DEPENDENCE PROPERTIES OF MULTIVARIATE MIXTURE DISTRIBUTIONS AND THEIR APPLICATIONS

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Abstract. Consider a multivariate mixture model where the random variables $X_1,\ldots,X_n$ given $(\Theta_1,\ldots,\Theta_n)$, are conditionally independent. Conditions are obtained under which different kinds of positive dependence hold among $X_i$'s. The results obtained are applied to a variety of problems including the concomitants of order statistics and of record values; and to frailty models.

Key words and phrases: Dependence by total positivity (DTP), dependence by reverse regular (DRR), MTP$_2$ dependence, associated random variables, concomitants of order statistics and record values, frailty models.

1. Introduction

Let $X_1,\ldots,X_n$ be $n$ random variables such that they are conditionally independent given some random vector $\Theta = (\Theta_1,\ldots,\Theta_m)$. It is of interest to know which kind of dependence arises among $X_i$'s when $\Theta$ is unknown. If $F_i(\cdot \mid \theta_1,\ldots,\theta_m)$ denotes the conditional distribution function of $X_i$ given $\Theta = (\theta_1,\ldots,\theta_m)$ and $G(\theta_1,\ldots,\theta_m)$ denotes the joint distribution function of $\Theta$, then the joint distribution function of $X$ is given by

$$F(x) = \int_{R^m} \prod_{i=1}^n F_i(x_i \mid \theta_1,\ldots,\theta_m) dG(\theta_1,\ldots,\theta_m).$$

If $F_i(\cdot \mid \theta_1,\ldots,\theta_m)$ is absolutely continuous with respect to the Lebesgue measure on $R$ for each $(\theta_1,\ldots,\theta_m)$ in the support of $\Theta$ with a density function $f_i(\cdot \mid \theta_1,\ldots,\theta_m)$, then the joint distribution of $X_1,\ldots,X_n$ is absolutely continuous with respect to the Lebesgue measure on $R^n$ and is given by

$$f(x) = \int_{R^n} \prod_{i=1}^n f_i(x_i \mid \theta_1,\ldots,\theta_m) dG(\theta_1,\ldots,\theta_m).$$

Such a model is known as a mixture model. An interesting special case of this model is,

$$f(x) = \int_{R^n} \prod_{i=1}^n f_i(x_i \mid \theta_i) dG(\theta_1,\ldots,\theta_n).$$

In this case $m = n$ and the conditional distribution of $X_i$ given $\Theta$ depends on $\Theta$ only through $\Theta_i$ for $i = 1,\ldots,n$. In Section 3, we give several examples of such models.

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Variants of the multivariate mixture model as given by (1.1) and (1.2) have been studied in the literature by many researchers including Shaked (1977), Jogdeo (1978), Lee (1985), Marshall and Olkin (1988,1991) and Shaked and Spizzichino (1998), beside others.

There are several notions of positive and negative dependence among random variables with varying degree of strength and these have been discussed in detail in Yanagimoto (1972), Barlow and Proschan (1975), Shaked (1977), Karlin and Rinott (1980a,1980b), Lee (1985) and Kimeldorf and Sampson (1989), among others. In this paper we identify conditions on $F_i$'s and $G$ under which the $X_i$'s possess positive dependence properties of various types. First we introduce some notations and review some of the concepts that will be used later in this paper.

**Definition 1.1.** (Karlin (1968)) We say that a function $h(x,y)$ is Sign-Regular of order 2 ($SR_2$) if $\varepsilon_1 h(x,y) \geq 0$ and

$$
\varepsilon_2 \left| \begin{array}{cc}
h(x_1,y_1) & h(x_1,y_2) \\
h(x_2,y_1) & h(x_2,y_2)
\end{array} \right| \geq 0,
$$

whenever $x_1 < x_2$, $y_1 < y_2$ for $\varepsilon_1$ and $\varepsilon_2$ equal to $+1$ or $-1$.

If the above relations hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = +1$, then $h$ is said to be Totally Positive of order 2 ($TP_2$); and if they hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = -1$ then $h$ is said to be Reverse Regular of order 2 ($RR_2$).

Let $X_1, \ldots, X_n$ be random variables with joint distribution function $F$ and density $f$. For $k > 0$, let $\gamma^{(k)}(t)$ be defined as follows:

$$
\gamma^{(k)}(t) = \begin{cases}
(-t)^{k-1}/\Gamma(k) & \text{if } t \leq 0, \\
0 & \text{if } t > 0.
\end{cases}
$$

Define $n$ fold integral $\psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n)$ by

$$
\psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \gamma^{(k_i)}(x_i - t_i) dF(t_1,\ldots,t_n)
$$

and define $\psi_{0,\ldots,0}(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)$. Also define $\psi_{0,\ldots,0,k_{i+1},\ldots,k_n}(x_1,\ldots,x_n)$ to be $(n-i)$ fold integral

$$
\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=i+1}^{n} \gamma^{(k_j)}(x_j - t_j) g_i(x_1,\ldots,x_i) dF(t_{i+1},\ldots,t_n \mid x_1,\ldots,x_i)
$$

where $g_i$ is joint density of $(X_1,\ldots,X_i)$ and $F(t_{i+1},\ldots,t_n \mid x_1,\ldots,x_i)$ is the conditional distribution function of $(X_{i+1},\ldots,X_n)$ given $X_1 = x_1,\ldots,X_i = x_i$, for $k_{i+1} > 0,\ldots,k_n > 0$. Similarly we can define $\psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n)$ with any subset of $k_1,\ldots,k_n$ consisting of zeros.

**Definition 1.2.** (Shaked (1977), Lee (1985)) A random vector $(X_1,\ldots,X_n)$ is said to be dependent by total positivity with degree $(k_1,\ldots,k_n)$, denoted by $DTP(k_1,\ldots,k_n)$, if $\psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n)$ is $TP_2$ in pairs of $\{x_1,\ldots,x_n\}$. 
We explain the concept of $DTP(k_1, \ldots, k_n)$ for the bivariate case. As observed in Shaked (1977), two random variables $X$ and $Y$ are likelihood ratio (or $TP_2$) dependent if and only if $X$ and $Y$ are $DTP(0,0)$ dependent. $(X,Y)$ are $DTP(1,0)$ dependent if the function $F(x \mid y) = P[X > x \mid Y = y]$ is $TP_2$. In this case the conditional hazard rate of $X$ given $Y = y$, $r(x \mid y)$, is decreasing in $y$. The random variables $X$ and $Y$ are also said to be right corner set increasing ($RCSI$) if they are $DTP(1,1)$ dependent. The random variables $X$ and $Y$ are $DTP(2,0)$ dependent if the conditional mean residual life function of $X$ given $Y = y$, $\mu(x \mid Y = y) = E[X - x \mid X > x, Y = y]$, is increasing in $y$. We say that $X$ is stochastically increasing in $Y$ (denoted by $SI(X \mid Y)$) if $P[X > x \mid Y = y]$ is increasing in $y$ for all $x$. Two random variables $X$ and $Y$ are said to be associated (denoted by $A(X,Y)$) if Cov$(u(X,Y),v(X,Y)) \geq 0$ for all increasing functions $u$ and $v$.

Figure 1 shows the chain of implications that hold among the above notions of positive dependence. See Lee (1985) for interpretation of $DTP(k_1, \ldots, k_n)$ distributions for other values of $n$.

One of the notions of positive dependence in multivariate setting is that of multivariate total positivity of order 2 (denoted by $MTP_2$). A function $\psi : R^n \rightarrow [0,\infty)$ is said to be $MTP_2$ if $\psi(x)\psi(y) \leq \psi(x \wedge y)\psi(x \vee y)$ for every $x$ and $y$ in $R^n$, where $x \wedge y = (\min(x_1,y_1),\ldots,\min(x_n,y_n))$ and $x \vee y = (\max(x_1,y_1),\ldots,\max(x_n,y_n))$. Random variables $(X_1,\ldots,X_n)$ are said to be $MTP_2$ dependent if their joint density function is $MTP_2$. If a set of random variables is $MTP_2$ dependent, then they are $TP_2$ in pairs (i.e. $DTP(0,0,0)$) and the converse is true if $X$ has a lattice support. See Karlin and Rinott (1980a) for this observation and for other properties of $MTP_2$ functions.

The corresponding concept of negative dependence for the case $n = 2$ is given in Shaked (1977).

**Definition 1.3.** We say that $(X,Y)$ is dependent by reverse regular rule of degree $k_1$ and $k_2$, denoted by $DRR(k_1,k_2)$, if $\psi_{k_1,k_2}(x,y)$ is $RR_2$.

Lee (1985) considered the model (1.2) when $\Theta$ is a continuous univariate random variable. She proved that if $(X_i,\Theta)$ is $DTP(k_i,0)$ for $i = 1,\ldots,n$, then $(X_1,\ldots,X_n)$ is $DTP(k_1,\ldots,k_n)$. In this paper we extend this problem to the case when $\Theta$ is a random vector with joint density $g(\theta_1,\ldots,\theta_n)$. We also consider the case when $X_i$ and $\Theta_i$ are negatively dependent for $i = 1,\ldots,n$. In particular, we prove that if $g$ is $TP_2$ in pairs and if either $(X_i,\Theta_i)$ are all $DTP(k_i,0)$ or are all $DRR(k_i,0)$, then $(X_1,\ldots,X_n)$ are $DTP(k_1,\ldots,k_n)$.

Jogdeo (1978) and Shaked and Spizzichino (1998) studied the dependence properties of a random vector $X$ satisfying the general mixture model (1.1). Jogdeo (1978) proved
that if \( X_i \) is stochastically increasing (decreasing) in \( \Theta \) for \( i = 1, \ldots, n \); and \( G \) is associated, then the random variables \( X_1, \ldots, X_n \) are associated. Shaked and Spizzichino (1998) proved that if \( f_i(x | \theta_1, \ldots, \theta_m) \) is \( MTP_2 \) in \( (x, \theta_1, \ldots, \theta_m) \) for \( i = 1, \ldots, n \) and \( \Theta \) is \( MTP_2 \), then \( X \) is \( MTP_2 \). We prove in the next section that if \( f_i(x | \theta_1, \ldots, \theta_m) \) is \( RR_2(TP_2) \) in \( (x, \theta_j) \) and is \( TP_2 \) in \( (\theta_j, \theta_k) \) for \( j, k \in \{1, \ldots, m\} \); and if \( g \) is \( TP_2 \) in pairs then the joint density function \( f(x) \) of \( X \) is \( TP_2 \) in pairs.

In Section 3 we give several applications of the results obtained in Section 2. In particular, we apply these results to study dependence among concomitants of order statistics and of record values for continuous bivariate distributions. An example concerning frailty models is also given. Throughout this paper we assume that expectations, whenever they are defined, exist and we can interchange the order of integration in multiple integrals.

2. Main Results

We shall need the following four lemmas to prove Theorem 2.1 which gives a general composition result for \( SR_2 \) functions. Lemmas 2.1 and 2.2 are due to Karlin (1968) and Lemma 2.3 has been proved recently in Khaledi and Kochar (2000). We state Lemmas 2.1–2.3 and prove Lemma 2.4 which may also be of independent interest. In the following \( \mu \) represents a \( \sigma \)-finite measure.

**Lemma 2.1.** Let \( A, B \) and \( C \) be subsets of the real line and let \( L(x, z) \) be \( SR_2 \) for \( x \in A, z \in B \) and \( M(z, y) \) be \( SR_2 \) for \( z \in B, y \in C \). Then \( K(x, y) = \int L(x, z)M(z, y) \ \mathrm{d}\mu(z) \) is \( SR_2 \) for \( x \in A, y \in C \) and \( \epsilon_i(K) = \epsilon_i(L) \times \epsilon_i(M) \ \forall \ i = 1, 2 \).

**Lemma 2.2.** Suppose \( \lambda, x, \zeta \) traverse the ordered sets \( \Lambda, X \) and \( Z \), respectively and consider the functions \( f(\lambda, x, \zeta) \) and \( g(\lambda, \zeta) \) satisfying the following conditions,

(a) \( f(\lambda, x, \zeta) > 0 \) and \( f \) is \( TP_2 \) in each pairs of variables when the third variable is held fixed; and
(b) \( g(\lambda, \zeta) \) is \( TP_2 \). Then the function

\[
h(\lambda, x) = \int_Z f(\lambda, x, \zeta)g(\lambda, \zeta) \ \mathrm{d}\mu(\zeta),
\]

defined on \( \Lambda \times X \) is \( TP_2 \) in \((\lambda, x)\).

**Lemma 2.3.** Suppose \( \lambda, x, \zeta \) traverse the ordered sets \( \Lambda, X \) and \( Z \), respectively and consider the function \( f(\lambda, x, \zeta) \) satisfying the following conditions,

(a) \( f(\lambda, x, \zeta) > 0 \) and \( f \) is \( TP_2 \) in \((\lambda, x)\),
(b) \( f(\lambda, x, \zeta) \) is \( RR_2 \) in \((\lambda, \zeta)\) as well as in \((x, \zeta)\).

Then the function

\[
h(\lambda, x) = \int_Z f(\lambda, x, \zeta) \ \mathrm{d}\mu(\zeta),
\]

defined on \( \Lambda \times X \) is \( TP_2 \) in \((\lambda, x)\).

**Lemma 2.4.** Suppose \( \lambda, x, \zeta \) traverse the ordered sets \( \Lambda, X \) and \( Z \), respectively and consider the functions \( f(\lambda, x, \zeta) \) and \( g(\lambda, \zeta) \) satisfying the following conditions,
(a) \( f(\lambda, x, \zeta) > 0 \), \( f \) and \( g \) are \( TP_2 \) in \((\lambda, \zeta)\).
(b) \( f(\lambda, x, \zeta) \) is \( RR_2 \) in \((\lambda, x)\) and \((x, \zeta)\).

Then the function
\[
h(\lambda, x) = \int_Z f(\lambda, x, \zeta) g(\lambda, \zeta) d\mu(\zeta),
\]
defined on \( \Lambda \times X \) is \( RR_2 \) in \((\lambda, x)\).

**Proof.** We have to prove that for \( \lambda_1 < \lambda_2 \) and \( x_1 < x_2 \),
\[
(2.1) \quad h(\lambda_2, x_2) h(\lambda_1, x_1) - h(\lambda_2, x_1) h(\lambda_1, x_2) \leq 0.
\]

Karlin ((1968), p. 123) showed that the L. H. S. in (2.1) is equal to,
\[
(2.2) \quad \int_{-\infty}^{+\infty} \int_{-\infty}^{\zeta} \left\{ \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} - \frac{f(\lambda_1, x_2, u)}{f(\lambda_1, x_1, u)} \right\} \{ f(\lambda_1, x_1, u) g(\lambda_1, u) f(\lambda_2, x_1, \zeta) g(\lambda_2, \zeta) \\
- f(\lambda_2, x_1, u) g(\lambda_2, u) f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta) \} d\mu(u) d\mu(\zeta) \]
\[
+ \int_{-\infty}^{+\infty} \int_{-\infty}^{\zeta} \left\{ \frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} - \frac{f(\lambda_1, x_2, u)}{f(\lambda_1, x_1, u)} \right\} \{ f(\lambda_1, x_1, u) f(\lambda_2, x_2, \zeta) g(\lambda_1, \zeta) g(\lambda_2, \zeta) \\
+ f(\lambda_2, x_1, u) f(\lambda_1, x_1, \zeta) g(\lambda_1, \zeta) g(\lambda_2, \zeta) \} d\mu(u) d\mu(\zeta).
\]

The first expression in (2.2) is non-positive since \( f \) is \( RR_2 \) in \((\lambda, x)\) and in \((x, \zeta)\).
The second expression in the first integral is non-negative since \( g \) and \( f \) both are \( TP_2 \) in \((\lambda, \zeta)\). Hence the first double integral in (2.2) is non-positive. The second integral is non-positive since \( f \) is \( RR_2 \) in \((\lambda, x)\). This proves the required result. \( \square \)

With the help of the above lemmas we prove Theorem 2.1 which is a mathematical tool used to prove dependence properties of mixtures of the type (1.1) and (1.2) in Theorems 2.2 and 2.3.

**Theorem 2.1.** Consider
\[
(2.3) \quad \phi(x) = \int_R \left\{ \prod_{i=1}^{n} h_i(x_i, \theta_1, \ldots, \theta_m) \right\} g(\theta_1, \ldots, \theta_m) \prod_{i=1}^{m} d\theta_i.
\]
Suppose that \( g(\theta_1, \ldots, \theta_m) \) is \( TP_2 \) in pairs and for each \( i \in \{1, \ldots, n\} \), \( h_i(x_i, \theta_1, \ldots, \theta_m) \) is \( TP_2 \) in \((\theta_j, \theta_k)\) for \( j, k \in \{1, \ldots, m\}\). If \( h_i(x_i, \theta_1, \ldots, \theta_m) \) is either
(a) \( RR_2 \) in \((x_i, \theta_j)\) for all \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, m\} \); or
(b) \( TP_2 \) in \((x_i, \theta_j)\) for all \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, m\} \),
then \( \phi(x) \) is \( TP_2 \) in pairs.

**Proof.** (a) Let \( I = \{1, \ldots, n\} \) and \( J_j = \{j, \ldots, m\} \) for \( j = 1, \ldots, m \). Define
\[
g_1(x_1, \ldots, x_n, \theta_2, \ldots, \theta_m) = \int_R \left\{ \prod_{i=1}^{n} h_i(x_i, \theta_1, \ldots, \theta_m) \right\} g(\theta_1, \ldots, \theta_m) d\theta_1.
\]

By Lemma 2.1, \( g_1 \) is \( TP_2 \) in \((x_i, x_j)\), \( i, j \in I \) and it is \( TP_2 \) in \((\theta_j, \theta_k)\), \( j, k \in J_2 \) by Lemma 2.2. For fixed \( i \in I \) and \( j \in J_2 \) let
\[
z(x_i, \theta_j, \theta_1) = h_i(x_i, \theta_1, \ldots, \theta_j, \ldots, \theta_m)
\]
and
\[
\mathbf{w}(\theta_1, \theta_j) = \left\{ \prod_{j=1}^{n} h_j(x_j, \theta_1, \ldots, \theta_j, \ldots, \theta_m) \right\} g(\theta_1, \ldots, \theta_j, \ldots, \theta_m).
\]

The function \( z \) defined above is \( \mathbf{RR}_2 \) in \((x_i, \theta_j)\), \( \mathbf{RR}_2 \) in \((x_i, \theta_1)\) and \( \mathbf{TP}_2 \) in \((\theta_1, \theta_j)\). The function \( \mathbf{w} \) is \( \mathbf{TP}_2 \) in \((\theta_1, \theta_j)\). Hence by Lemma 2.4 the function

\[
g_1(x_1, \ldots, x_n, \theta_2, \ldots, \theta_m) = \int_{R} z(x_i, \theta_j; \theta_1) \mathbf{w}(\theta_1, \theta_j) d\theta_1
\]

is \( \mathbf{RR}_2 \) in \((x_i, \theta_j)\). Define

\[
g_i(x_1, \ldots, x_n, \theta_{i+1}, \ldots, \theta_m) = \int_{R} g_{i-1}(x_1, \ldots, x_n, \theta_i, \ldots, \theta_m) d\theta_i, \quad \text{for} \quad i = 2, \ldots, m-1.
\]

We prove the required result by induction. Suppose \( g_{i-1} \) is \( \mathbf{TP}_2 \) in \((x_k, x_j)\), \( \mathbf{TP}_2 \) in \((\theta_p, \theta_l)\) and \( \mathbf{RR}_2 \) in \((x_k, \theta_p)\) for \( k, j \in I \) and \( p, l \in J_i \). By Lemma 2.3, \( g_i \) is \( \mathbf{TP}_2 \) in \((x_k, x_j)\), and by Lemma 2.2, it is \( \mathbf{TP}_2 \) in \((\theta_p, \theta_l)\) for \( p, l \in J_{i+1} \). From Lemma 2.4, \( g_i \) is \( \mathbf{RR}_2 \) in \((x_k, \theta_p)\), \( k \in I \) and \( p \in J_{i+1} \). Thus \( g_{m-1}(x_1, \ldots, x_n, \theta_m) \) is \( \mathbf{TP}_2 \) in \((x_k, x_j)\) and \( \mathbf{RR}_2 \) in \((x_j, \theta_m)\) for \( k, j \in I \). Using Lemma 2.3 we find that,

\[
\phi(x_1, \ldots, x_n) = \int_{R} g_{m-1}(x_1, \ldots, x_n, \theta_m) d\theta_m
\]

is \( \mathbf{TP}_2 \) in pairs of its arguments.

(b) The proof follows from Lemmas 2.1 and 2.2 and using arguments similar to those used for proving part (a). \(\Box\)

In the next theorem we prove that for the mixture model (1.2), under appropriate conditions, both positive as well as negative monotone dependence between \( X_i \) and \( \Theta_i \), \( i = 1, \ldots, n \), imply positive dependence among \( X_1, \ldots, X_n \).

**THEOREM 2.2.** Consider the mixture model (1.2) and suppose that the density function \( g \) is \( \mathbf{TP}_2 \) in pairs. If either

(a) \((X_i, \Theta_i)\) is \( \mathbf{DRR}(k_i, 0) \) for all \( i = 1, \ldots, n \); or

(b) \((X_i, \Theta_i)\) is \( \mathbf{DTP}(k_i, 0) \) for all \( i = 1, \ldots, n \),

then \( X_1, \ldots, X_n \) is \( \mathbf{DTP}(k_1, \ldots, k_n) \).

**PROOF.** (a) By definition, for \( k_i > 0 \) we have

\[
\psi_{k_1, \ldots, k_n}(x_1, \ldots, x_n) = \int_{R^n} \prod_{i=1}^{n} \gamma^{(k_i)}(x_i - t_i) f(t_1, \ldots, t_n) \prod_{i=1}^{n} dt_i
\]

\[
= \int_{R^n} \int_{R^n} \left\{ \prod_{i=1}^{n} \gamma^{(k_i)}(x_i - t_i) f_i(t_i \mid \theta_i) \right\} \times g(\theta_1, \ldots, \theta_n) \prod_{i=1}^{n} d\theta_i \prod_{i=1}^{n} dt_i
\]

\[
= \int_{R^n} \left\{ \prod_{i=1}^{n} \int_{R} \gamma^{(k_i)}(x_i - t_i) f_i(t_i \mid \theta_i) dt_i \right\} g(\theta_1, \ldots, \theta_n) \prod_{i=1}^{n} d\theta_i.
\]
Since \((X_i, \Theta_i)\) being \(\text{DRR}(k_i, 0)\) is equivalent to \(\int_R \gamma^{(k_i)}(x_i - t_i)f_i(t_i \mid \theta_i)dt_i\) being \(\text{RR}_2\) in \((x_i, \theta_i)\), taking \(m = n\) and replacing the function \(h_i(x_i, \theta_1, \ldots, \theta_m)\) by this function in Theorem 2.1 (a), it follows that \(\psi_{k_1, \ldots, k_n}(x_1, \ldots, x_n)\) is \(\text{TP}_2\) in pairs.

Let us now consider the case when \(k_i = 0\) for each \(i\). In this case the function

\[
\psi_{0, \ldots, 0}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)
\]

\[
= \int_{R^n} \left\{ \prod_{i=1}^n f_i(x_i \mid \theta_i) \right\} g(\theta_1, \ldots, \theta_n) \prod_{i=1}^n d\theta_i
\]

is clearly seen to be \(\text{DTP}(0, \ldots, 0)\). The proof follows from Theorem 2.1 (a) on replacing \(h_i(x_i, \theta_1, \ldots, \theta_m)\) with \(f_i(x_i \mid \theta_i)\) and taking \(m = n\).

(b) In this case, by assumption, the function \(\int_R \gamma^{(k_i)}(x_i - t_i)f_i(t_i \mid \theta_i)dt_i\) is \(\text{TP}_2\) in \((x_i, \theta_i)\) for each \(i\). The required result follows on the same lines using part (b) of Theorem 2.1. □

The following results are immediate consequences of the above theorem.

**Corollary 2.1.** Suppose that the density function \(g\) in the mixture model (1.2) is \(\text{TP}_2\) in pairs. Then if

(i) \(X_i\) and \(\Theta_i\) are either all \(\text{TP}_2\) or all \(\text{RR}_2\) dependent, then the joint density of \((X_1, \ldots, X_n)\) is \(\text{TP}_2\) in pairs;

(ii) the conditional hazard rate of \(X_i\) given \(\Theta_1 = \theta_i\) is nondecreasing (nonincreasing) in \(\theta_i\), for \(i = 1, \ldots, n\); then the random variables \((X_1, \ldots, X_n)\) are \(\text{DTP}(1, \ldots, 1)\). In particular \(X_i\) and \(X_j\) are \(\text{RCSI}\) for \(i \neq j \in \{1, \ldots, n\}\) in this case.

**Remarks.** 1. Shaked and Spizzichino (1998) established a different type of dependence result among \((X_1, \ldots, X_n)\) when \(\theta_1 = \theta_2 = \cdots = \theta_m\). They proved that if either all \((X_i, \Theta)\) are \(\text{DTP}(1, 0)\) or all are \(\text{DRR}(1, 0)\), then \((X_1, \ldots, X_n)\) is \(\text{WBF}\) (weakened by failure) dependent. It is not clear whether there is any relation between \(\text{DTP}(1, \ldots, 1)\) and \(\text{WBF}\) dependence.

2. Marshall and Olkin (1991) proved a related result that if each \(f_i(x_i \mid \theta_i)\) in (1.2) is \(\text{TP}_2\) in \((x_i, \theta_i)\) and \(g\) is \(\text{MTP}_2\), then the function \(f(x_1, \ldots, x_n)\) is \(\text{MTP}_2\).

For the general model (1.1), Shaked and Spizzichino (1998) proved that if \(\Theta\) is \(\text{MTP}_2\) and for each \(i \in \{1, \ldots, n\}\), \(f_i(x \mid \theta_1, \ldots, \theta_m)\) is \(\text{MTP}_2\) in \((x, \theta_1, \ldots, \theta_m)\), then the random vector \(X\) is \(\text{MTP}_2\). In Theorem 2.3 below we extend this result to prove positive dependence among \(X_1, \ldots, X_n\) when \(X_i\) and \(\Theta_j\) are even \(\text{RR}_2\) dependent for all \(i \in \{1, \ldots, n\}\) and \(j \in \{1, \ldots, m\}\).

**Theorem 2.3.** Let \(X_1, \ldots, X_n\) follow the mixture model (1.1). Suppose \(g\), the joint density function of \(\Theta\) is \(\text{TP}_2\) in pairs. Then for \(i \in \{1, \ldots, n\}\) and \(j, k \in \{1, \ldots, m\}\),

(a) \(f_i(x \mid \theta_1, \ldots, \theta_m)\) being \(\text{RR}_2(\text{TP}_2)\) in \((x, \theta_j)\) and \(\text{TP}_2\) in \((\theta_j, \theta_k)\) implies that \(f(x_1, \ldots, x_n)\) is \(\text{TP}_2\) in pairs,

(b) \(\tilde{f}_i(x \mid \theta_1, \ldots, \theta_m)\) being \(\text{RR}_2(\text{TP}_2)\) in \((x, \theta_j)\) and \(\text{TP}_2\) in \((\theta_j, \theta_k)\) implies that \(\tilde{f}(x_1, \ldots, x_n)\) is \(\text{TP}_2\) in pairs.

**Proof.** The results follow immediately from Theorem 2.1.

As mentioned earlier model (1.2) is a special case of model (1.1) when the conditional distribution of \(X_i\) given \(\Theta\) depends on \(\Theta\) only through \(\Theta_i\) for \(i = 1, \ldots, n\). Theorem 2.3
establishes \( DTP \) type results for the general mixture model (1.1) only for \( k = 0 \) and \( 1 \) whereas Theorem 2.2 gives more general results for the restricted model (1.2). □

3. Examples and applications

**Application 3.1. Dependence among concomitants of record values:** Let \( \{(X_i, Y_i), i \geq 1\} \) be a sequence of independent and identically distributed random variables from a continuous bivariate distribution. \( X_n \) is called a (upper) record value of the sequence \( \{X_i, i \geq 1\} \) if \( X_n > X_i \) for \( i = 1, \ldots, n-1 \). By convention \( X_1 \) is a record value. The serial numbers at which record values occur are given by the \( \{T_n, n \geq 1\} \), defined recursively by \( T_1 = 1, T_n = \min \{k; k > T_{n-1}, X_k > X_{T_{n-1}}\}, n \geq 2 \). The sequence \( \{T_n, n \geq 1\} \) is called the sequence of (upper) record times and \( \{X_{T_n}, n \geq 1\} \) the sequence of (upper) record values corresponding to \( \{X_n, n \geq 1\} \). For convenience of notation we shall denote \( X_{T_n} \) by \( R_n \) so that \( \{R_n, n \geq 1\} \) is the sequence of record values. The \( Y \)-variate associated with \( R_n \) is denoted by \( Y_{[n]} \) and is called the concomitant of the \( n \)-th record value. That is, the sequence \( \{Y_{i}, i \geq 1\} \) is the sequence of concomitants of \( \{R_n, n \geq 1\} \).

As discussed in Ahsanullah (1994) the joint pfd of the \( n \) record values \( (R_1, \ldots, R_n) \), is

\[
f_{1, \ldots, n}(x_1, \ldots, x_n) = \prod_{i=1}^{n-1} \left\{ f(x_i)/\bar{F}(x_i) \right\} f(x_n), \quad \text{for } x_1 < \cdots < x_n.
\]

From this we obtain the joint pfd of the concomitants \( (Y_{[1]}, \ldots, Y_{[n]}) \) as

\[
f_{Y_{[1]}, \ldots, Y_{[n]}}(y_1, \ldots, y_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_3} \cdots \int_{-\infty}^{x_n} \left\{ \prod_{i=1}^{n} f(y_i \mid x_i) \right\} f_{1, \ldots, n}(x_1, \ldots, x_n) \prod_{i=1}^{n} dx_i,
\]

which can also be written as

\[
f_{Y_{[1]}, \ldots, Y_{[n]}}(y_1, \ldots, y_n) = \int_{R^n} \left\{ \prod_{i=1}^{n} f(y_i \mid x_i) \right\} \left\{ \prod_{i=1}^{n-1} f(x_i)/\bar{F}(x_i) \right\} f(x_n) \prod_{i=1}^{n} k(x_i, x_{i+1}) \prod_{i=1}^{n} dx_i,
\]

where

\[
k(x, y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x \geq y \end{cases}
\]

and \( x_{n+1} = +\infty \). Since the function \( k \) in (3.1) is \( TP_2 \) in \((x, y)\), it follows that the function

\[
\prod_{i=1}^{n} k(x_i, x_{i+1}) \prod_{i=1}^{n-1} \left\{ f(x_i)/\bar{F}(x_i) \right\} f(x_n)
\]

is \( TP_2 \) in pairs. By replacing the function \( g \) in Theorem 2.2 by the one given by (3.2) and assuming that \( X \) and \( Y \) are \( DTP(0, k) \) or \( DRR(0, k) \) dependent we find that \( (Y_{[1]}, \ldots, Y_{[n]}) \), the concomitants of record values, are \( DTP(k, \ldots, k) \) dependent. In particular if

(i) \( X \) and \( Y \) are either \( TP_2 \) or \( RR_2 \) dependent, then the joint density of \((Y_{[1]}, \ldots, Y_{[n]}), \) the concomitants of records, is \( TP_2 \) in pairs.
(ii) \( X \) and \( Y \) are \( DTP(0,1) \) or \( DRR(0,1) \), then the concomitants of record values are dependent according to \( DTP(1,\ldots,1) \) criteria.

**Remark.** Similar results can be obtained for dependence among concomitants of order statistics. It can be proved that if \( (X,Y) \) are \( DTP(0,k) \) or \( DRR(0,k) \), then the concomitants of order statistics are \( DTP(k,\ldots,k) \). See Khaledi and Kochar (2000) for details.

**Application 3.2. The frailty model:** Marshall and Olkin (1988) studied the family of multivariate distributions generated by mixture in which marginals were considered as parameters. Suppose that \( \bar{H}_1,\ldots,\bar{H}_n \) are univariate survival functions and let \( G \) be an \( n \)-variate distribution function such that \( \bar{G}(0,\ldots,0) = 1 \) with univariate marginals \( G_i, \ i = 1,\ldots,n \). Denote the Laplace transforms of \( G \) and \( G_i \) by \( \phi \) and \( \phi_i \), respectively. Let for each \( i, \bar{F}_i(x) = \exp[-\phi_i^{-1}\bar{H}_i(x)] \), where the function \( \phi_i^{-1} \) is inverse of \( \phi_i \), then

\[
\bar{H}(x_1,\ldots,x_n) = \int_{R^n} K(F_1^{\theta_1}(x_1),\ldots,F_n^{\theta_n}(x_n))dG(\theta_1,\ldots,\theta_n),
\]

is an \( n \)-variate survival function with marginals survival functions \( \bar{H}_1,\ldots,\bar{H}_n \). Here \( K \) denotes the distribution function of an \( n \)-dimensional vector with marginals as uniform \((0,1)\) distributions. Different choices of \( K \) and \( G \) lead to a variety of distributions with marginals as specified parameters. If \( K \) is the joint distribution function of \( n \) independent uniform \((0,1)\) random variables and \( g \) is the density function corresponding to distribution function \( G \), then \( \bar{H} \) in (3.3) can be written as,

\[
\bar{H}(x_1,\ldots,x_n) = \int_{R^n} \left\{ \prod_{i=1}^{n} \bar{F}_i^{\theta_i}(x_i) \right\} g(\theta_1,\ldots,\theta_n) \prod_{i=1}^{n} d\theta_i,
\]

with density function as

\[
h(x_1,\ldots,x_n) = \int_{R^n} \left\{ \prod_{i=1}^{n} \theta_i \bar{F}_i^{\theta_i-1}(x_i)f_i(x_i) \right\} g(\theta_1,\ldots,\theta_n) \prod_{i=1}^{n} d\theta_i,
\]

which is of the form (1.2). It is easy to see that the function \( \theta_i \bar{F}_i^{\theta_i-1}(x_i)f_i(x_i) \) is \( RR_2 \) in \((x_i,\theta_i)\) for \( i = 1,\ldots,n \). Using Theorem 2.2 with \( k_i = 0, \ i = 1,\ldots,n \), it follows that if \( g \) is \( TP_2 \) in pairs, then the joint density of \( X \) is \( TP_2 \) in pairs.

The model described in (3.4), for \( n = 2 \) is known as a bivariate frailty model and it has been studied intensively in the literature. See Oakes (1989) for more details. If we assume that \( \Theta \) is univariate in this, then \( h(x) \) in (3.5) is always \( TP_2 \) in pairs, a result which is stronger than the result of Marshall and Olkin (1988) who proved that \( X \) is \( DTP(1,\ldots,1) \) under the given conditions.

**Example 3.3.** Let \( Z = (Z_1,\ldots,Z_n) \) be a random vector of independent components and let \( \Theta = (\Theta_1,\ldots,\Theta_n) \) be a random vector with joint pdf \( g \). Let \( X = Z + \Theta \) and assume \( Z \) and \( \Theta \) are independent. Karlin and Rinott (1980a) showed that if the joint pdf of \( \Theta \) is \( MTP_2 \) and if \( Z_i \) has log-concave density \( f_i \) for \( i = 1,\ldots,n \), then the random vector \( X \) is \( MTP_2 \). Recall that the \( MTP_2 \) property implies \( TP_2 \) in pairs property. We
show here that the joint pdf \( \phi \) of \( X \) is \( TP_2 \) in pairs when \( Z_i \)'s have log-convex densities and the function \( g \) is \( TP_2 \) in pairs. The joint pdf of \( X \) is

\[
\phi(x_1, \ldots, x_n) = \int_{R^n} \prod_{i=1}^{n} f_i(x_i - \theta_i) g(\theta_1, \ldots, \theta_n) \prod_{i=1}^{n} d\theta_i,
\]

which is of the form (1.2). By replacing the function \( h_i \) in Theorem 2.1 with \( f_i \) and using this fact that \( f_i(x_i - \theta_i) \) is \( RR_2 \) in \( (x_i, \theta_i) \) if \( f_i \) is log-convex for \( i = 1, \ldots, n \), it follows that joint pdf of \( X \) is \( TP_2 \) in each pairs of its arguments.

**Example 3.4.** Let \( X_1, X_2, \ldots, X_n \) have the joint density function given by (1.2) with

\[
f_i(x \mid \theta) = \left( \frac{1}{\sqrt{x}} + \frac{1}{\theta} \right) \exp \left\{ -2\sqrt{x} - \frac{x}{\theta} \right\}, \quad x \geq 0, \quad \text{for } i = 1, 2, \ldots, n,
\]

where the random variable \( \Theta \) is univariate and positive. The corresponding hazard rate functions are given by

\[
r_i(x \mid \theta) = \frac{1}{\sqrt{x}} + \frac{1}{\theta}, \quad x > 0,
\]

for \( i = 1, 2, \ldots, n \). Clearly, \( r_i(x \mid \theta) \) is decreasing in \( \theta \) for all \( x > 0 \), \( i = 1, 2, \ldots, n \). On the other hand, \( f_i(x \mid \theta) \) is neither \( TP_2 \) nor \( RR_2 \). It follows from Theorem 2.2 (b) \((X_1, \ldots, X_n)\) is \( DTP(1, \ldots, 1)\).

There are other interesting applications and examples given in Shaked and Spizzichino (1998) to which the results of Section 2 can be applied.

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**References**


