\right\} \left\{\prod_{i=1}^{n-1} f(x_i)/\bar{F}(x_i)\right\} f(x_n) \\ &\quad \cdot \prod_{i=1}^{n} k(x_i, x_{i+1}) \prod_{i=1}^{n} dx_i \end{aligned}$$

where

(3.1) $$k(x, y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x \geq y \end{cases}$$

and $x_{n+1} = +\infty$. Since the function k in (3.1) is TP_2 in (x, y), it follows that the function

(3.2) $$\prod_{i=1}^{n} k(x_i, x_{i+1}) \prod_{i=1}^{n-1} \{f(x_i)/\bar{F}(x_i)\} f(x_n)$$

is TP_2 in pairs. By replacing the function g in Theorem 2.2 by the one given by (3.2) and assuming that X and Y are $DTP(0, k)$ or $DRR(0, k)$ dependent we find that $(Y_{[1]}, \ldots, Y_{[n]})$, the concomitants of record values, are $DTP(k, \ldots, k)$ dependent. In particular if

(i) X and Y are either TP_2 or RR_2 dependent, then the joint density of $(Y_{[1]}, \ldots, Y_{[n]})$, the concomitants of records, is TP_2 in pairs.

(ii) X and Y are $DTP(0,1)$ or $DRR(0,1)$, then the concomitants of record values are dependent according to $DTP(1,\ldots,1)$ criteria.

Remark. Similar results can be obtained for dependence among concomitants of order statistics. It can be proved that if (X,Y) are $DTP(0,k)$ or $DRR(0,k)$, then the concomitants of order statistics are $DTP(k,\ldots,k)$. See Khaledi and Kochar (2000) for details.

Application 3.2. *The frailty model*: Marshall and Olkin (1988) studied the family of multivariate distributions generated by mixture in which marginals were considered as parameters. Suppose that $\bar{H}_1,\ldots,\bar{H}_n$ are univariate survival functions and let G be an n-variate distribution function such that $\bar{G}(0,\ldots,0)=1$ with univariate marginals G_i, $i=1,\ldots,n$. Denote the Laplace transforms of G and G_i by ϕ and ϕ_i, respectively. Let for each i, $\bar{F}_i(x)=\exp[-\phi_i^{-1}\bar{H}_i(x)]$, where the function ϕ_i^{-1} is inverse of ϕ_i, then

$$\bar{H}(x_1,\ldots,x_n)=\int_{R^n}K(\bar{F}_1^{\theta_1}(x_1),\ldots,\bar{F}_n^{\theta_n}(x_n))dG(\theta_1,\ldots,\theta_n), \tag{3.3}$$

is an n-variate survival function with marginals survival functions $\bar{H}_1,\ldots,\bar{H}_n$. Here K denotes the distribution function of an n-dimensional vector with marginals as uniform (0,1) distributions. Different choices of K and G lead to a variety of distributions with marginals as specified parameters. If K is the joint distribution function of n independent uniform (0,1) random variables and g is the density function corresponding to distribution function G, then $\bar{H}$ in (3.3) can be written as,

$$\bar{H}(x_1,\ldots,x_n)=\int_{R^n}\left\{\prod_{i=1}^n\bar{F}_i^{\theta_i}(x_i)\right\}g(\theta_1,\ldots,\theta_n)\prod_{i=1}^n d\theta_i, \tag{3.4}$$

with density function as

$$h(x_1,\ldots,x_n)=\int_{R^n}\left\{\prod_{i=1}^n\theta_i\bar{F}_i^{\theta_i-1}(x_i)f_i(x_i)\right\}g(\theta_1,\ldots,\theta_n)\prod_{i=1}^n d\theta_i, \tag{3.5}$$

which is of the form (1.2). It is easy to see that the function $\theta_i\bar{F}_i^{\theta_i-1}(x_i)f_i(x_i)$ is RR_2 in (x_i,θ_i) for $i=1,\ldots,n$. Using Theorem 2.2 with $k_i=0$, $i=1,\ldots,n$, it follows that if g is TP_2 in pairs, then the joint density of $\boldsymbol{X}$ is TP_2 in pairs.

The model described in (3.4), for $n=2$ is known as a bivariate *frailty model* and it has been studied intensively in the literature. See Oakes (1989) for more details. If we assume that Θ is univariate in this, then $h(\boldsymbol{x})$ in (3.5) is always TP_2 in pairs, a result which is stronger than the result of Marshall and Olkin (1988) who proved that $\boldsymbol{X}$ is $DTP(1,\ldots,1)$ under the given conditions.

Example 3.3. Let $\boldsymbol{Z}=(Z_1,\ldots,Z_n)$ be a random vector of independent components and let $\boldsymbol{\Theta}=(\Theta_1,\ldots,\Theta_n)$ be a random vector with joint pdf g. Let $\boldsymbol{X}=\boldsymbol{Z}+\boldsymbol{\Theta}$ and assume $\boldsymbol{Z}$ and $\boldsymbol{\Theta}$ are independent. Karlin and Rinott (1980*a*) showed that if the joint pdf of Θ is MTP_2 and if Z_i has log-concave density f_i for $i=1,\ldots,n$, then the random vector $\boldsymbol{X}$ is MTP_2. Recall that the MTP_2 property implies TP_2 in pairs property. We show here that the joint pdf ϕ of $\boldsymbol{X}$ is TP_2 in pairs when Z_i's have log-convex densities and the function g is TP_2 in pairs. The joint pdf of $\boldsymbol{X}$ is

$$\phi(x_1,\ldots,x_n) = \int_{R^n} \prod_{i=1}^{n} f_i(x_i - \theta_i) g(\theta_1,\ldots,\theta_n) \prod_{i=1}^{n} d\theta_i,$$

which is of the form (1.2). By replacing the function h_i in Theorem 2.1 with f_i and using this fact that $f_i(x_i - \theta_i)$ is RR_2 in (x_i, θ_i) if f_i is log-convex for $i = 1,\ldots,n$, it follows that joint pdf of $\boldsymbol{X}$ is TP_2 in each pairs of its arguments.

Example 3.4. Let $X_1, X_2, \ldots, X_n$ have the joint density function given by (1.2) with

$$f_i(x \mid \theta) = \left(\frac{1}{\sqrt{x}} + \frac{1}{\theta}\right) \exp\left\{-2\sqrt{x} - \frac{x}{\theta}\right\}, \qquad x \geq 0, \qquad \text{for} \quad i = 1, 2, \ldots, n,$$

where the random variable Θ is univariate and positive. The corresponding hazard rate functions are given by

$$r_i(x \mid \theta) = \frac{1}{\sqrt{x}} + \frac{1}{\theta}, \qquad x > 0,$$

for $i = 1, 2, \ldots, n$. Clearly, $r_i(x \mid \theta)$ is decreasing in θ for all $x > 0$, $i = 1, 2, \ldots, n$. On the other hand, $f_i(x \mid \theta)$ is neither TP_2 nor RR_2. It follows from Theorem 2.2 (b) $(X_1, \ldots, X_n)$ is $DTP(1, \ldots, 1)$.

There are other interesting applications and examples given in Shaked and Spizzichino (1998) to which the results of Section 2 can be applied.

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