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DISTRIBUTIONS OF NUMBERS OF SUCCESS RUNS OF FIXED LENGTH IN MARKOV DEPENDENT TRIALS

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Abstract. Let $\{Z_n, n \geq 1\}$ be a time-homogeneous $\{0,1\}$-valued Markov chain, and let N_n be a random variable denoting the number of runs of "1" of length k in the first n trials. In this article we conduct a systematic study of N_n by establishing formulae for the evaluation of its probability generating function, probability mass function and moments. This is done in three different enumeration schemes for counting runs of length k, the "non-overlapping", the "overlapping" and the "at least" scheme. In the special case of i.i.d. trials several new results are established.

Key words and phrases: Binomial/negative binomial distribution of order k, success runs, Markov chain, probability generating function, probability mass function, moments.

1. Introduction

The concept of runs has been used in various areas of statistical analysis as a reasonable criterion for detecting changes in sequences involving experimental trials with two possible outcomes. In the early 1940s it was used in the area of hypothesis testing (run-test) by Mood (1940) and in statistical quality control problems by Mosteller (1941). Recently, it has been successfully employed in many other areas such as reliability (see Chao *et al.* (1995)), start-up demonstration tests (Balakrishnan *et al.* (1995)), DNA sequence matching (Goldstein (1990)), analysis of the "maximum drawdown" in stochastic finance (Binswager and Embrechts (1994)), psychology, ecology, radar astronomy (Schwager (1983)), etc.

We note, however, that there can be different ways of counting runs. It depends on the statistical problem which way of counting should be adopted. Let $\{Z_n, n \geq 1\}$ be a sequence of repeated trials with two possible outcomes, success (1) and failure (0). Any uninterrupted sequence of k consecutive successes will be called *success run of length* k (k is a positive integer). The classical scheme for enumerating runs of length k is the one proposed by Feller (1968). According to this, we start counting from scratch each time a succession of k consecutive successes is observed (*non-overlapping counting*). Ling (1988) proposed an alternative enumeration scheme in which a success run of length $m \geq k$ accounts for $m - k + 1$ runs (*overlapping counting*). Finally, a third scheme can be initiated by counting a succession of *at least* k successes as a single run (see, e.g. Goldstein (1990)). We mention the following illustrative example to fix the distinction between the aforementioned enumeration schemes: for the sequence 11101111011 and $k = 2$ we have that it contains 4, 6, 3 non-overlapping, overlapping and "at least" runs of length 2, respectively.

The distributions of the number of success runs of length k in a fixed number of trials have been termed as *binomial distributions of order* k, while the corresponding waiting time distributions until the first, or in general the r-th occurrence of a success run of length k, as *geometric and negative binomial distributions of order* k, respectively, and this is due to the work of Philippou *et al.* (1983) which initiated the systematic study of these distributions. The combination of enumeration schemes of counting runs along with the probabilistic behaviour of the sequence $\{Z_n, n \geq 1\}$ (i.i.d. trials, non i.i.d. trials, Markov dependent trials, etc.), resulted in a number of papers dealing with binomial distributions of order k. We mention Philippou and Makri (1986), Ling (1988), Aki and Hirano (1988, 1993), Godbole (1990, 1992), Hirano and Aki (1993), Chryssaphinou *et al.* (1993), Mohanty (1994), Koutras and Alexandrou (1995), Koutras (1997*a*), and references therein. The main effort of the aforementioned authors was to establish formulae for the evaluation of the probability mass function (p.m.f.) and of the probability generating function (p.g.f.) of the binomial distributions of order k. The formulae obtained were usually complicated, and there is an absence of results regarding higher order moments of these distributions.

In the present manuscript we systematically develop formulae for the study of the binomial distributions of order k (viz., N_n) in the case of $\{Z_n, n \geq 1\}$ being a time homogeneous two-state Markov chain in all three aforementioned enumeration schemes. To achieve this goal we shall make use of a result of Koutras (1997*a*), where the double p.g.f. of N_n is expressed in terms of the p.g.f. of the distribution of the waiting times. More specifically, the present paper is organized as follows. In Section 2, some preliminary results and the necessary notations are introduced. In Section 3, we present recursive schemes and/or exact formulae for the evaluation of the p.g.f. and the p.m.f. of N_n. In Section 4, we develop formulae for the evaluation of the (descending) factorial moments and moments about zero of N_n. In the special case of the i.i.d. framework several new results are also established.

2. Preliminary results and notation

The waiting time for the r-th occurrence of a success run of length k (r, k positive integers) in a sequence of repeated trials with two possible outcomes will be denoted by $T_r^{(a)}$, with the superscript pointing out the enumeration scheme employed; $a = I$ indicates the "non-overlapping" counting scheme, $a = II$ indicates the "at least" counting scheme, and $a = III$ indicates the "overlapping" one. Let

$$h_r^{(a)}(m) = P[T_r^{(a)} = m], \qquad m = 1, 2, \ldots,$$

be the p.m.f. of $T_r^{(a)}$. By imposing the convention $h_r^{(a)}(0) = \delta_{r,0}$, where δ_{ij} is the Kronecker's delta function, $h_r^{(a)}(m)$ becomes meaningful for all non-negative integers r and m. Let

$$H_r^{(a)}(z) = \sum_{m=0}^{\infty} h_r^{(a)}(m) z^m, \qquad r \geq 0,$$

be the p.g.f. of $T_r^{(a)}$.

The distribution of the numbers of success runs of length k in n trials (n positive integer) will be denoted by $N_n^{(a)}$. Let

$$g_n^{(a)}(x) = P[N_n^{(a)} = x], \qquad x = 0, 1, \dots,$$

be the p.m.f. of $N_n^{(a)}$. By imposing the convention $g_0^{(a)}(x) = \delta_{x,0}$ the p.m.f. of $N_n^{(a)}$ becomes meaningful for $n = 0$. Let

$$G_n^{(a)}(w) = \sum_{x=0}^{\infty} g_n^{(a)}(x) w^x, \qquad n \geq 0, \tag{2.1}$$

and

$$G^{(a)}(z, w) = \sum_{n=0}^{\infty} \sum_{x=0}^{\infty} g_n^{(a)}(x) w^x z^n,$$

be its single and double p.g.f.

Suppose that successive occurrences of success runs of length k constitute a *delayed recurrent event* (see Feller (1968)), and therefore

$$H_r^{(a)}(z) = H^{(a)}(z)[A^{(a)}(z)]^{r-1}, \qquad r \geq 1, \tag{2.2}$$

where $H^{(a)}(z)$ and $A^{(a)}(z)$ are proper p.g.fs. In this case, Koutras (1997a) showed that

$$G^{(a)}(z, w) = \frac{1 - wA^{(a)}(z) - (1 - w)H^{(a)}(z)}{(1 - z)[1 - wA^{(a)}(z)]}. \tag{2.3}$$

Now, let $\{Z_n, n \geq 1\}$ be a time homogeneous two-state Markov chain with transition probabilities defined by

$$p_{ij} = P(Z_{n+1} = j \mid Z_n = i), \qquad n \geq 1, \quad 0 \leq i, \ j \leq 1,$$

and initial probabilities $p_j = P(Z_1 = j)$, $j = 0, 1$. Under this framework and for $k \geq 2$, we have that $H_r^{(I)}(z)$, $H_r^{(II)}(z)$ and $H_r^{(III)}(z)$ obey relation (2.2), with

$$\begin{cases} H^{(I)}(z) = H^{(II)}(z) = H^{(III)}(z) = \dfrac{P(z)}{Q(z)}, \quad \text{and} \\ A^{(I)}(z) = \dfrac{R^{(I)}(z)}{Q(z)}, \qquad A^{(II)}(z) = \dfrac{p_{10}z}{1 - p_{11}z} \cdot \dfrac{R^{(II)}(z)}{Q(z)}, \\ A^{(III)}(z) = p_{11}z + \dfrac{R^{(III)}(z)}{Q(z)}, \end{cases} \tag{2.4}$$

where

$$\begin{cases} P(z) = \left[p_1 z + (p_0 p_{01} - p_1 p_{00})z^2\right] (p_{11}z)^{k-1}, \\ Q(z) = (1 - p_{11}z)^{-1}[1 - (p_{00} + p_{11})z + (p_{11} - p_{01})z^2 \\ \qquad\qquad + (p_{01}z)(p_{10}z)(p_{11}z)^{k-1}], \\ R^{(I)}(z) = [p_{11}z + (p_{01} - p_{11})z^2](p_{11}z)^{k-1}, \\ R^{(II)}(z) = (p_{01}z)(p_{11}z)^{k-1}, \\ R^{(III)}(z) = (p_{01}z)(p_{10}z)(p_{11}z)^{k-1}. \end{cases} \tag{2.5}$$

The p.g.fs $H_r^{(II)}(z)$ and $H_r^{(III)}(z)$ are consistent with the ones derived by Koutras (1997b). An extra factor z appearing here in the p.g.fs reflects the different set-up used for the evolution of the Markov dependent trials. The formula of $H_r^{(I)}(z)$ coincides with the respective one in Antzoulakos (1999) and it is different to the one derived by Koutras (1997*b*). The difference appears in the formula of $A^{(I)}(z)$. Both Koutras (1997*b*) and Antzoulakos (1999) concluded that the p.g.f. of $A^{(I)}(z)$ may be obtained from $H^{(I)}(z)$ by modifying properly the initial probability vector $[p_0, p_1]$. The methodology employed by Koutras (1997*b*) yielded the vector $[1, 0]$, while the methodology of Antzoulakos (1999) yielded the vector $[p_{10}, p_{11}]$, as one would expected. Analogues p.g.fs may be found in Aki and Hirano (1993), Balasubramanian *et al.* (1993), Hirano and Aki (1993), Mohanty (1994) and Uchida and Aki (1995). Obviously, for $p_1 = p_{01} = p_{11} = p$, the above p.g.fs reduce to respective ones regarding the case of i.i.d. trials with success probability p. Also, we mention here that when no confusion is likely to arise, formulae encompassing all three enumeration schemes will be presented without the use of superscripts I, II and III.

3. Probability generating functions and probability mass functions

In this section we establish recursive schemes for the evaluation of the p.g.f. and p.m.f. of $N_n^{(a)}$, $a = I, II, III$. Additionally, the p.g.f. and p.m.f. of the r.v. $N_n^{(II)}$ is expressed in terms of binomial coefficients. In the special case of i.i.d. trials several new results are established.

3.1 *Non-overlapping scheme*

(a) Markov dependent trials

Relations (2.3) and (2.4) imply that the double p.g.f. of N_n is given by

$$G(z,w) = \frac{Q(z) - wR(z) - (1-w)P(z)}{(1-z)[Q(z) - wR(z)]}.$$

Multiplying both numerator and denominator of $G(z,w)$ by $(1 - p_{11}z)$, using relation (2.5), and carrying out some elementary but involved algebra, we obtain that

$$G(z,w) = \frac{1 - a_1 z - a_2 z^2 - (\beta + a_3)z^k + (\beta p_{11} - \gamma - a_4)z^{k+1} + (\gamma p_{11} - a_5)z^{k+2}}{(1-z)(1 - a_1 z - a_2 z^2 - a_3 z^k - a_4 z^{k+1} - a_5 z^{k+2})},$$

where

$$\begin{aligned} &a_1 = p_{00} + p_{11}, \quad a_2 = 1 - a_1, \quad a_3 = w p_{11}^k, \\ &a_4 = -[p_{01}p_{10}p_{11}^{k-1}(1-w) + a_1 a_3], \quad a_5 = a_3(a_1 - 1), \\ &\beta = (1-w)p_1 p_{11}^{k-1}, \quad \gamma = (1-w)(p_0 p_{01} - p_1 p_{00})p_{11}^{k-1}. \end{aligned}$$

The numerator of $G(z,w)$ may be written as $(1-z)[1 + a_2 z - (\beta + a_3)z^k - (\gamma p_{11} - a_5)z^{k+1}]$, and therefore

$$G(z,w) = \frac{G_1(z,w)}{G_2(z,w)} = \frac{1 + a_2 z - (\beta + a_3)z^k - (\gamma p_{11} - a_5)z^{k+1}}{1 - a_1 z - a_2 z^2 - a_3 z^k - a_4 z^{k+1} - a_5 z^{k+2}}. \tag{3.1}$$

In the following theorem we derive a recursive scheme for the evaluation of $G_n(w)$.

THEOREM 3.1. *The p.g.f. $G_n(w)$ of the r.v. N_n satisfies the following recursive scheme:*

$$\begin{aligned} &G_n(w) = 1, \quad \text{for} \quad 0 \le n < k, \quad G_k(w) = 1-\beta, \; G_{k+1}(w) = 1-\beta(1+p_{00})-\gamma, \\ &G_n(w) = a_1 G_{n-1}(w) + a_2 G_{n-2}(w) \\ &\qquad + a_3 G_{n-k}(w) + a_4 G_{n-k-1}(w) + a_5 G_{n-k-2}(w), \quad \text{for} \quad n \ge k+2. \end{aligned}$$

PROOF. It follows by equating the coefficients of z^n on both sides of

$$G_2(z,w)\sum_{n=0}^{\infty} G_n(w) z^n = G_1(z,w),$$

where $G_i(z,w)$, $i = 1, 2$ are given by (3.1). □

An efficient recursive scheme for the evaluation of the p.m.f. of N_n can be obtained on manipulating over Theorem's 3.1 outcome. More specifically we have the next.

THEOREM 3.2. *The p.m.f. $g_n(x)$ of the r.v. N_n satisfies the following relations:*

$$\begin{aligned} &g_n(x) = 0, \quad \text{for} \quad x < 0 \text{ or } x > \left[\frac{n}{k}\right], \\ &g_n(x) = \delta_{x,0}, \quad \text{for} \quad 0 \le n < k, \\ &g_k(0) = 1 - p_1 p_{11}^{k-1}, \quad g_k(1) = p_1 p_{11}^{k-1}, \quad g_k(x) = 0 \text{ for } x \ge 2, \\ &g_{k+1}(0) = 1 - (p_1 + p_0 p_{01}) p_{11}^{k-1}, \quad g_{k+1}(1) = (p_1 + p_0 p_{01}) p_{11}^{k-1}, \\ &g_{k+1}(x) = 0, \text{ for } x \ge 2, \\ &g_n(x) = a_1 g_{n-1}(x) + a_2 g_{n-2}(x) \\ &\qquad + p_{11}^k [g_{n-k}(x-1) - a_1 g_{n-k-1}(x-1) - a_2 g_{n-k-2}(x-1)] \\ &\qquad - p_{01} p_{10} p_{11}^{k-1} [g_{n-k-1}(x) - g_{n-k-1}(x-1)], \quad \text{for} \quad n \ge k+2. \end{aligned}$$

PROOF. The proof may be established by replacing $G_m(w)$, $m \ge 0$, in Theorem 3.1 with its expansion provided by relation (2.1), and then equating the coefficients of w^x on both sides of the resulting equalities. □

(b) I.i.d. trials

In the case of i.i.d. trials ($p_1 = p_{01} = p_{11} = p, p_0 = p_{00} = p_{10} = q$ *and* $p + q = 1$) the double p.g.f. of N_n may be easily deduced from relation (3.1) and it is given by

$$G(z,w) = \frac{1-(pz)^k}{1 - z - w(pz)^k + p^k(q+wp)z^{k+1}}. \tag{3.2}$$

In the following proposition we derive an exact formula of $G_n(w)$ in terms of binomial coefficients.

PROPOSITION 3.1. *The p.g.f. $G_n(w)$ of the r.v. N_n is given by*

$$G_n(w) = \sum_{x=0}^{[n/k]} \sum_{r=0}^{u(0)} \sum_{s=\max\{0,x-r\}}^{\nu(0)} \psi(0) w^x - \sum_{x=0}^{[n/k]-1} \sum_{r=0}^{u(k)} \sum_{s=\max\{0,x-r\}}^{\nu(k)} \psi(k) w^x,$$

where for $i=0,k$

$$\psi(i)=(-1)^r\binom{s+r}{r}\binom{r}{x-s}\binom{n-k(s+r)+s-i}{s+r}q^{r+s-x}p^{k(s+r)+x-s+i},$$

$$u(i)=\min\left\{\left[\frac{n-i}{k+1}\right],n-kx-i\right\},\qquad \nu(i)=\min\left\{x,\left[\frac{n-r-i}{k}\right]-r\right\}.$$

PROOF. Using relation (3.2) and the multinomial theorem we get

$$\begin{aligned}G(z,w) &= [1-(pz)^k]\sum_{n=0}^{\infty}\sum_{n_1+n_2+n_3=n}\binom{n_1+n_2+n_3}{n_1,n_2,n_3}(wp^k)^{n_2}\\ &\quad\cdot[-p^k(q+wp)]^{n_3}z^{n_1+kn_2+(k+1)n_3}\\ &= [1-(pz)^k]\sum_{n=0}^{\infty}\sum_{n_1+kn_2+(k+1)n_3=n}\binom{n_1+n_2+n_3}{n_1,n_2,n_3}(wp^k)^{n_2}\\ &\quad\cdot[-p^k(q+wp)]^{n_3}z^n.\end{aligned}$$

Equating the coefficients of z^n on both sides of the above relation we obtain that the p.g.f. of N_n is given by

$$G_n(w)=\varphi_0(w)-p^k\varphi_k(w),$$

where

$$\varphi_i(w)=\sum_{n_1+kn_2+(k+1)n_3=n-i}\binom{n_1+n_2+n_3}{n_1,n_2,n_3}(wp^k)^{n_2}[-p^k(q+wp)]^{n_3},\qquad i=0,k.$$

Next, we observe that

$$\begin{aligned}\varphi_0(w) &= \sum_{r=0}^{[n/(k+1)]}\sum_{s=0}^{[(n-(k+1)r)/k]}(-1)^r\binom{s+r}{r}\binom{n-k(s+r)+s}{s+r}(wp^k)^s[p^k(q+wp)]^r\\ &= \sum_{r=0}^{[n/(k+1)]}\sum_{s=0}^{[(n-(k+1)r)/k]}(-1)^r\binom{s+r}{r}\binom{n-k(s+r)+s}{s+r}p^{k(s+r)}\\ &\quad\cdot\sum_{x=s}^{s+r}\binom{r}{x-s}q^{r+s-x}p^{x-s}w^x\\ &= \sum_{r=0}^{[n/(k+1)]}\sum_{s=0}^{[(n-r)/k]-r}\sum_{x=s}^{s+r}\psi(0)w^x.\end{aligned}$$

Interchanging the order of summations we get

$$\varphi_0(w)=\sum_{r=0}^{[n/(k+1)]}\sum_{x=0}^{[(n-r)/k]}\sum_{s=\max\{0,x-r\}}^{\nu(0)}\psi(0)w^x=\sum_{x=0}^{[n/k]}\sum_{r=0}^{u(0)}\sum_{s=\max\{0,x-r\}}^{\nu(0)}\psi(0)w^x.$$

Working as above for $p^k\varphi_k(w)$ we obtain that

$$p^k\varphi_k(w) = \sum_{r=0}^{[(n-k)/(k+1)]} \sum_{s=0}^{[(n-r-k)/k]-r} \sum_{x=s}^{s+r} \psi(k)w^x = \sum_{x=0}^{[n/k]-1} \sum_{r=0}^{u(k)} \sum_{s=\max\{0,x-r\}}^{\nu(k)} \psi(k)w^x,$$

and this completes the proof of the proposition. □

A direct inspection in Proposition 3.1 implies the following formula for the p.m.f. of N_n.

PROPOSITION 3.2. *The p.m.f. $g_n(x)$ of the r.v. N_n is given by*

$$g_n(x) = \sum_{r=0}^{u(0)} \sum_{s=\max\{0,x-r\}}^{\nu(0)} \psi(0) - \sum_{r=0}^{u(k)} \sum_{s=\max\{0,x-r\}}^{\nu(k)} \psi(k), \qquad 0 \le x \le \left[\frac{n}{k}\right] - 1,$$

and

$$g_n(x) = \sum_{r=0}^{u(0)} \sum_{s=\max\{0,x-r\}}^{\nu(0)} \psi(0), \qquad x = \left[\frac{n}{k}\right],$$

where $u(i)$, $\nu(i)$ and $\psi(i)$ $(i = 0, k)$ are as in Proposition 3.1.

An alternative formula of $g_n(x)$ in terms of binomial coefficients can be found in Godbole (1990), which was his main result and it was derived by combinatorial methods. Also, efficient recursive schemes for the evaluation of $G_n(w)$ and $g_n(x)$ follow immediately from Theorem 3.1 and Theorem 3.2, respectively, which are simpler compared with respective recursive schemes obtained by Aki and Hirano (1988).

3.2 *At least scheme*

Relations (2.3) and (2.4) imply that the double p.g.f. of N_n is given by

$$G(z,w) = \frac{(1-p_{11}z)Q(z) - w(p_{10}z)R(z) - (1-w)(1-p_{11}z)P(z)}{(1-z)[(1-p_{11}z)Q(z) - w(p_{10}z)R(z)]}.$$

Using relation (2.5) and carrying out some algebra we get

$$G(z,w) = \frac{1 - b_1z - b_2z^2 - \beta z^k + (\beta p_{11} - \gamma - b_3)z^{k+1} + \gamma p_{11}z^{k+2}}{(1-z)(1 - b_1z - b_2z^2 - b_3z^{k+1})},$$

where β and γ are as in Subsection 3.1, and

$$b_1 = p_{00} + p_{11}, \qquad b_2 = 1 - b_1, \quad b_3 = -(1-w)p_{01}p_{10}p_{11}^{k-1}.$$

The numerator of $G(z,w)$ may be written as $(1-z)(1 + b_2z - \beta z^k - \gamma p_{11}z^{k+1})$, and therefore

$$G(z,w) = \frac{G_1(z,w)}{G_2(z,w)} = \frac{1 + b_2z - \beta z^k - \gamma p_{11}z^{k+1}}{1 - b_1z - b_2z^2 - b_3z^{k+1}}. \tag{3.3}$$

Using relation (3.3) and the multinomial theorem, we arrive at the following theorem.

THEOREM 3.3. *The p.g.f. $G_n(w)$ of the r.v. N_n is given by*

$$G_n(w) = \varphi_0(w) + b_2\varphi_1(w) - \beta\varphi_k(w) - \gamma p_{11}\varphi_{k+1}(w),$$

where

$$\varphi_i(w) = \sum_{n_1+2n_2+(k+1)n_3=n-i} \binom{n_1+n_2+n_3}{n_1, n_2, n_3} b_1^{n_1} b_2^{n_2} b_3^{n_3}, \qquad i = 0, 1, k, k+1.$$

An expansion of $G_n(w)$ in terms of binomial coefficients is given in the following theorem. The proof may be established using the methodology of the proof of Proposition 3.1.

THEOREM 3.4. *The p.g.f. $G_n(w)$ of the r.v. N_n is given by*

$$\begin{aligned} G_n(w) = &\sum_{x=0}^{[n/(k+1)]} \sum_{r=x}^{u(0)} \sum_{s=0}^{\nu(0)} \binom{r}{x} \psi(0) w^x + \sum_{x=0}^{[(n-1)/(k+1)]} \sum_{r=x}^{u(1)} \sum_{s=0}^{\nu(1)} (p_{01} - p_{11}) \binom{r}{x} \psi(1) w^x \\ &- \sum_{x=0}^{[(n+1)/(k+1)]} \sum_{r=\max\{0,x-1\}}^{u(k)} \sum_{s=0}^{\nu(k)} p_1 p_{11}^{k-1} \binom{r+1}{x} \psi(k) w^x \\ &- \sum_{x=0}^{[n/(k+1)]} \sum_{r=\max\{0,x-1\}}^{u(k+1)} \sum_{s=0}^{\nu(k+1)} (p_0 p_{01} - p_1 p_{00}) p_{11}^k \binom{r+1}{x} \psi(k+1) w^x, \end{aligned}$$

where for $i = 0, 1, k, k+1$, we have that $u(i) = [\frac{n-i}{k+1}]$, $\nu(i) = [\frac{n-i-(k+1)r}{2}]$, and

$$\begin{aligned} \psi(i) = &(-1)^{r+x} \binom{s+r}{r} \binom{n-kr-s-i}{s+r} \\ &\cdot (p_{00} + p_{11})^{n-(k+1)r-2s-i} (p_{01} - p_{11})^s \left(p_{01} p_{10} p_{11}^{k-1}\right)^r. \end{aligned}$$

A careful look in Theorem 3.4 reveals the following formula for the p.m.f. of N_n.

THEOREM 3.5. *For $x = 0, 1, \ldots, [\frac{n+1}{k+1}]$, the p.m.f. $g_n(x)$ of the r.v. N_n is given by*

$$\begin{aligned} g_n(x) = &\sum_{r=x}^{u(0)} \sum_{s=0}^{\nu(0)} \binom{r}{x} \psi(0) + \sum_{r=x}^{u(1)} \sum_{s=0}^{\nu(1)} (p_{01} - p_{11}) \binom{r}{x} \psi(1) \\ &- \sum_{r=\max\{0,x-1\}}^{u(k)} \sum_{s=0}^{\nu(k)} p_1 p_{11}^{k-1} \binom{r+1}{x} \psi(k) \\ &- \sum_{r=\max\{0,x-1\}}^{u(k+1)} \sum_{s=0}^{\nu(k+1)} (p_0 p_{01} - p_1 p_{00}) p_{11}^k \binom{r+1}{x} \psi(k+1), \end{aligned}$$

where $u(i)$, $\nu(i)$ and $\psi(i)$ $(i = 0, 1, k, k+1)$ are as in Theorem 3.4.

In the following two theorems we present effective recursive schemes for the evaluation of the p.g.f. and p.m.f. of N_n. Equating the coefficients of z^n on both sides of

$$G_2(z,w)\sum_{n=0}^{\infty} G_n(w)z^n = G_1(z,w),$$

where $G_i(z,w)$, $i=1,2$ are given by (3.3), we obtain the following theorem.

THEOREM 3.6. *The p.g.f. of the r.v. N_n satisfies the following recursive scheme:*

$$G_n(w)=1, \quad \textit{for} \quad 0\le n<k, \quad G_k(w)=1-\beta, \quad G_{k+1}(w)=1-\beta(1+p_{00})-\gamma,$$
$$G_n(w)=b_1G_{n-1}(w)+b_2G_{n-2}(w)+b_3G_{n-k-1}(w), \qquad \textit{for} \quad n\ge k+2.$$

Replacing $G_m(w)$, $m\ge 0$, in Theorem 3.6 with its expansion provided by relation (2.1), and then equating the coefficients of w^x on both sides of the resulting equalities we obtain the following theorem.

THEOREM 3.7. *The p.m.f. $g_n(x)$ of the r.v. N_n satisfies the following recursive scheme:*

$$g_n(x)=0, \qquad \textit{for} \quad x<0 \ \textit{or} \ x>\left[\frac{n+1}{k+1}\right],$$
$$g_n(x)=b_1g_{n-1}(x)+b_2g_{n-2}(x)-p_{01}p_{10}p_{11}^{k-1}[g_{n-k-1}(x)-g_{n-k-1}(x-1)], \qquad \textit{for} \quad n\ge k+2,$$

with initial conditions for $g_n(x)$ for $0\le n\le k+1$ as those in Theorem 3.2.

The results of this paragraph may be instantly adjusted (by a proper modification of the initial distribution of the Markov dependent trials) in order to cover the set-up for the evolution of the Markov dependent trials used by Hirano and Aki (1993). Also, the results of this paragraph may be reduced to ones covering the case of i.i.d. trials.

3.3 *Overlapping scheme*

(a) Markov dependent trials

Relations (2.3) and (2.4) imply that the double p.g.f. of N_n is given by

$$G(z,w)=\frac{(1-wp_{11}z)Q(z)-wR(z)-(1-w)P(z)}{(1-z)[(1-wp_{11}z)Q(z)-wR(z)]}.$$

Multiplying both numerator and denominator of $G(z,w)$ by $(1-p_{11}z)$, using relation (2.5) and carrying out some algebra, we get

$$G(z,w)=\frac{1-c_1z-c_2z^2-c_3z^3-\beta z^k+(\beta p_{11}-\gamma-c_4)z^{k+1}+\gamma p_{11}z^{k+2}}{(1-z)(1-c_1z-c_2z^2-c_3z^3-c_4z^{k+1})},$$

where β and γ are as in Subsection 3.1, and

$$c_1=(1+w)p_{11}+p_{00}, \qquad c_2=p_{01}-p_{11}-wp_{11}(p_{00}+p_{11})$$
$$c_3=wp_{11}(p_{11}-p_{01}), \qquad c_4=-(1-w)p_{01}p_{10}p_{11}^{k-1}.$$

The numerator of $G(z,w)$ may be written as $(1-z)[1+(1-c_1)z+c_3z^2-\beta z^k-\gamma p_{11}z^{k+1}]$, and therefore

$$G(z,w)=\frac{1+(1-c_1)z+c_3z^2-\beta z^k-\gamma p_{11}z^{k+1}}{1-c_1z-c_2z^2-c_3z^3-c_4z^{k+1}}. \tag{3.4}$$

Using relation (3.4) we arrive at the following recursive scheme which permits the evaluation of $G_n(w)$.

THEOREM 3.8. *The p.g.f. $G_n(w)$ of the r.v. N_n satisfies the following recursive scheme*:

$$\begin{aligned}
&G_n(w)=1, \quad \textit{for} \quad 0\le n<k, \quad G_k(w)=1-\beta,\\
&G_{k+1}(w)=1-\beta(1+p_{00}+wp_{11})-\gamma,\\
&G_n(w)=c_1G_{n-1}(w)+c_2G_{n-2}(w)+c_3G_{n-3}(w)+c_4G_{n-k-1}(w), \qquad \textit{for} \quad n\ge k+2.
\end{aligned}$$

Theorem 3.8 implies the following recursive scheme satisfied by $g_n(x)$.

THEOREM 3.9. *The p.m.f. $g_n(x)$ of the r.v. N_n satisfies the following recursive scheme*:

$$\begin{aligned}
&g_n(x)=0, \quad \textit{for} \quad x<0 \ \textit{or}\ x>n-k+1, \qquad g_n(x)=\delta_{x,0}, \qquad \textit{for} \quad 0\le n<k,\\
&g_k(0)=1-p_1p_{11}^{k-1}, \qquad g_k(1)=p_1p_{11}^{k-1}, \quad g_k(x)=0, \ \textit{for}\ x\ge 2,\\
&g_{k+1}(0)=1-(p_1+p_0p_{01})p_{11}^{k-1}, \qquad g_{k+1}(1)=(p_1p_{10}+p_0p_{01})p_{11}^{k-1},\\
&g_{k+1}(2)=p_1p_{11}^{k}, \qquad g_{k+1}(x)=0, \quad \textit{for} \quad x\ge 3,\\
&g_n(x)=(p_{00}+p_{11})[g_{n-1}(x)-p_{11}g_{n-2}(x-1)]+(p_{01}-p_{11})[g_{n-2}(x)-p_{11}g_{n-3}(x-1)]\\
&\qquad +p_{11}g_{n-1}(x-1)-p_{01}p_{10}p_{11}^{k-1}[g_{n-k-1}(x)-g_{n-k-1}(x-1)],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \quad n\ge k+2.
\end{aligned}$$

(b) I.i.d. trials

The double p.g.f. of N_n follows directly from relation (3.4) and it is given by

$$G(z,w)=\frac{1-wpz-(1-w)(pz)^k}{1-(1+wp)z+wpz^2+(1-w)qp^kz^{k+1}}. \tag{3.5}$$

In the following proposition we give a formula for the p.g.f. of N_n in terms of binomial coefficients. First, we give a lemma which is important for our derivations.

LEMMA 3.1. *For any integers n and m such that $n\le m$ it holds that*

$$\sum_{s=0}^{\min\{n,m-n\}}(-1)^s\binom{m-s}{s}\binom{m-2s}{n-s}=1.$$

PROOF. Using relation (12.15) on page 65 in Feller (1968), we have that

$$\begin{aligned}
&\sum_{s=0}^{\min\{n,m-n\}}(-1)^s\binom{m-s}{s}\binom{m-2s}{n-s}\\
&\quad=\sum_{s=0}^{\min\{n,m-n\}}(-1)^s\binom{n}{s}\binom{m-s}{n}=\binom{m-n}{m-n}=1,
\end{aligned}$$

which completes the proof of the lemma. □

PROPOSITION 3.3. *The p.g.f. $G_n(w)$ of the r.v. N_n is given by*

$$G_n(w) = \sum_{x=0}^{n-k+1} \sum_{r=0}^{\rho(0)} \sum_{s=0}^{\min\{x,n-kr-x\}} \sum_{y=\max\{0,h\}}^{\min\{x-s,r\}} \binom{r}{y} \binom{n-(k+1)r-2s}{x-s-y} \psi(0) w^x$$
$$- \sum_{x=1}^{n-k+1} \sum_{r=0}^{\rho(1)} \sum_{s=0}^{\min\{x-1,n-kr-x\}} \sum_{y=\max\{0,h\}}^{\min\{x-s-1,r\}} \binom{r}{y} \binom{n-1-(k+1)r-2s}{x-s-y-1} \psi(1) w^x$$
$$- \sum_{x=0}^{n-k+1} \sum_{r=0}^{\rho(k)} \sum_{s=0}^{\min\{x,n-k(r+1)-x+1\}} \sum_{y=\max\{0,h+k\}}^{\min\{x-s,r+1\}} p^k \binom{r+1}{y}$$
$$\cdot \binom{n-k-(k+1)r-2s}{x-s-y} \psi(k) w^x,$$

where

$$\psi(i) = (-1)^{s+r+y} \binom{s+r}{r} \binom{n-i-kr-s}{s+r} q^r p^{kr+x-y}, \qquad i = 0, 1, k,$$
$$\rho(0) = \min\left\{\left[\frac{n}{k+1}\right], \left[\frac{n-x}{k}\right]\right\}, \qquad \rho(1) = \min\left\{\left[\frac{n-1}{k+1}\right], \left[\frac{n-x}{k}\right]\right\},$$
$$\rho(k) = \min\left\{\left[\frac{n-k}{k+1}\right], \left[\frac{n-x-k+1}{k}\right]\right\}, \qquad h = x - [n-(k+1)r-s].$$

PROOF. Using relation (3.5) and the multinomial theorem we obtain that the p.g.f. $G_n(w)$ of the r.v. N_n is given by

$$G_n(w) = \varphi_0(w) - wp\varphi_1(z) - (1-w)p^k\varphi_k(w),$$

where for $i = 0, 1, k$, we have that

$$\varphi_i(w) = \sum_{n_1+2n_2+(k+1)n_3=n-i} \binom{n_1+n_2+n_3}{n_1, n_2, n_3} (-1)^{n_2+n_3}(1+wp)^{n_1}(wp)^{n_2}[(1-w)qp^k]^{n_3}.$$

Next, using the identity

$$w^s(1-w)^r(1+wp)^n = \sum_{x=s}^{s+r+n} \sum_{y=\max\{0,x-s-n\}}^{\min\{x-s,r\}} (-1)^y \binom{r}{y} \binom{n}{x-s-y} p^{x-s-y} w^x,$$

we get

$$G_n(w) = \sum_{r=0}^{u(0)} \sum_{s=0}^{\nu(0)} \sum_{x=s}^{n-kr-s} \sum_{y=\max\{0,h\}}^{\min\{x-s,r\}} \binom{r}{y} \binom{n-(k+1)r-2s}{x-s-y} \psi(0) w^x$$
$$- \sum_{r=0}^{u(1)} \sum_{s=0}^{\nu(1)} \sum_{x=s+1}^{n-kr-s} \sum_{y=\max\{0,h\}}^{\min\{x-s-1,r\}} \binom{r}{y} \binom{n-1-(k+1)r-2s}{x-s-y-1} \psi(1) w^x$$
$$- \sum_{r=0}^{u(k)} \sum_{s=0}^{\nu(k)} \sum_{x=s}^{n-k(r+1)-s+1} \sum_{y=\max\{0,h+k\}}^{\min\{x-s,r+1\}} p^k \binom{r+1}{y} \binom{n-k-(k+1)r-2s}{x-s-y} \psi(k) w^x,$$

where

$$u(i) = \left[\frac{n-i}{k+1}\right], \qquad \nu(i) = \left[\frac{n-i-(k+1)r}{2}\right], \qquad i = 0, 1, k.$$

Interchanging the order of summations we obtain

$$\begin{aligned}(3.6)\ G_n(w) = &\sum_{x=0}^{n}\sum_{r=0}^{\rho(0)}\sum_{s=0}^{\min\{x,n-kr-x\}}\sum_{y=\max\{0,h\}}^{\min\{x-s,r\}}\binom{r}{y}\binom{n-(k+1)r-2s}{x-s-y}\psi(0)w^x\\ &-\sum_{x=1}^{n}\sum_{r=0}^{\rho(1)}\sum_{s=0}^{\min\{x-1,n-kr-x\}}\sum_{y=\max\{0,h\}}^{\min\{x-s-1,r\}}\binom{r}{y}\binom{n-1-(k+1)r-2s}{x-s-y-1}\psi(1)w^x\\ &-\sum_{x=0}^{n-k+1}\sum_{r=0}^{\rho(k)}\sum_{s=0}^{\min\{x,n-k(r+1)-x+1\}}\sum_{y=\max\{0,h+k\}}^{\min\{x-s,r+1\}}p^k\binom{r+1}{y}\binom{n-k-(k+1)r-2s}{x-s-y}\psi(k)w^x.\end{aligned}$$

Relation (3.6) implies the existence of $\xi_1(w,x)$ and $\xi_2(w,x)$ such that $G_n(w)$ can be written in the form

$$G_n(w) = \sum_{x=0}^{n-k+1}\xi_1(w,x) + \sum_{x=n-k+2}^{n}\xi_2(w,x).$$

But for $n-k+2 \le x \le n$ we observe that $\left[\frac{n-x}{k}\right] = 0$, which implies that $\rho(0) = \rho(1) = 0$, and therefore

$$\begin{aligned}\sum_{x=n-k+2}^{n}\xi_2(w,x) = &\sum_{x=n-k+2}^{n}p^x w^x\sum_{s=0}^{\min\{x,n-x\}}(-1)^s\binom{n-s}{s}\binom{n-2s}{x-s}\\ &-\sum_{x=n-k+2}^{n}p^x w^x\sum_{s=0}^{\min\{x-1,n-x\}}(-1)^s\binom{n-1-s}{s}\binom{n-1-2s}{x-s-1}.\end{aligned}$$

Using Lemma 3.1 we get that $\sum_{x=n-k+2}^{n}\xi_2(w,x) = 0$, which clearly implies that the upper limits of x in relation (3.6) may be restricted to $x \le n-k+1$, and this completes the proof of the proposition. □

Proposition 3.3 reveals the following formula for the p.m.f. of N_n.

PROPOSITION 3.4. *For $0 < x < n-k+1$, the p.m.f. $g_n(x)$ of the r.v. N_n is given by*

$$\begin{aligned}g_n(x) = &\sum_{r=0}^{\rho(0)}\sum_{s=0}^{\min\{x,n-kr-x\}}\sum_{y=\max\{0,h\}}^{\min\{x-s,r\}}\binom{r}{y}\binom{n-(k+1)r-2s}{x-s-y}\psi(0)\\ &-\sum_{r=0}^{\rho(1)}\sum_{s=0}^{\min\{x-1,n-kr-x\}}\sum_{y=\max\{0,h\}}^{\min\{x-s-1,r\}}\binom{r}{y}\binom{n-1-(k+1)r-2s}{x-s-y-1}\psi(1)\\ &-\sum_{r=0}^{\rho(k)}\sum_{s=0}^{\min\{x,n-k(r+1)-x+1\}}\sum_{y=\max\{0,h+k\}}^{\min\{x-s,r+1\}}p^k\binom{r+1}{y}\end{aligned}$$

$$\cdot \binom{n-k-(k+1)r-2s}{x-s-y} \psi(k),$$

and

$$g_n(0) = \sum_{r=0}^{[n/(k+1)]} (-1)^r \binom{n-kr}{r} q^r p^{kr} - \sum_{r=0}^{[(n-k)/(k+1)]} (-1)^r \binom{n-k(r+1)}{r} q^r p^{k(r+1)},$$

$$g_n(n-k+1) = p^n,$$

where h, $\rho(i)$ *and* $\psi(i)$ $(i = 0, 1, k)$ *are as in Proposition* 3.3.

PROOF. Using Proposition 3.3 we may easily derive $g_n(x)$ for $0 \le x < n-k+1$, by collecting the coefficients of w^x in $G_n(w)$. For $x = n-k+1$, we observe that

$$g_n(n-k+1) = p^{n-k+1} \left\{ \sum_{s=0}^{\min\{n-k+1,k-1\}} (-1)^s \binom{n-s}{s} \binom{n-2s}{n-k+1-s} - \sum_{s=0}^{\min\{n-k,k-1\}} (-1)^s \binom{n-1-s}{s} \binom{n-1-2s}{n-k-s} \right\} + p^n.$$

Lemma 3.1 implies that the coefficient of p^{n-k+1} in the above equality is zero, and this completes the proof of the proposition. □

Alternative formulae for $g_n(x)$ have been obtained by Godbole (1992) and Charalambides (1997). In the special case of i.i.d. trials, the results of Theorems 3.8 and 3.9 reduce to the ones derived by Chryssaphinou *et al.* (1993).

4. Evaluation of moments

Let $\mu_{n,[m]} = E[N_n(N_n-1)\cdots(N_n-m+1)]$ and $\mu_{n,m} = E[(N_n)^m]$ denote, respectively, the m-th factorial moment and the m-th order moment about zero $(m \ge 1)$ of the r.v. N_n, with $\mu_{n,[0]} = \mu_{n,0} = 1$. In this section we establish simple recursive formulae for the evaluation of $\mu_{n,[m]}$ and $\mu_{n,m}$ via the p.g.fs derived in Section 3. Also, several new results regarding the case of i.i.d. trials are also established.

4.1 *Non-overlapping scheme*

(a) Markov dependent trials

For $n \ge k+2$ and for any integer $m \ge 1$, Theorem 3.1 implies that

$$\begin{aligned} G_n^{(m)}(w) &= a_1 G_{n-1}^{(m)}(w) + a_2 G_{n-2}^{(m)}(w) + p_{11}^k [w G_{n-k}^{(m)}(w) + m G_{n-k}^{(m-1)}(w)] \\ &\quad + a_4 G_{n-k-1}^{(m)}(w) + m p_{11}^{k-1}(p_{01}p_{10} - a_1 p_{11}) G_{n-k-1}^{(m-1)}(w) \\ &\quad - a_2 p_{11}^k [w G_{n-k-2}^{(m)}(w) + m G_{n-k-2}^{(m-1)}(w)], \end{aligned}$$

where $G_n^{(s)}(w)$ denotes the s-th order derivative of $G_n(w)$ at w. Evaluating also the m-th order derivative of $G_n(w)$, $0 \le n \le k+1$, on both sides of the initial relations of Theorem 3.1 and then setting $w = 1$, we get the following theorem.

THEOREM 4.1. *The m-th factorial moment $\mu_{n,[m]}$, $m \geq 1$, of the r.v. N_n satisfies the following recursive scheme:*

$$\begin{aligned}
&\mu_{n,[m]} = 0, \qquad \textit{for} \quad 0 \leq n < k, \\
&\mu_{k,[1]} = p_1 p_{11}^{k-1}, \qquad \mu_{k,[m]} = 0, \quad \textit{for } m \geq 2, \\
&\mu_{k+1,[1]} = (p_1 + p_0 p_{01}) p_{11}^{k-1}, \qquad \mu_{k+1,[m]} = 0, \quad \textit{for } m \geq 2, \\
&\mu_{n,[m]} = a_1 \mu_{n-1,[m]} + a_2 \mu_{n-2,[m]} + p_{11}^k (\mu_{n-k,[m]} + m \mu_{n-k,[m-1]}) \\
&\qquad - a_1 p_{11}^k \mu_{n-k-1,[m]} + m p_{11}^k (p_{01} p_{10} - a_1 p_{11}) \mu_{n-k-1,[m-1]} \\
&\qquad - a_2 p_{11}^k (\mu_{n-k-2,[m]} + m \mu_{n-k-2,[m-1]}), \qquad \textit{for} \quad n \geq k+2.
\end{aligned}$$

Let $M_n(w)$ be the m.g.f. of the r.v. N_n. Since $M_n(w) = G_n(e^w)$, it follows from Theorem 3.1 that for $n \geq k+2$ and for any $m \geq 1$, the m-th order derivative of $M_n(w)$ at w satisfies the recursive relation

$$\begin{aligned}
M_n^{(m)}(w) = {} & a_1 M_{n-1}^{(m)}(w) + a_2 M_{n-2}^{(m)}(w) + e^w p_{11}^k \sum_{j=0}^{m} \binom{m}{j} M_{n-k}^{(m-j)}(w) \\
& -[(1-e^w) p_{01} p_{10} p_{11}^{k-1} + a_1 e^w p_{11}^k] M_{n-k-1}^{(m)}(w) \\
& + e^w p_{11}^{k-1} (p_{01} p_{10} - a_1 p_{11} \sum_{j=1}^{m} \binom{m}{j} M_{n-k-1}^{(m-j)}(w) \\
& - a_2 e^w p_{11}^k \sum_{j=0}^{m} \binom{m}{j} M_{n-k-2}^{(m-j)}(w).
\end{aligned}$$

Using the above relation and taking also the m-th order derivative of $M_n(w)$, $0 \leq n \leq k+1$, on both sides of the initial relations of Theorem 3.1 (with w replaced by e^w), we obtain the following theorem.

THEOREM 4.2. *The m-th order moment $\mu_{n,m}$, $m \geq 1$, of the r.v. N_n satisfies the following recursive scheme:*

$$\begin{aligned}
&\mu_{n,m} = 0, \qquad \textit{for} \quad 0 \leq n < k, \\
&\mu_{k,m} = p_1 p_{11}^{k-1}, \qquad \mu_{k+1,m} = (p_1 + p_0 p_{01}) p_{11}^{k-1}, \\
&\mu_{n,m} = a_1 \mu_{n-1,m} + a_2 \mu_{n-2,m} - a_1 p_{11}^k \mu_{n-k-1,m} \\
&\qquad + p_{11}^k \sum_{j=0}^{m} \binom{m}{j} [\mu_{n-k,m-j} - a_2 \mu_{n-k-2,m-j}] \\
&\qquad + p_{11}^{k-1} (p_{01} p_{10} - a_1 p_{11}) \sum_{j=1}^{m} \binom{m}{j} \mu_{n-k-1,m-j}, \qquad \textit{for} \quad n \geq k+2.
\end{aligned}$$

PROPOSITION 4.1. *The mean μ_n of the r.v. N_n satisfies the following recursive scheme:*

$$\begin{aligned}
&\mu_n = 0, \qquad \textit{for} \quad 0 \leq n < k, \;\; \mu_k = p_1 p_{11}^{k-1}, \\
&\mu_n = (1 - p_{00} - p_{11}) \left(p_{11}^{k-1} \mu_{n-k-1} - \mu_{n-1} \right) \\
&\qquad + p_{11}^k \mu_{n-k} + [p_1 p_{10} + p_{01} + (n-k-1) p_{01} p_{10}] p_{11}^{k-1}, \qquad \textit{for} \quad n \geq k+1.
\end{aligned}$$

PROOF. Theorem 4.1 (and/or Theorem 4.2) for $m = 1$ and $n \geq k+2$, implies that

$$\mu_n - \mu_{n-1} = a_2(\mu_{n-2} - \mu_{n-1}) + p_{11}^k(\mu_{n-k} - \mu_{n-k-1}) + a_2 p_{11}^k(\mu_{n-k-1} - \mu_{n-k-2}) + p_{01}p_{10}p_{11}^{k-1},$$

from which by setting $n = k+2, k+3, \ldots$, and then adding all the resulting relations we get

$$\mu_n = \mu_{k+1} - a_2(\mu_{n-1} - \mu_{n-k}) + p_{11}^k \mu_{n-k} + a_2 p_{11}^k \mu_{n-k-1} + (n-k-1)p_{01}p_{10}p_{11}^{k-1}.$$

Using the initial conditions $\mu_n = 0$, for $0 \leq n < k$, $\mu_k = p_1 p_{11}^{k-1}$ and $\mu_{k+1} = (p_1 + p_0 p_{01})p_{11}^{k-1}$, we easily arrive at the required recursion by observing that it is also valid for $n = k+1$. □

(b) I.i.d. trials

Theorems 4.1 and 4.2 provide recursive schemes for the evaluation of the m-th factorial and the m-th order moments of the r.v. N_n by replacing p_1, p_{01} and p_{11} by p. Proposition 4.1 reduces to the following recursive scheme regarding the mean μ_n of the r.v. N_n

$$\mu_n = 0, \quad \text{for} \quad 0 \leq n < k, \quad \mu_n = p^k[\mu_{n-k} + 1 + (n-k)q], \text{ for } n \geq k,$$

which easily leads to the exact relation for μ_n, given in Proposition 2.4 by Aki and Hirano (1988).

In the following theorem we derive an exact formula for $\mu_{n,[m]}$.

THEOREM 4.3. *The m-th factorial moment $\mu_{n,[m]}$, $m \geq 1$, of the r.v. N_n is given by*

$$\mu_{n,[m]} = 0, \quad \textit{for} \quad 0 \leq n < km$$

$$\mu_{n,[m]} = m! p^{km} \sum_{j=0}^{\min\{n-km,m\}} (-1)^j \binom{m}{j} p^j d(n-km-j, m), \quad \textit{for} \quad n \geq km,$$

where

$$d(n,m) = \sum_{i=0}^{[n/k]} \binom{m+i-1}{i} \binom{m+n-ki}{m} p^{ki}.$$

PROOF. Let

$$f_n(w) = \sum_{m=0}^{\infty} \mu_{n,[m]} \frac{w^m}{m!} \quad \text{and} \quad f(z,w) = \sum_{n=0}^{\infty} f_n(w) z^n.$$

Since $f_n(w) = G_n(w+1)$, it follows that $f(z,w) = G(z, w+1)$, and thus from relation (3.2) we get

$$\begin{aligned} f(z,w) &= \frac{1 - p^k z^k}{(1-z)(1-p^k z^k) - w(1-pz)p^k z^k} \\ &= \frac{1}{1-z} \cdot \left[1 + \frac{(1-pz)p^k z^k w}{(1-z)(1-p^k z^k) - w(1-pz)p^k z^k}\right] \\ &= \frac{1}{1-z} + \frac{(1-pz)p^k z^k w}{(1-z)^2(1-p^k z^k)} \left[1 - \frac{(1-pz)p^k z^k w}{(1-z)(1-p^k z^k)}\right]^{-1}. \end{aligned}$$

Expanding the RHS of the above equality in powers of w, we get

$$\begin{aligned} f(z,w) &= \frac{1}{1-z} + \frac{(1-pz)p^k z^k}{(1-z)^2(1-p^k z^k)} \sum_{m=0}^{\infty} \left[\frac{(1-pz)p^k z^k}{(1-z)(1-p^k z^k)}\right]^m w^{m+1} \\ &= \frac{1}{1-z} + \sum_{m=1}^{\infty} \frac{(1-pz)^m p^{km} z^{km}}{(1-z)^{m+1}(1-p^k z^k)^m} w^m. \end{aligned}$$

Consider now the following generating function of the moments $\mu_{n,[m]}$

$$M_{[m]}(z) = \sum_{n=0}^{\infty} \mu_{n,[m]} z^n = \left[\frac{\partial^m G(z,w)}{\partial w^m}\right]_{w=1}.$$

We have that

$$f(z,w) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mu_{n,[m]} \frac{w^m}{m!} z^n = \sum_{m=0}^{\infty} M_{[m]}(z) \frac{w^m}{m!},$$

and hence the aforementioned expansion of $f(z,w)$ yields that

$$M_{[0]}(z) = (1-z)^{-1} \quad \text{and} \quad M_{[m]}(z) = \frac{m!(1-pz)^m p^{km} z^{km}}{(1-z)^{m+1}(1-p^k z^k)^m}, \quad m \geq 1.$$

Since

$$(1-z)^{-(m+1)} = \sum_{i=0}^{\infty} \binom{m+i}{i} z^i \quad \text{and} \quad (1-p^k z^k)^{-m} = \sum_{i=0}^{\infty} \binom{m+i-1}{i} p^{ki} z^{ki},$$

it is true that

$$(1-z)^{-(m+1)}(1-p^k z^k)^{-m} = \sum_{n=0}^{\infty} d(n,m) z^n.$$

Therefore, the expression for $M_{[m]}(z)$, $m \geq 1$, becomes

$$\begin{aligned} M_{[m]}(z) &= m! p^{km} \sum_{j=0}^{m}\sum_{n=0}^{\infty} (-1)^j \binom{m}{j} p^j d(n,m) z^{n+km+j} \\ &= m! p^{km} \sum_{j=0}^{m}\sum_{n=km+j}^{\infty} (-1)^j \binom{m}{j} p^j d(n-km-j,m) z^n. \end{aligned}$$

The proof of the theorem then follows by collecting the coefficient of z^n in the RHS of the last equality. □

Setting $d(n,m) = 0$ for all $n < 0$, we have that Theorem 4.3 provides the following exact formula for the mean μ_n, $n \geq 0$,

$$\mu_n = p^k \left\{ \sum_{j=0}^{[(n-k)/k]} [n+1-k(j+1)] p^{kj} - \sum_{j=0}^{[(n-k-1)/k]} [n-k(j+1)] p^{kj+1} \right\}.$$

4.2 *At least scheme*

(a) Markov dependent trials

Using Theorem 3.6 in order to evaluate the m-th order derivative of $G_n(w)$ at $w=1$, we led to the following theorem.

THEOREM 4.4. *The m-th factorial moment $\mu_{n,[m]}$, $m \geq 1$, of the r.v. N_n satisfies the following recursive scheme:*

$$\mu_{n,[m]} = 0, \quad \textit{for} \quad 0 \leq n < k,$$
$$\mu_{k,[1]} = p_1 p_{11}^{k-1}, \qquad \mu_{k,[m]} = 0, \quad \textit{for } m \geq 2,$$
$$\mu_{k+1,[1]} = (p_1 + p_0 p_{01}) p_{11}^{k-1}, \qquad \mu_{k+1,[m]} = 0, \quad \textit{for} \quad m \geq 2,$$
$$\mu_{n,[m]} = b_1 \mu_{n-1,[m]} + b_2 \mu_{n-2,[m]} + m p_{01} p_{10} p_{11}^{k-1} \mu_{n-k-1,[m-1]}, \qquad \textit{for} \quad n \geq k+2.$$

Now, let $M_n(w)$ be the m.g.f of the r.v. N_n. Using Theorem 3.6 along with the fact that $M_n(w) = G_n(e^w)$, and evaluating the m-th order derivative of $M_n(w)$ at $w=0$ we obtain the following theorem.

THEOREM 4.5. *The m-th order moment $\mu_{n,m}$, $m \geq 1$, of the r.v. N_n satisfies the following recursive scheme:*

$$\mu_{n,m} = 0, \quad \textit{for} \quad 0 \leq n < k,$$
$$\mu_{k,m} = p_1 p_{11}^{k-1}, \qquad \mu_{k+1,m} = (p_{11} + p_0 p_{01}) p_{11}^{k-1},$$
$$\mu_{n,m} = b_1 \mu_{n-1,m} + b_2 \mu_{n-2,m} + p_{01} p_{10} p_{11}^{k-1} \sum_{j=1}^{m} \binom{m}{j} \mu_{n-k-1,m-j}, \qquad \textit{for} \quad n \geq k+2.$$

PROPOSITION 4.2. *The mean μ_n of the r.v. N_n is given by*

$$\mu_n = 0, \quad \textit{for} \quad 0 \leq n < k,$$
$$\mu_n = p_{11}^{k-1} \left\{ p_1 (p_{11} - p_{01})^{n-k} + \frac{p_1 p_{10} + p_{01}}{p_{10} + p_{01}} [1 - (p_{11} - p_{01})^{n-k}] \right.$$
$$\left. + \frac{p_{01} p_{10}}{(p_{10} + p_{01})^2} \left[(n-k-1) - (n-k)(p_{11} - p_{01}) + (p_{11} - p_{01})^{n-k} \right] \right\},$$
$$\textit{for} \quad n \geq k.$$

PROOF. Using Theorem 4.4 (and/or Theorem 4.5) and working as in the proof of Proposition 4.1 we obtain that

$$(4.1) \quad \begin{cases} \mu_n = 0, \quad \text{for} \quad 0 \leq n < k, \quad \mu_k = p_1 p_{11}^{k-1}, \\ \mu_n = -b_2 \mu_{n-1} + p_{11}^{k-1} [p_1 p_{10} + p_{01} + (n-k-1) p_{01} p_{10}], \qquad \text{for} \quad n \geq k+1. \end{cases}$$

It can be shown by induction that for $n \geq k+1$ it is true that

$$\mu_n = (p_{11} - p_{01})^{n-k} \mu_k + \frac{p_{11}^{k-1}(p_1 p_{10} + p_{01})}{1 - (p_{11} - p_{01})} [1 - (p_{11} - p_{01})^{n-k}]$$
$$+ p_{11}^{k-1} p_{01} p_{10} \sum_{i=0}^{n-k-2} (n-k-1-i)(p_{11} - p_{01})^i.$$

Now, using the identity

$$\sum_{i=0}^{r}(r-i+1)x^i = \frac{(r+1)-(r+2)x+x^{r+2}}{(1-x)^2},$$

the proof of the proposition may be easily established. □

A similar formula for μ_n has been obtained by Koutras (1997a). He also derived a recurrence relation for the evaluation of μ_n but the one given above in (4.1) is simpler.

(b) I.i.d. trials

Proposition 4.2 reduces to the following exact formula for the mean μ_n of the r.v. N_n

$$\mu_n = 0, \quad \text{for} \quad 0 \le n < k, \;\; \mu_n = p^k[1+(n-k)q], \text{ for } n \ge k,$$

which has also been obtained by Goldstein (1990) (see, also Hirano and Aki (1993) and Koutras (1997a)).

In the following proposition we obtain a simple exact formula for the evaluation of $\mu_{n,[m]}$.

PROPOSITION 4.3. *The m-th factorial moment $\mu_{n,[m]}$, $m \ge 1$, of the r.v. N_n is given by*

$$\mu_{n,[m]} = m! q^{m-1} p^{km}\left\{\binom{n-km+1}{m} - p\binom{n-km}{m}\right\}.$$

PROOF. The double p.g.f. $G(z,w)$ of the r.v. N_n follows from relation (3.3), and it is given by

$$G(z,w) = \frac{1-(1-w)(pz)^k}{1-z+(1-w)qp^k z^{k+1}}.$$

Expanding $G(z,w)$ in powers of z we obtain that

$$\begin{aligned}(4.2)\qquad G_n(w) = &\sum_{r=0}^{[n/(k+1)]}(qp^k)^r\binom{n-kr}{r}(w-1)^r \\ &+ \sum_{r=0}^{[(n-k)/(k+1)]} p^k(qp^k)^r\binom{n-k(r+1)}{r}(w-1)^{r+1}.\end{aligned}$$

The proof of the proposition then follows by evaluating the m-th order derivative of $G_n(w)$ at $w=1$. □

Using relation (4.2) along with the fact that $M_n(w) = G_n(e^w)$, and evaluating the m-th order derivative of $M_n(w)$ at $w=0$, we obtain the following proposition.

PROPOSITION 4.4. *The m-th order moment $\mu_{n,m}$, $m \ge 1$, of the r.v. N_n is given by*

$$\mu_{n,m} = \sum_{r=1}^{m} r! q^{r-1} p^{kr}\binom{m-1}{r-1}\left\{\binom{n-kr+1}{r} - p\binom{n-kr}{r}\right\}.$$

4.3 *Overlapping scheme*

(a) Markov dependent trials

Using Theorem 3.8, and evaluating the m-th order derivative of $G_n(w)$ at $w = 1$, we obtain the following theorem.

THEOREM 4.6. *The m-th factorial moment $\mu_{n,[m]}$, $m \geq 1$, of the r.v. N_n satisfies the following recursive scheme:*

$$\begin{aligned}
&\mu_{n,[m]} = 0, \quad \text{for} \quad 0 \leq n < k,\\
&\mu_{k,[1]} = p_1 p_{11}^{k-1}, \qquad \mu_{k,[m]} = 0, \quad \text{for } m \geq 2,\\
&\mu_{k+1,[1]} = [p_1(1+p_{11}) + p_0 p_{01}]p_{11}^{k-1}, \qquad \mu_{k+1,[2]} = 2p_1 p_{11}^{k}, \qquad \mu_{k+1,[m]} = 0, \ \text{for } m \geq 3,\\
&\mu_{n,[m]} = (2p_{11} + p_{00})\mu_{n-1,[m]} + [p_{01} - p_{11}(1 + p_{00} + p_{11})]\mu_{n-2,[m]}\\
&\qquad + p_{11}(p_{11} - p_{01})\mu_{n-3,[m]} + mp_{11}\mu_{n-1,[m-1]} - mp_{11}(p_{00} + p_{11})\mu_{n-2,[m-1]}\\
&\qquad + mp_{11}(p_{11} - p_{01})\mu_{n-3,[m-1]} + mp_{01}p_{10}p_{11}^{k-1}\mu_{n-k-1,[m-1]}, \qquad \text{for} \quad n \geq k+2.
\end{aligned}$$

The m-th order moments $\mu_{n,m}$, of the r.v. N_n can be obtained on manipulating over Theorem's 3.8 outcome. More specifically we have the next.

THEOREM 4.7. *The m-th order moment $\mu_{n,m}$, $m \geq 1$, of the r.v. N_n satisfies the following recursive scheme:*

$$\begin{aligned}
&\mu_{n,m} = 0, \quad \text{for} \quad 0 \leq n < k,\\
&\mu_{k,m} = p_1 p_{11}^{k-1}, \qquad \mu_{k+1,m} = [p_1(p_{10} + 2^m p_{11}) + p_0 p_{01}]p_{11}^{k-1},\\
&\mu_{n,m} = p_{11}\sum_{j=1}^{m}\binom{m}{j}[\mu_{n-1,m-j} - (p_{00} + p_{11})\mu_{n-2,m-j} + p_{01}p_{10}p_{11}^{k-2}\mu_{n-k-1,m-j}]\\
&\qquad + p_{11}(p_{00} + p_{11})\sum_{j=0}^{m}\binom{m}{j}\mu_{n-3,m-j} + (2p_{11} + p_{00})\mu_{n-1,m}\\
&\qquad + [p_{00} - p_{11}(1 + p_{00} + p_{11})]\mu_{n-2,m}, \qquad \text{for} \quad n \geq k+2.
\end{aligned}$$

Using Theorem 4.6 (and/or Theorem 4.7) and working as in the proof of Proposition 4.1 we obtain the following Proposition.

PROPOSITION 4.5. *The mean μ_n of the r.v. N_n satisfies the following recursive scheme*

$$\begin{aligned}
&\mu_n = 0, \quad \text{for} \quad 0 \leq n < k, \ \mu_k = p_1 p_{11}^{k-1},\\
&\mu_n = (2p_{11} - p_{01})\mu_{n-1} + p_{11}(p_{01} - p_{11})\mu_{n-2}\\
&\qquad + p_{11}^{k-1}[p_1 p_{10} + p_{01} + (n-k-1)p_{01}p_{10}], \qquad \text{for} \quad n \geq k+1.
\end{aligned}$$

We mention that Koutras (1997a) obtained an exact formula for the evaluation of μ_n.

(b) I.i.d. trials

Theorems 4.6 and 4.7 provide recursive schemes for the evaluation of the m-th factorial and the m-th order moments of the r.v. N_n by replacing p_1, p_{01}, p_{11} by p.

Proposition 4.5 reduces to the following recursive scheme for the evaluation of the mean μ_n of the r.v. N_n

$$\mu_n = 0, \quad \text{for} \quad 0 \leq n < k, \; \mu_n = p\mu_{n-1} + p^k[1 + (n-k)q], \; \text{for } n \geq k.$$

Furthermore, following the methodology of the proof of Theorem 4.3 we may obtain an exact formula for the evaluation of $\mu_{n,[m]}$, which is

$$\mu_{n,[m]} = m! \sum_{i=1}^{s} \binom{m-1}{i-1} \sum_{j=m}^{n-ki+1} \binom{j-2}{j-m} \binom{n-ki+i-j+1}{i} p^{ki-i+j} q^{i-1}, \quad m \geq 1, \quad n \geq k,$$

where $s = \min\{m, [(n-m+1)/k]\}$. An explicit proof of the above result may be found in Charalambides (1997).

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