Ann. Inst. Statist. Math.
Vol. 53, No. 3, 576–598 (2001)

DISTRIBUTIONS OF THE NUMBERS OF FAILURES AND SUCCESSES IN A WAITING TIME PROBLEM

STATHIS CHADJICONSTANTINIDIS AND MARKOS V. KOUTRAS

Department of Statistics and Insurance Science, University of Piraeus, Piraeus 18534, Greece

(Received February 8, 1999; revised July 8, 1999)

Abstract. Abstract. In this article we consider infinite sequences of Bernoulli trials and study the exact and asymptotic distribution of the number of failures and the number of successes observed before the r-th appearance of a pair of successes separated by a pre-specified number of failures. Several formulae are provided for the probability mass function, probability generating function and moments of the distribution along with some asymptotic results and a Poisson limit theorem. A number of interesting applications in various areas of applied science are also discussed.

Key words and phrases: Waiting times, scan statistics, success runs, unimodality, partial fraction expansions, Poisson limit theorems.

1. Introduction

The motivation of the present paper stems from several areas of applied science such as actuarial science, educational psychology, quality control, engineering etc. Many problems encountered in these areas can be described through dichotomous (binary) variables $X_1, X_2, \ldots$ taking on the values 1 (success, S) or 0 (failure,F) and the interest focuses on random variables related to the time a predetermined criterion is satisfied (stopping rule).

As an example, let us consider the following model which is of special importance for educational transfer and learning studies. Assume that a subject is asked to try finishing several tasks one after the other and each time he manages to complete successfully two tasks separated by at most two unsuccessful trials (i.e. he achieves an outcome of the form SS, SFS or $SFFS$) he gains one point. The test terminates when the total number of points collected by the subject reaches a predetermined level r. Two variables that are significant in such a model are the number of successfully and unsuccessfully completed tasks by the end of the test. Thus, if we denote these variables by $Y^{(1)}$, $Y^{(0)}$ respectively and assume that the subject is awarded a mark a, $-b$ for a successful, unsuccessful task the total score achieved at the time of termination of the test equals $aY^{(1)} + (-b)Y^{(0)}$.

The aforementioned model can be accommodated in the following general set-up which will be the subject of this paper. Let $X_1, X_2 \ldots$ be an infinite sequence of binary outcomes (success, S-failure, F) and denote by $T_{k,r}$ the waiting time for the r-th occurrence of two successes which lie at most k places apart from each other, i.e. they are separated by at most $k-2$ failures. Clearly, $T_{k,1}$ counts the number of trials required to observe for the first time one of the patterns

$$SS, SFS, \dots, S\overbrace{F\dots F}^{k-2}S; \tag{1.1}$$

these patterns will be refered hereafter as "detection patterns". It should be stressed that the enumeration scheme employed here is the non-overlapping one, i.e. once a detection pattern is registered the sequence of outcomes ending up with this pattern is disregarded and we start our search for a new pattern from scratch. In the present paper we shall proceed to a systematic study of the number $Y_{k,r}^{(0)}$ of failures and $Y_{k,r}^{(1)}$ of successes appearing among $X_1, X_2, \dots, X_{T_{k,r}}$. To make the above definitions transparent to the reader we mention by way of example that, for the sequence of outcomes $SFFFSFSSSFFSFS$ we have

$$T_{3,1} = 7, \quad Y_{3,1}^{(0)} = 4, \quad Y_{3,1}^{(1)} = 3, \; T_{3,2} = 9, \; Y_{3,2}^{(0)} = 4, \; Y_{3,2}^{(1)} = 5,$$
$$T_{3,3} = 14, \quad Y_{3,3}^{(0)} = 7, \quad Y_{3,3}^{(1)} = 7.$$

For $r = 1$, $k \geq 2$ the random variable $T_{k,1}$ is a special case of a *scan statistic*, see e.g. Glaz (1983, 1989), Glaz and Naus (1991), Greenberg (1970), Saperstein (1973) and Chen and Glaz (1997) for a review. For $r > 1$, $k \geq 2$, since we are looking at multiple occurrences of a scan statistic we could make use of the term *multiple scan statistic*.

Note that, for $k = 2$ we are in fact enumerating success runs of length $k = 2$ and $T_{2,1}$, $T_{2,r}$ follow a *geometric distribution of order* k and a *negative binomial distribution of order* k respectively; the interested reader may consult the upcoming book by Balakrishnan and Koutras (2000) for a lucid and elaborate account of developments relating to waiting times for runs.

The distribution of the number of failures and successes until the first occurrence of a success run of length k was recently studied by Aki and Hirano (1994) and Balakrishnan (1997). Similar problems have been addressed by Antzoulakos and Philippou (1996) for the more general case where the stopping rule is associated with the r-th occurrence of success runs (non-overlapping and overlapping schemes).

In the present paper we conduct a systematic study of the distribution of numbers of successes and failures until the r-th occurrence of a detection pattern. Section 2 introduces the necessary definitions and notations. In Section 3 we consider the case $r = 1$ (first appearance of a detection pattern) and establish recurrence relations and non-recursive formulae for the probability mass function of the number of successes and failures until $T_{k,r}$. Recursive schemes for the moments and explicit expressions for the probability generating functions are also provided. Section 4 deals with the general case ($r \geq 1$) and presents recurrence schemes and alternative expressions through appropriately defined sequences of numbers and polynomials. In Section 5 we accomplish several asymptotic results, whereas Section 6 discusses in brief several interesting applications in actuarial science, reliability engineering and sampling inspection.

2. Definitions and notations

Let X_1, $X_2, \dots$ be a sequence of (independent and identical) Bernoulli trials with success probabilities $p = \Pr(X_i = 1)$, and failure probabilities $q = \Pr(X_i = 0)$ ($p + q = 1, 0 < p < 1$). If $k \geq 2$, $r \geq 1$ are two positive integers, we shall denote by $T_{k,r}$ the waiting time for the r-th occurrence of "two successes which lie at most k places apart", (i.e. two successes separated by at most $k-2$ failures) by $Y_{k,r}^{(0)}$ the number of failures among $X_1, X_2, \ldots, X_{T_{k,r}}$, and $Y_{k,r}^{(1)}$ the number of successes among $X_1, X_2, \ldots, X_{T_{k,r}}$.

For $a \in \{0,1\}$, let

$$f_{k,r}^{(a)}(m,n) = \Pr(Y_{k,r}^{(a)} = m, T_{k,r} = n), \qquad \Phi_{k,r}^{(a)}(z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{k,r}^{(a)}(m,n) z^m t^n$$

denote the joint probability mass function and the joint probability generating function of $(Y_{k,r}^{(a)}, T_{k,r})$; moreover, let

$$g_{k,r}^{(a)}(m) = \Pr(Y_{k,r}^{(a)} = m), \qquad G_{k,r}^{(a)}(z) = E(z^{Y_{k,r}^{(a)}}) = \sum_{m=0}^{\infty} g_{k,r}^{(a)}(m) z^m$$

be the marginal probability mass function and probability generating function of $Y_{k,r}^{(a)}$.

When no confusion is likely to arise we shall suppress the indices k, r using T, $Y^{(a)}$, $f^{(a)}$, $\Phi^{(a)}$, $g^{(a)}$, $G^{(a)}$ instead of $T_{k,1}$, $Y_{k,1}^{(a)}$, $f_{k,1}^{(a)}$, $\Phi_{k,1}^{(a)}$, $g_{k,1}^{(a)}$, $G_{k,1}^{(a)}$ and T_r, $Y_r^{(a)}$, f_r, $\Phi_r^{(a)}$, $g_r^{(a)}$, $G_r^{(a)}$ instead of $T_{k,r}$, $Y_{k,r}^{(a)}$, $f_{k,r}^{(a)}$, $\Phi_{k,r}^{(a)}$, $g_{k,r}^{(a)}$, $G_{k,r}^{(a)}$, respectively.

3. Distributions of the numbers of successes and failures until the first appearance of a detection pattern

3.1 *Number of failures*

In this subsection we study the number $Y^{(0)}$ of failures until the first appearance of a detection pattern. The exact distribution of $Y^{(0)}$ is examined in some detail and several interesting properties of it are given.

Let us start with the development of an effective recursive scheme for the evaluation of the joint probability mass function of $(Y^{(0)}, T)$.

THEOREM 3.1. *The joint probability mass function $f^{(0)}(m,n) = \Pr(Y^{(0)} = m, T = n)$ of the (bivariate) random variable $(Y^{(0)}, T)$, satisfies the recurrence relation*

$$f^{(0)}(m,n) = qf^{(0)}(m-1,n-1) + pq^{k-1}f^{(0)}(m-k+1,n-k), \qquad n > k,\ m \geq k-1 \tag{3.1}$$

with initial conditions

$$f^{(0)}(m,n) = \begin{cases} (n-1)p^2q^m, & 2 \leq n \leq k, \quad m = n-2 \\ 0, & \textit{otherwise}. \end{cases} \tag{3.2}$$

PROOF. The derivation of the initial condition is straightforward. Assume next that $n > k$, $m \geq k-1$. Manifestly

$$f^{(0)}(m,n) = \Pr(Y^{(0)} = m, T = n, X_1 = 0) + \Pr(Y^{(0)} = m, T = n, X_1 = 1)$$

with the first term in the RHS taking on the form

$$\begin{aligned} \Pr(Y^{(0)} = m, T = n, X_1 = 0) &= \Pr(X_1 = 0)\Pr(Y^{(0)} = m, T = n \mid X_1 = 0) \\ &= qf^{(0)}(m-1,n-1). \end{aligned}$$

In order to evaluate the second summand of the RHS observe that the event $\{Y^{(0)} = m, T = n, X_1 = 1\}$ can be alternatively expressed as $\{Y^{(0)} = m, T = n\} \cap A$, where $A = \{X_1 = 1 \text{ and } X_i = 0 \text{ for all } i = 2, 3, \ldots, k\}$, and therefore

$$\begin{aligned}\Pr(Y^{(0)} = m, T = n, X_1 = 1) &= \Pr(A)\Pr(Y^{(0)} = m, T = n \mid A)\\ &= pq^{k-1} f^{(0)}(m-k+1, n-k).\end{aligned}$$

In the special case $k = 2$, formulae (3.1) and (3.2) reduce to

$$f^{(0)}(m,n) = qf^{(0)}(m-1,n-1) + pqf^{(0)}(m-1,n-2), \qquad n > 2, \quad m \geq 1,$$

with initial conditions

$$f^{(0)}(m,n) = \begin{cases} p^2, & n = 2,\ m = 0 \\ 0, & \text{otherwise} \end{cases}.$$

These are in concordance with a special case of formula (2.1) in Aki and Hirano (1994).

The evaluation of the joint probability generating function of $(Y^{(0)}, T)$ can be easily established by exploiting Theorem 3.1. More specifically, we have the next.

THEOREM 3.2. *The joint probability generating function of* $(Y^{(0)}, T)$

$$\Phi^{(0)}(z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f^{(0)}(m,n) z^m t^n$$

is given by

$$\Phi^{(0)}(z,t) = \frac{p^2t^2A(zt)}{1 - qzt - pq^{k-1}z^{k-1}t^k}, \qquad |z| \leq 1, \quad |t| \leq 1 \tag{3.3}$$

where

$$A(x) = \sum_{i=0}^{k-2}(qx)^i = \frac{1-(qx)^{k-1}}{1-qx}.$$

PROOF. For $2 \leq n \leq k$, $m = n-2$, the initial condition (3.2) is equivalent to

$$f^{(0)}(m,n) = qf^{(0)}(m-1,n-1) + p^2q^m. \tag{3.4}$$

Writing $\Phi^{(0)}(z,t)$ as

$$\Phi^{(0)}(z,t) = \sum_{m=0}^{\infty}\sum_{n=2}^{k} f^{(0)}(m,n)z^mt^n + \sum_{m=0}^{\infty}\sum_{n=k+1}^{\infty} f^{(0)}(m,n)z^mt^n$$

and replacing $f^{(0)}(m,n)$ by the aid of (3.4) and (3.1) respectively, we deduce

$$\begin{aligned}\Phi^{(0)}(z,t) &= \sum_{n=2}^{k} qf^{(0)}(n-3,n-1)z^{n-2}t^n + \sum_{n=2}^{k} p^2t^2(qzt)^{n-2}\\ &+ \sum_{m=k-1}^{\infty}\sum_{n=k+1}^{\infty} qf^{(0)}(m-1,n-1)z^mt^n + pq^{k-1}\sum_{m=k-1}^{\infty}\sum_{n=k+1}^{\infty} f^{(0)}(m-k+1,n-k)z^mt^n.\end{aligned}$$

Observe next that

$$\sum_{n=2}^{k} f^{(0)}(n-3,n-1)z^{n-2}t^n + \sum_{m=k-1}^{\infty}\sum_{n=k+1}^{\infty} f^{(0)}(m-1,n-1)z^m t^n$$
$$= \sum_{m=1}^{\infty}\sum_{n=2}^{\infty} f^{(0)}(m-1,n-1)z^m t^n$$

and substitute in the previous expression to get

$$\Phi^{(0)}(z,t) = p^2t^2A(zt) + qzt\Phi^{(0)}(z,t) + pq^{k-1}z^{k-1}t^k\Phi^{(0)}(z,t).$$

This completes the proof.

It is worth mentioning that the computation of $\Phi^{(0)}(z,t)$ can also be accomplished by direct generating function arguments. To this end, we observe first that a typical sequence of outcomes ending up with the first occurrence of a detection pattern is given by

$$\underbrace{FF\ldots F}_{\geq 0 \text{ times}} S \underbrace{\underbrace{F\ldots F}_{\geq k-1} S \ldots S \underbrace{F\ldots F}_{\geq k-1} S}_{l-2 \text{ times}, l\geq 2} \underbrace{F\ldots F}_{\leq k-2} S.$$

Clearly, the contribution of $\overbrace{FF\ldots F}^{i\geq 0 \text{ times}} S$ is

$$\sum_{i=0}^{\infty} pq^i z^i t^{i+1} = \frac{pt}{1-qzt}$$

whereas the contribution of the last part $\overbrace{F\ldots F}^{i\leq k-2} S$ is

$$\sum_{i=0}^{k-2} pq^i z^i t^{i+1} = pt\sum_{i=0}^{k-2}(qzt)^i = ptA(zt).$$

Each of the $l-2$ terms $\overbrace{F\ldots F}^{i\geq k-1} S$ contributes to the typical sequence by

$$\sum_{i=k-1}^{\infty} pq^i z^i t^{i+1} = pt\sum_{i=k-1}^{\infty}(qzt)^i = \frac{pq^{k-1}z^{k-1}t^k}{1-qzt}$$

and the overall contribution of the sequence to the probability generating function is computed as

$$\frac{pt}{1-qzt}\left(\frac{pq^{k-1}z^{k-1}t^k}{1-qzt}\right)^{l-2} ptA(zt).$$

Summing up the above expression for all possible values of l, i.e. for $l = 2, 3, \ldots$, we obtain the probability generating function of $(Y^{(0)}, T)$ as

$$\Phi^{(0)}(z,t) = \frac{(pt)^2A(zt)}{1-qzt}\sum_{l=2}^{\infty}\left(\frac{pq^{k-1}z^{k-1}t^k}{1-qzt}\right)^{l-2}$$

which reestablishes formula (3.3). The direct algebraic method used here was first suggested by Balakrishnan (1997), (see also Koutras and Balakrishnan (1999)), who exploited it to establish joint generating functions for run-related problems. It is interesting to note that, after some trivial modifications in the arguments presented above, the direct algebraic method could be used to capture the respective generating functions for dependent sequences of binary outcomes, e.g. for Markov dependent trials or for binary sequences of order k (c.f. Aki (1985)).

Setting $t = 1$ in (3.3) we may easily gain the probability generating function of $Y^{(0)}$ as described in the next corollary.

COROLLARY 3.1. *The probability generating function $G^{(0)}(z)$ of $Y^{(0)}$, is given by*

$$G^{(0)}(z) = \frac{p^2 A(z)}{1 - qz - pq^{k-1}z^{k-1}}, \qquad |z| \le 1. \tag{3.5}$$

For the special case $k = 2$, (3.3) and (3.5) reduce to

$$\Phi^{(0)}(z,t) = \frac{p^2t^2}{1 - qzt - pqzt^2}, \qquad G^{(0)}(z) = \frac{p^2}{1-(1-p^2)z}$$

respectively (compare with Proposition 2.1 and Corollary 2.1 in Aki and Hirano (1994)).

From the form of the probability generating function $G^{(0)}(z)$ it is clear that $Y^{(0)}$ follows a geometric distribution with parameter p^2.

The probability generating function of the random variable T can be easily confered from expression (3.3) by setting . This yields the formula

$$\sum_{n=0}^{\infty} \Pr(T = n)t^n = \frac{p^2t^2 A(t)}{1 - qt - pq^{k-1}t^k}, \qquad |t| \le 1$$

which coincides with the one established by Koutras (1996) and Leslie (1967).

The next corollary provides an efficient recursive scheme for the evaluation of the probability mass function $g^{(0)}(m)$ of $Y^{(0)}$. Its proof is easily carried out by multiplying both sides of (3.5) by $1 - qz - pq^{k-1}z^{k-1}$ and picking up the coefficients of z^m in the resulting power series.

COROLLARY 3.2. *The probability mass function $g^{(0)}(m) = \Pr(Y^{(0)} = m)$ of $Y^{(0)}$ satisfies the recurrence relations*

$$g^{(0)}(m) = \begin{cases} (m+1)p^2q^m, & \text{if } 0 \le m \le k-2 \\ qg^{(0)}(m-1) + pq^{k-1}g^{(0)}(m-k+1), & \text{if } m \ge k-1. \end{cases} \tag{3.6}$$

Corollary 3.2 can be used to prove that the distribution of $Y^{(0)}$ is unimodal. For $1 \le m \le k-2$, we have $g^{(0)}(m) = (m+1)p^2q^m$ and therefore

$$\frac{g^{(0)}(m)}{g^{(0)}(m-1)} = \frac{(m+1)q}{m}$$

which ascertains that

$$g^{(0)}(m) \ge g^{(0)}(m-1) \quad \text{for} \quad m \le \frac{q}{p}, \qquad g^{(0)}(m) \le g^{(0)}(m-1) \quad \text{for} \quad m \ge \frac{q}{p}.$$

For $m \geq k-1$, it is not difficult to verify that $g^{(0)}(m-1) \geq q^{k-1}g^{(0)}(m-k+1)$ and since $g^{(0)}(m) - g^{(0)}(m-1) = p[q^{k-1}g^{(0)}(m-k+1) - g^{(0)}(m-1)]$ we may state that

$$g^{(0)}(m) \leq g^{(0)}(m-1) \quad \text{for all} \quad m \geq k-1.$$

The aforementioned inequalities guarantee that the distribution of $Y^{(0)}$ is unimodal, attaining its maximum value for $m_0 = \min\{k-2, [q/p]\}+1$. Moreover, using Corollary 3.2 for $k \geq 3$ we conclude that

$$(g^{(0)}(m))^2 - g^{(0)}(m-1)g^{(0)}(m+1) = \begin{cases} p^4q^{2m}, & 1 \leq m \leq k-3 \\ p^4q^{2(k-2)}[1+q(k-2)], & m = k-2; \end{cases}$$

therefore $g^{(0)}(m)$ satisfies the strong unimodality condition (characterization) of Keilson and Gerber (1971)

$$(g^{(0)}(m))^2 \geq g^{(0)}(m-1)g^{(0)}(m+1)$$

for the range $1 \leq m \leq k-2$, $k \geq 3$. However, for $m = k-1$, the inequality is reversed and thus the distribution of $Y^{(0)}$ is not strongly unimodal, for $k \geq 3$; as a consequence, its convolution with other unimodal distributions is not necessarily unimodal. It goes without saying that for $k = 2$ the distribution of $Y^{(0)}$ is strongly unimodal (in this case, the last inequality holds true as an equality). This is not surprising since in this special case, the distribution of $Y^{(0)}$ is an (ordinary) geometric distribution.

Let us now proceed to the development of non-recursive formulae for the probability mass function $g^{(0)}(m)$. The following theorem provides an exact formula for $g^{(0)}(m)$ in terms of binomial coefficients.

THEOREM 3.3. *The probability mass function of $Y^{(0)}$ is given by*

$$g^{(0)}(m) = p^2q^m \sum_{i=0}^{\min\{k-2,m\}} \sum_{x=0}^{\left[\frac{m-i}{k-1}\right]} \binom{m-i-x(k-2)}{x} p^x, \qquad m \geq 0, \quad k \geq 2. \tag{3.7}$$

PROOF. For $|z| \leq 1$, $0 < q < 1$, $k \geq 2$ we have $|qz + pq^{k-1}z^{k-1}| \leq q + pq^{k-1} < 1$, and expanding $G^{(0)}(z)$ of (3.5) in a geometric series we may write

$$G^{(0)}(z) = p^2A(z)\sum_{y=0}^{\infty}(qz + pq^{k-1}z^{k-1})^y.$$

Recalling the binomial formula we get

$$G^{(0)}(z) = p^2\sum_{i=0}^{k-2}(qz)^i \sum_{y=0}^{\infty}\sum_{j=0}^{y}\binom{y}{j}(qz)^{y-j}(pq^{k-1}z^{k-1})^j$$

and the result follows immediately by straightforward algebraic manipulation.

In the next corollary, formula (3.7) is further simplified by replacing the double summation involved therein by a single sum.

COROLLARY 3.3. *The probability mass function of $Y^{(0)}$ can be written as*

$$(3.8)\quad g^{(0)}(m) = p^2 q^m \sum_{x=0}^{[\frac{m}{k-1}]} p^x \left\{ \binom{m - x(k-2) + 1}{x+1} - \binom{m - x(k-2) - u(x,m)}{x+1} \right\}, \qquad m \geq 0; \quad k \geq 2$$

where $u(x,m) = \min\{\min(k-2, m), m - x(k-1)\}$, $x \geq 0$.

PROOF. Interchanging the order of summation in (3.7) we get

$$g^{(0)}(m) = p^2 q^m \sum_{x=0}^{[\frac{m}{k-1}]} p^x \sum_{i=0}^{u(x,m)} \left\{ \binom{m - x(k-2) - i}{x} \right\}$$

and the desired result follows immediately by making use of the well known combinatorial identity

$$\sum_{i=0}^{n} \binom{a-i}{x} = \binom{a+1}{x+1} - \binom{a-n}{x+1}.$$

Note, that for $0 \leq m \leq k-2$, the above formula leads to (c.f. Corollary 3.2)

$$g^{(0)}(m) = p^2 q^m (u(0,m) + 1) = (m+1) p^2 q^m.$$

In the sequel we turn our attention to the problem of evaluating the moments of $Y^{(0)}$. To begin with, let us write down two expressions for the mean and variance of $Y^{(0)}$, which follow immediately from Corollary 3.1 by considering the first two derivatives of $G^{(0)}(z)$ at $z = 1$.

COROLLARY 3.4. *The mean and variance of $Y^{(0)}$ are given by*

$$\mu = E[Y^{(0)}] = \frac{q(2 - q^{k-1})}{p(1 - q^{k-1})}, \qquad \sigma^2 = \mathrm{Var}[Y^{(0)}] = \frac{2q}{p^2} + \frac{[(2k-3)p + 2]q^k}{p^2(1 - q^{k-1})^2}.$$

Since the evaluation of higher order moments of $Y^{(0)}$ via the derivatives of $G^{(0)}(z)$ at $z = 1$ becomes rather cumbersome, we shall proceed to the development of effective recursive schemes for both factorial moments and moments about zero.

THEOREM 3.4. *The s-th (descending) factorial moments of the random variable $Y^{(0)}$, namely*

$$\mu_{(s)} = E[Y^{(0)}(Y^{(0)} - 1) \cdots (Y^{(0)} - s + 1)], \qquad s \geq 1; \quad \mu_{(0)} = 1$$

satisfy the recurrence relation

$$\mu_{(s)} = \frac{1}{p(1 - q^{k-1})} \left\{ p^2 \sum_{i=s}^{k-2} (i)_{(s)} q^i + sq[1 + (k-1)pq^{k-2}]\mu_{(s-1)} + \sum_{i=2}^{\min\{s,k-1\}} \binom{s}{i} (k-1)_{(i)} \mu_{(s-i)} \right\},$$

where $(a)_{(s)} = a(a-1) \cdots (a-s+1)$, $(a)_{(0)} = 1$ (*Convention*: $\sum_{i=a}^{b} f(i) = 0$ *for* $b < a$).

PROOF. Differentiating the equality (which results immediately from (3.5))

$$G^{(0}(z) = p^2 A(z) + qzG^{(0)}(z) + pq^{k-1} z^{k-1} G^{(0)}(z)$$

s-times and applying Leibnitz's formula, we obtain

$$\frac{d^s(G^{(0)}(z))}{dz^s} = p^2 \frac{d^s(A(z))}{dz^s} + q \sum_{j=0}^{s} \binom{s}{j} \frac{d^j z}{dz^j} \cdot \frac{d^{s-j}(G^{(0)}(z))}{dz^{s-j}}$$

$$+pq^{k-1} \sum_{j=k_0}^{s} \binom{s}{j} \frac{d^j(z^{k-1})}{dz^j} \cdot \frac{d^{s-j}(G^{(0)}(z))}{dz^{s-j}} = p^2 \sum_{i=s}^{k-2} q^i \frac{d^s(z^i)}{dz^s}$$

$$+q \left[z \frac{d^s(G^{(0)}(z))}{dz^s} + s \frac{d^{s-1}(G^{(0)}(z))}{dz^{s-1}} \right] + pq^{k-1}$$

$$\cdot \left[z^{k-1} \frac{d^s(G^{(0)}(z))}{dz^s} + s(k-1) z^{k-2} \frac{d^{s-1}(G^{(0)}(z))}{dz^{s-1}} \right.$$

$$\left. + \sum_{j=2}^{\min\{s,k-1\}} \binom{s}{j} \frac{d^j(z^{k-1})}{dz^j} \cdot \frac{d^{s-j}(G^{(0)}(z))}{dz^{s-j}} \right].$$

The proof is now easily concluded by setting $z = 1$ and taking into account that

$$\mu_{(s)} = \left[\frac{d^s(G^{(0)}(z))}{dz^s} \right]_{z=1}.$$

It is well known that the factorial moments of a random variable can be expressed in terms of its moments about 0 and vice versa as (see e.g. Johnson *et al.* (1992))

$$\mu_{(i)} = \sum_{j=0}^{i} s(i,j) \mu'_j, \qquad \mu'_i = \sum_{j=0}^{i} S(i,j) \mu_{(j)} \tag{3.9}$$

where $s(i,j)$ and $S(i,j)$ are the Stirling numbers of the first and second kind respectively. Consequently the evaluation of the s-th order moments about zero of $Y^{(0)}$ could be performed by a combined use of Theorem 3.4 and the second of the above identities. An alternative scheme is offered by the next theorem which establishes direct recurrence relations for μ'_s, $s = 1, 2, \ldots$.

THEOREM 3.5. *The s-th order moments about zero of* $Y^{(0)}$

$$\mu'_s = E[(Y^{(0)})^s], \qquad s \geq 0$$

satisfy the recurrence relation

$$\mu'_s = \frac{1}{p(1-q^{k-1})} \left\{ p^2 \sum_{m=1}^{k-2} m^s q^m + q \sum_{i=0}^{s-1} \binom{s}{i} [1 + pq^{k-2}(k-1)^{s-i}] \mu'_i \right\}, \qquad s \geq 1.$$

PROOF. Note first that (3.6) can be restated as

$$g^{(0)}(m)=\begin{cases}0 & \text{for } m<0\\ qg^{(0)}(m-1)+p^2q^m & \text{for } 0\le m\le k-2\\ qg^{(0)}(m-1)+pq^{k-1}g^{(0)}(m-k+1) & \text{for } m\ge k-1\end{cases}$$

and implug it in the obvious formula

$$\mu_s'=\sum_{m=1}^{k-2}m^sg^{(0)}(m)+\sum_{m=k-1}^{\infty}m^sg^{(0)}(m)$$

to gain the expression

$$\mu_s'=p^2\sum_{m=1}^{k-2}m^sq^m+q\sum_{m=1}^{\infty}m^sg^{(0)}(m-1)+pq^{k-1}\sum_{m=k-1}^{\infty}m^sg^{(0)}(m-k+1).$$

Taking into account that

$$\sum_{m=0}^{\infty}(m+1)^sg^{(0)}(m)=\sum_{m=0}^{\infty}\sum_{i=0}^{s}\binom{s}{i}m^ig^{(0)}(m)=\sum_{i=0}^{s}\binom{s}{i}\mu_i',\qquad\text{and}$$

$$\sum_{m=0}^{\infty}(m+k-1)^sg^{(0)}(m)=\sum_{m=0}^{\infty}\sum_{i=0}^{s}\binom{s}{i}m^i(k-1)^{s-i}g^{(0)}(m)=\sum_{i=0}^{s}\binom{s}{i}(k-1)^{s-i}\mu_i',$$

we may write

$$\mu_s'=p^2\sum_{m=1}^{k-2}m^sq^n+q\sum_{i=0}^{s-1}\binom{s}{i}\mu_i'+pq^{k-1}\sum_{i=0}^{s-1}\binom{s}{i}(k-1)^{s-i}\mu_i'+(q+pq^{k-1})\mu_s',$$

and the desired result is easily deduced by solving the last formula with respect to μ_s'.

Closing this subsection, we shall discuss in brief the problem of estimating the unknown parameter p. More precisely, we shall examine how the moment estimator and the maximum likelihood estimator (MLE) of p can be computed.

We shall prove first that the mean of $Y^{(0)}$ is a monotonically decreasing function in p. To this end write $\mu=\mu(q)=E[Y^{(0)}]$ as (c.f. Corollary 3.4)

$$\mu(q)=\frac{q(2-q^{k-1})}{(1-q)(1-q^{k-1})}$$

and observe that

$$\frac{d\mu(q)}{dq}=\frac{h_k(q)}{(1-q)^2(1-q^{k-1})^2}$$

where $h_k(q)=2+(k-4)q^{k-1}-(k-1)q^k+q^{2(k-1)}$. Since

$$h_k(q)>2+(k-4)q^{k-1}-(k-1)q^{k-1}+q^{2(k-1)}=(q^{k-1}-1)(q^{k-1}-2)>0$$

(for $0<q<1$, $k>1$) it follows that $\mu(q)$ is monotonically increasing in q. Therefore, $E[Y^{(0)}]$ is a monotonically decreasing function of p, with

$$\lim_{p\to 0}E[Y^{(0)}]=+\infty\qquad\text{and}\qquad\lim_{p\to 1}E[Y^{(0)}]=0.$$

Let now $Y_1^{(0)}, Y_2^{(0)}, \ldots, Y_N^{(0)}$ be a random sample from the distribution of $Y^{(0)}$. Then

$$\bar{Y}^{(0)} = \frac{1}{N}\sum_{i=1}^{N} Y_i^{(0)} \geq 0$$

and the equation $\mu(q) = \bar{Y}^{(0)}$ will have a unique admissible root, which is the moment estimator $\tilde{p}$ of p.

Regarding the MLE of p, the loglikelihood function, for a realization $y_1^{(0)}, y_2^{(0)}, \ldots, y_N^{(0)}$ of the random sample, can be easily evaluated through Corollary 3.3. Indeed, on introducing the notation

$$a(x;m) = \binom{m - x(k-2) + 1}{x+1} - \binom{m - x(k-2) - u(x,m)}{x+1},$$

the loglikelihood function will be given by

$$\ln L(p) = \sum_{i=1}^{N} \ln(g^{(0)}(y_i^{(0)})) = 2N \ln p + N\bar{y}^{(0)} \ln(1-p) + \sum_{i=1}^{N} \ln\left(\sum_{x=0}^{[y_i^{(0)}/(k-1)]} p^x a(x, y_i^{(0)})\right)$$

and the computation of its first and second derivatives (with respect to p) is straightforward. In general, no explicit analytic solution for the MLE equation $d\ln L(p)/dp = 0$ can be computed and the calculation of the MLE of p has to be done numerically. The details are left to the reader.

It is worth mentioning that Corollary 3.2 offers an efficient scheme for computing the probability mass function $g^{(0)}(m)$ and its derivatives with respect to p (direct differentiation on (3.6) will initiate computationally powerful recursive schemes for them as well). Therefore one could resort to them and solve the MLE equation by the aid of an iterative algorithm (e.g. Newton-Raphson) instead of working with Corollary 3.3 as described earlier.

3.2 *Number of successes*

Here, we study the distribution of the number $Y^{(1)}$ of successes until the first occurrence of a detection pattern.

As in Subsection 3.1, our analysis commences with an effective recursive scheme for the evaluation of the joint probability distribution function of $(Y^{(1)}, T)$.

THEOREM 3.6. *The joint probability mass function* $f^{(1)}(m,n) = \Pr(Y^{(1)} = m, T = n)$ *of* $(Y^{(1)}, T)$, *satisfies the following recurrence relation*

$$f^{(1)}(m,n) = qf^{(1)}(m,n-1) + pq^{k-1}f^{(1)}(m-1,n-k), \qquad n > k, \quad m \geq 1$$

with initial conditions

$$f^{(1)}(m,n) = \begin{cases} (n-1)p^m q^{n-2}, & 2 \leq n \leq k, \quad m = 2 \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. The derivation of the initial conditions is straightforward. Assume now that $n > k$, $m \geq 1$. Following an exact parallel of the arguments employed in the proof of Theorem 3.1, we write $f^{(1)}(m,n)$ as

$$f^{(1)}(m,n) = \Pr(Y^{(1)} = m, T = n, X_1 = 0) + \Pr(Y^{(1)} = m, T = n, X_1 = 1)$$

and use once more the event $A = \{X_1 = 1 \text{ and } X_i = 0 \text{ for all } i = 2, 3, \ldots, k\}$ to capture the expressions

$$\begin{aligned}\Pr(Y^{(1)} = m, T = n, X_1 = 0) &= \Pr(X_1 = 0)\Pr(Y^{(1)} = m, T = n \mid X_1 = 0)\\ &= qf^{(1)}(m, n-1),\\ \Pr(Y^{(1)} = m, T = n, X_1 = 1) &= \Pr(A)\Pr(Y^{(1)} = m, T = n \mid A)\\ &= pq^{k-1}f^{(1)}(m-1, n-k).\end{aligned}$$

This completes the proof.

The joint probability generating function of $(Y^{(1)}, T)$ can now be easily calculated by the aid of Theorem 3.6. This is described in the next theorem.

THEOREM 3.7. *The joint probability generating function*

$$\Phi^{(1)}(z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f^{(1)}(m,n)z^m t^n$$

of $(Y^{(1)}, T)$ *is given by*

$$\Phi^{(1)}(z,t) = \frac{(pzt)^2 A(t)}{1 - qt - pq^{k-1}zt^k}, \qquad |z| \leq 1, \quad |t| \leq 1.$$

PROOF. By virtue of Theorem 3.6 we may write

$$\begin{aligned}\Phi^{(1)}(z,t) &= \sum_{n=2}^{k} f^{(1)}(2,n)z^2t^n + q\sum_{m=1}^{\infty}\sum_{n=k+1}^{\infty} f^{(1)}(m,n-1)z^m t^n\\ &\quad + pq^{k-1}\sum_{m=1}^{\infty}\sum_{n=k+1}^{\infty} f^{(1)}(m-1,n-k)z^m t^n.\end{aligned}$$

Observe next that, for $2 \leq n \leq k$, $m = 2$, the initial conditions of Theorem 3.6 can be alternatively expressed as $f^{(1)} = (m,n) = qf^{(1)}(m,n-1) + p^m q^{n-2}$ and therefore the first summand of the RHS of $\Phi^{(1)}(z,t)$ reads

$$\sum_{n=2}^{k} f^{(1)}(2,n)z^2t^n = q\sum_{m=1}^{\infty}\sum_{n=2}^{k} f^{(1)}(m,n-1)z^m t^n + \sum_{n=2}^{k} p^2 q^{n-2} z^2 t^n.$$

It is now immediate that $\Phi^{(1)}(z,t)$ takes on the form

$$\begin{aligned}\Phi^{(1)}(z,t) &= (pzt)^2\sum_{i=0}^{k-2}(qt)^i + q\sum_{m=1}^{\infty}\sum_{n=2}^{\infty} f^{(1)}(m,n-1)z^m t^n\\ &\quad + pq^{k-1}\sum_{m=1}^{\infty}\sum_{n=k+1}^{\infty} f^{(1)}(m-1,n-k)z^m t^n,\end{aligned}$$

which yields

$$\Phi^{(1)}(z,t) = (pzt)^2 \sum_{i=2}^{k-2} (qt)^i + qt\Phi^{(1)}(z,t) + pq^{k-1}zt^k\Phi^{(1)}(z,t)$$

and the proof is completed by solving the last equation in terms of $\Phi^{(1)}(z,t)$.

It goes without saying that, a direct algebraic proof similar to the one presented after Theorem 3.2, can be used here as well to accomplish the outcome of Theorem 3.7. The details are left to the reader.

The next corollary is an immediate by-product of Theorem 3.7.

COROLLARY 3.5. *The probability generating function $G^{(1)}(z)$ of $Y^{(1)}$, is given by*

$$G^{(1)}(z) = \frac{z^2(1-q^{k-1})}{1-q^{k-1}z}, \qquad |z| \le 1.$$

Corollary 3.5 indicates that $Y^{(1)}$ follows a geometric distribution with parameter $1 - q^{k-1}$ shifted to the support $\{2, 3, 4, \ldots\}$. Due to this we shall not proceed to a detailed examination of the distribution of $Y^{(1)}$ as we did for $Y^{(0)}$.

For the special case $k = 2$, Theorem 3.7 and Corollary 3.5 yield

$$\Phi^{(1)}(z,t) = \frac{(pzt)^2}{1-qt(1+pzt)}, \qquad G^{(1)}(z) = \frac{pz^2}{1-qz} = \frac{p(1-pz)z^2}{1-z+pqz^2}.$$

Since for this case the detection pattern is a run of length 2, the second formula results also as a special case of Aki and Hirano's (1994) Proposition 2.4 pertaining to number of successes until a success run of specified length is obtained.

4. Distribution of the number of failures until the r-th appearance of a detection pattern

In this section we focus on the distribution of the total number of failures until the r-th occurrence of two successes separated by at most $k-2$ failures. The enumeration scheme used here is the non-overlapping one, i.e. each time a detection pattern occurs (and is counted) the observed sequence is disregarded and we start from scratch. As a consequence, if ν_1 denotes the number of failures until the first appearance of a detection pattern, ν_2 the number of failures until the next appearance and so on, the total number of failures, say $Y_r^{(0)} = Y_{k,r}^{(0)}$, until the r-th occurrence can be expressed as

$$Y_r^{(0)} = \nu_1 + \nu_2 + \cdots + \nu_r. \tag{4.1}$$

Therefore, the random variable $Y_r^{(0)}$ can be decomposed in a sum of identical and independently distributed (iid) random variables ν_i , $1 \le i \le r$, each with probability mass function $g^{(0)}$. This fact, in conjunction with Corollary 3.1, yields the next theorem.

THEOREM 4.1. *The probability generating function $G_r^{(0)}(z) = \sum_{m=0}^{\infty} g_r^0(m)z^m$ of $Y_r^{(0)}$, is given by*

$$G_r^{(0)}(z) = \left[\frac{p^2A(z)}{1-qz-pq^{k-1}z^{k-1}}\right]^r, \qquad |z| \le 1.$$

Manipulating on Theorem's 4.1 outcome, we can easily accomplish a recursive scheme for the numerical evaluation of the probability distribution function $g_r^{(0)}$ of $Y_r^{(0)}$. More precisely, we have the following

COROLLARY 4.1. *The probability mass function* $g_r^{(0)}(m) = \Pr(Y_r^{(0)} = m)$ *of* $Y_r^{(0)}$ *satisfies the recurrence relations*

$$g_{r+1}^{(0)}(0) = p^2 g_r^{(0)}(0)$$

$$g_{r+1}^{(0)}(m) = q g_{r+1}^{(0)}(m-1) + p^2 \sum_{i=0}^{m} g_r^{(0)}(m-i) q^i, \qquad 1 \le m \le k-2$$

$$g_{r+1}^{(0)}(m) = q g_{r+1}^{(0)}(m-1) + p q^{k-1} g_{r+1}^{(0)}(m-k+1) + p^2 \sum_{i=0}^{k-2} g_r^{(0)}(m-i) q^i, \qquad m \ge k-1$$

for any integer $r \ge 1$.

PROOF. It is clear that

$$G_{r+1}^{(0)}(z) = G_r^{(0)}(z) G^{(0)}(z) \tag{4.2}$$

and making use of (3.5) we deduce

$$\sum_{m=0}^{\infty} g_{r+1}^{(0)}(m) z^m = q \sum_{m=1}^{\infty} g_{r+1}^{(0)}(m-1) z^m + p q^{k-1} \sum_{m=k-1}^{\infty} g_{r+1}^{(0)}(m-k+1) z^m$$
$$+ p^2 \sum_{i=0}^{k-2} \sum_{m=i}^{\infty} g_r^{(0)}(m-i) q^i z^m.$$

The desired recurrences are now easily established by a careful inspection of the coefficients of z^m in both sides.

Note that, the quantities $g_1^{(0)}(m)$ which are necessary to initiate the recursive scheme described in Corollary 4.1 can be fetched from Corollary 3.2 ($g_1^{(0)}(m) = g^{(0)}(m)$).

Needless to say, an exact formula of $g_r^{(0)}(m)$ can also be established by expanding $G_r^{(0)}(z)$ of Theorem 4.1, into a power series. Nevertheless, the resulting formula is not very attractive, so we shall not further pursue this matter.

The mean and variance of $Y_r^{(0)}$ are (c.f. Corollary 3.4 and (4.1))

$$E[Y_r^{(0)}] = r E[Y^{(0)}] = \frac{rq(2-q^{k-1})}{p(1-q^{k-1})},$$

$$\operatorname{Var}[Y_r^{(0)}] = r \operatorname{Var}[Y^{(0)}] = \frac{2rq}{p^2} + \frac{r[(2k-3)p+2]q^k}{p^2(1-q^{k-1})^2}.$$

For the evaluation of higher order moments of $Y_r^{(0)}$ it seems reasonable to resort to recursive schemes, since the establishment of neat and computationally tractable exact formulae is rather infeasible. This problem is addressed in the next two theorems.

THEOREM 4.2. *The s-th factorial moments of* $Y_r^{(0)}$

$$\mu_{r,(s)} = E[Y_r^{(0)}(Y_r^{(0)}-1)\cdots(Y_r^{(0)}-s+1)], \qquad s \geq 1; \quad \mu_{r,(0)} = 1$$

satisfy the following recurrence relation

$$\mu_{r+1,(s)} = \frac{1}{p(1-q^{k-1})}\left\{ p^2 \sum_{j=0}^{s} \binom{s}{j} \mu_{r,(s-j)} \sum_{i=j}^{s} (i)_{(j)} q^i + sq[1+(k-1)pq^{k-2}]\mu_{r+1,(s+1)} + \sum_{j=2}^{\min\{s,k-1\}} \binom{s}{j} (k-1)_{(j)} \mu_{r+1,(s-j)} \right\}, \qquad r \geq 1$$

($\mu_{1,(s)} \equiv \mu_{(s)}$ *is given in Theorem* 3.4).

PROOF. Substituting $G^{(0)}(z)$(c.f. Corollary 3.1) into (4.2) we obtain

$$G_{r+1}^{(0)}(z) = p^2 A(z) G_r^{(0)}(z) + qzG_{r+1}^{(0)}(z) + pq^{k-1}z^{k-1}G_{r+1}^{(0)}(z).$$

Applying Leibnitz's formula in the last equality we get

$$\begin{aligned}
\frac{d^s(G_{r+1}^{(0)}(z))}{dz^s} &= p^2 \sum_{j=0}^{s} \binom{s}{j} \frac{d^j(A(z))}{dz^j} \cdot \frac{d^{s-j}(G_r^{(0)}(z))}{dz^{s-j}} \\
&\quad + q\left[z\frac{d^s(G_{r+1}^{(0)}(z))}{dz^s} + s\frac{d^{s-1}(G_{r+1}^{(0)}(z))}{dz^{s-1}} \right] \\
&\quad + pq^{k-1} \sum_{j=0}^{s} \binom{s}{j} \frac{d^j(z^{k-1})}{dz^j} \cdot \frac{d^{s-j}(G_{r+1}^{(0)}(z))}{dz^{s-j}} \\
&= p^2 \sum_{j=0}^{s} \binom{s}{j} \sum_{i=j}^{k-2} q^i \frac{d^j(z^i)}{dz^j} \cdot \frac{d^{s-j}(G_r^{(0)}(z))}{dz^{s-j}} \\
&\quad + q\left[z\frac{d^s(G_{r+1}^{(0)}(z))}{dz^s} + s\frac{d^{s-1}(G_{r+1}^{(0)}(z))}{dz^{s-1}} \right] \\
&\quad + pq^{k-1}\left[z^{k-1}\frac{d^s(G_{r+1}^{(0)}(z))}{dz^s} + s(k-1)z^{k-2}\frac{d^{s-1}(G_{r+1}^{(0)}(z))}{dz^{s-1}} \right. \\
&\quad \left. + \sum_{j=2}^{\min\{s,k-1\}} \binom{s}{j} \frac{d^j(z^{k-1})}{dz^j} \cdot \frac{d^{s-j}(G_{r+1}^{(0)}(z))}{dz^{s-j}} \right],
\end{aligned}$$

which, on setting $z = 1$, reduces to the recurrences described in the theorem.

Clearly, the evaluation of the moments about 0 could be accomplished by a combined use of Theorem 4.2 and the formulae relating factorial and ordinary moments (c.f. (3.9)). Nevertheless, a direct (recursive) scheme is still of some interest since it leads to less computationally demanding procedures. This is the subject of the next theorem.

THEOREM 4.3. *The s-th order moments about zero of* $Y_r^{(0)}$

$$\mu'_{r,s} = E[(Y_r^{(0)})^s], \qquad s \geq 0$$

satisfy the recurrence relation

$$\mu'_{r+1,s} = \frac{1}{p(1-q^{k-1})}\left\{q\sum_{i=0}^{s-1}\binom{s}{i}[1+pq^{k-2}(k-1)^{s-i}]\mu'_{r+1,i} + p^2\sum_{x=0}^{k-2}\sum_{i=0}^{s}\binom{s}{i}x^{s-i}q^x\mu'_{r,i}\right\}$$

for $r, s \geq 1$ $(\mu'_{1,s} \equiv \mu'_s$ *is given by Theorem* 3.5).

PROOF. Substituting the second and the third recurrences of Corollary 4.1, into the obvious formula

$$\mu'_{r+1,s} = \sum_{m=1}^{k-2} m^s g_{r+1}^{(0)}(m) + \sum_{m=k-1}^{\infty} m^s g_{r+1}^{(0)}(m)$$

we get

$$\begin{aligned}\mu'_{r+1,s} &= q\sum_{m=1}^{\infty} m^s g_{r+1}^{(0)}(m-1) + pq^{k-1}\sum_{m=k-1}^{\infty} m^s g_{r+1}^{(0)}(m-k+1) \\ &+p^2\left\{\sum_{m=1}^{k-2} m^s \sum_{i=0}^{m} g_r^{(0)}(m-i)q^i + \sum_{m=k-1}^{\infty} m^s \sum_{i=0}^{k-2} g_r^{(0)}(m-i)q^i\right\}\end{aligned}$$

and taking into account that the bracketed term equals $\sum_{x=0}^{k-2} q^x \sum_{m=x}^{\infty} m^s g_r^{(0)}(m-x)$, we gain the expression

$$\begin{aligned}\mu'_{r+1,s} &= q\sum_{m=0}^{\infty}(m+1)^s g_{r+1}^{(0)}(m) + pq^{k-1}\sum_{m=0}^{\infty}(m+k-1)^s g_{r+1}^{(0)}(m) \\ &+p^2\sum_{x=0}^{k-2} q^x \sum_{m=0}^{\infty}(m+x)^s g_r^{(0)}(m).\end{aligned}$$

Observe next that

$$\sum_{m=0}^{\infty}(m+a)^s g_r^{(0)}(m) = \sum_{m=0}^{\infty}\sum_{i=0}^{s}\binom{s}{i} m^i a^{s-i} g_r^{(0)}(m) = \sum_{i=0}^{s}\binom{s}{i} a^{s-i}\mu'_{r,i},$$

and substitute in the previous formula to get

$$\begin{aligned}\mu'_{r+1,s} &= q\sum_{i=0}^{s-1}\binom{s}{i}\mu'_{r+1,i} + pq^{k-1}\sum_{i=0}^{s-1}\binom{s}{i}(k-1)^{s-i}\mu'_{r+1,i} + (q+pq^{k-1})\mu'_{r+1,s} \\ &+p^2\sum_{x=0}^{k-2}\sum_{i=0}^{s}\binom{s}{i}x^{s-i}q^x\mu'_{r,i}.\end{aligned}$$

The desired result can be easily established now by solving the last formula with respect to $\mu'_{r+1,s}$.

It is evident that the mean of $Y_r^{(0)}$ is a monotonically decreasing function in p, and therefore, there is a unique moment estimator of p. On the other hand, the numerical calculations needed for performing the maximum likelihood estimation of p (i.e. the calculation of the first and the second derivatives of the probability mass function of $Y_r^{(0)}$) are highly facilitated by the recurrences for $g_r^{(0)}(m)$ established in Corollary 4.1. We are not going to further details on these topics, leaving them to the interested reader.

Before closing this section, we shall discuss in brief how the distribution of the random variable $Y_{k,r}^{(0)}$ can be expressed in terms of properly defined sequences of numbers and polynomials.

For $k \geq 2$, a fixed positive integer, let us introduce the sequence of numbers $\{F_{k,m}\}_{m\geq 0}$, or simply $\{F_m\}_{m\geq 0}$, by the following recursive scheme

$$F_m = \begin{cases} m+1, & \text{if } \ 0 \leq m \leq k-2 \\ F_{m-1} + \frac{1}{2}F_{m-k+1}, & \text{if } \ m \geq k-1. \end{cases}$$

In the next theorem we consider a sequence of symmetric Bernoulli trials and express the probability mass function of $Y_r^{(0)}$ in terms of convolutions of the numbers F_m .

THEOREM 4.4. *If $p = q = 1/2$, the probability mass function of $Y_r^{(0)}$ is given by*

$$g_r^{(0)}(m) = \frac{F_m^{(r)}}{2^{m+2r}}$$

where $F_m^{(i)}$ is the i-th convolution of the numbers $\{F_m\}_{m\geq 0}$, i.e.

$$F_m^{(i)} = \sum_{j=0}^{m} F_j^{(i-1)} F_{m-j}, \qquad i \geq 2; \quad F_m^{(1)} \equiv F_m.$$

PROOF. Clearly, the initial conditions $F_m = m+1$, $0 \leq m \leq k-2$, can be equivalently described through the recursive scheme

$$F_0 = 1; \qquad F_m = F_{m-1} + 1, \quad 1 \leq m \leq k-2.$$

Substituting now the F_m's in the expression

$$\sum_{m=0}^{\infty} F_m z^m = 1 + \sum_{m=1}^{k-2} F_m z^m + \sum_{m=k-1}^{\infty} F_m z^m$$

we can easily calculate the generating function of the numbers $\{F_n\}_{m\geq 0}$ as

$$\sum_{m=0}^{\infty} F_m z^m = \frac{\sum_{i=0}^{k-2} z^i}{1 - z - \frac{1}{2}z^{k-1}} = 4G^{(0)}(2z).$$

The generating function of the r-th convoluted numbers $\{F_m^{(r)}\}_{m\geq 0}$ will be given by

$$\sum_{m=0}^{\infty} F_m^{(r)} z^m = \left(\sum_{m=0}^{\infty} F_m z^m\right)^r = 2^{2r}(G^{(0)}(2z))^r$$

and replacing $z/2$ for z, we get

$$\sum_{m=0}^{\infty} \frac{F_m^{(r)}}{2^m} z^m = 2^{2r}(G^{(0)}(z))^r = 2^{2r} G_r^{(0)}(z) = 2^{2r} \sum_{m=0}^{\infty} g_r^{(0)}(m) z^m$$

which completes the proof.

In order to capture a similar result for $g_r^{(0)}(m)$ in the general case $0 < p < 1$, let us introduce a sequence of polynomials $\{F_{k,m}(x)\}_{m\geq 0} \equiv \{F_m(x)\}_{m\geq 0}$ defined by the recurrence relations

$$(4.3) \qquad F_m(x) = \begin{cases} (m+1)x^2, & \text{if } \ 0 \leq m \leq k-2 \\ F_{m-1}(x) + xF_{m-k+1}(x), & \text{if } \ m \geq k-1 \end{cases}$$

and denote by $F_m^{(i)}(x)$ the i-th convolution of the sequence $\{F_m(x)\}_{m\geq 0}$, i.e.

$$F_m^{(i)}(x) = \sum_{j=0}^{m} F_j^{(i-1)}(x) F_{m-j}(x), \qquad i \geq 2; \quad F_m^{(1)}(x) \equiv F_m(x).$$

THEOREM 4.5. *The probability mass function of* $Y_r^{(0)}$ *is given by*

$$g_r^{(0)}(m) = q^m F_m^{(r)}(p).$$

PROOF. Clearly, the definition of $F_m(x)$, $0 \leq m \leq k-2$ is equivalent to

$$F_0(x) = x^2; \qquad F_m(x) = F_{m-1}(x) + x^2, \quad 1 \leq m \leq k-2$$

and employing the same reasoning as before, we may express the generating function of the polynomials $\{F_m(x)\}_{m\geq 0}$ as

$$\sum_{m=0}^{\infty} F_m(x) z^m = \frac{x^2 \sum_{i=0}^{k-2} z^i}{1 - z - xz^{k-1}}.$$

Therefore

$$\sum_{m=0}^{\infty} F_m^{(r)}(x) z^m = \left(\sum_{m=0}^{\infty} F_m(x) z^m\right)^r = \left(\frac{x^2 \sum_{i=0}^{k-2} z^i}{1 - z - xz^{k-1}}\right)^r$$

and replacing p,qz for x, z respectively we deduce, by virtue of Theorem 4.1

$$\sum_{m=0}^{\infty} F_m^{(r)}(p) q^m z^m = \sum_{m=0}^{\infty} g_r^{(0)}(m) z^m.$$

This completes the proof.

Remark. If ξ_1 denotes the number of successes until the first appearance of a detection pattern, ξ_2 the number of successes until the second appearance of a detection pattern and so on (with the non-overlapping enumeration scheme), then

$$Y_r^{(1)} = \xi_1 + \xi_2 + \cdots + \xi_r;$$

recalling the discussion following Corollary 3.5 we may state that $Y_r^{(1)}$ has a negative binomial distribution with parameter r and $1 - q^{k-1}$, shifted to the support $\{2r, 2r + 1, 2r + 2, \ldots\}$.

5. Asymptotic results

Let us consider first the problem of approximating the distribution of the number of failures until the first appearance of a detection pattern in a prolonged sequence of trials.

THEOREM 5.1. *For the probability mass function $g^{(0)}(m)$ of $Y^{(0)}$, we have*

$$g^{(0)}(m) \sim \frac{pq(x-1)}{(1-qx)[k-(k-2)qx-1]} \cdot \frac{1}{x^m}, \qquad \text{as} \quad m \to \infty \tag{5.1}$$

where $0 < x < 1/q$ is the smallest in absolute value root of $V(z) = 1 - qz - pq^{k-1}z^{k-1}$ (the sign $\sim$ indicating that the ratio of the two sides tends to 1).

PROOF. By Corollary 3.1, the probability generating function of $Y^{(0)}$ can be expressed as a rational function, as follows

$$G^{(0)}(z) = \frac{p^2[1 + qz + (qz)^2 + \cdots + (qz))^{k-2}]}{1 - qz - pq^{k-1}z^{k-1}} = \frac{U(z)}{V(z)}.$$

According to the partial fraction expansions method (see Feller (1968), p. 277), the coefficient of z^m in $G^{(0)}(z)$ equals (approximately, for large m) $p_1 x^{-(m+1)}$, where x is a simple root of $V(z) = 0$ which is smaller, in absolute value, than any other root, and

$$p_1 = -U(x) \left\{ \left[\frac{dV(z)}{dz} \right]_{z=x} \right\}^{-1}.$$

Since $V(z)$ is a decreasing function for $z > 0$ and $V(0) = 1$, $\lim_{z\to\infty} V(z) = -\infty$, there is a unique positive root of $V(z)$, say $z = x$. For all real or complex numbers z with $|z| < x$, we have

$$|qz(1 + pq^{k-2}z^{k-2})| < qx(1 + pq^{k-2}x^{x-2}) = 1 \tag{5.2}$$

and accordingly

$$|V(z)| \geq 1 - |qz(1 + pq^{k-2}z^{k-2})| > 0, \qquad \text{for} \quad |z| < x$$

i.e., there exist no roots of $V(z)$ with $|z| < x$. Now, if there exists a real or complex number z_0, $|z_0| = x$, such that, for $z = z_0$, inequality (5.2) reduces to equality, then

$z_0 = x$. Therefore, x is smaller in absolute value than any other root of $V(z)$. Finally, we observe that $x > 1$ (note that $V(1) = p(1 - q^{k-1}) > 0$) is a single root of $V(z)$ (because $V'(x) = -q[1 + (k-1)pq^{k-2}x^{k-2}] \neq 0$) and satisfies the condition $qx < 1$, the last one being an immediate consequence of $V(1/q) = -p < 0$.

With all these in mind we may state the following asymptotic expression for $g^{(0)}(m)$

$$g^{(0)}(m) \sim -\frac{p^2\dfrac{1-q^{k-1}x^{k-1}}{1-qx}}{\dfrac{d}{dz}(1-qz-pq^{k-1}z^{k-1})|_{z=x}} \cdot \frac{1}{x^{m+1}}$$
$$= \frac{p^2(1-q^{k-1}x^{k-1})}{q(1-qx)\left[1+(k-1)\dfrac{p}{qx}q^{k-1}x^{k-1}\right]x^{m+1}}.$$

Formula (5.1) is now easily established if we replace $q^{k-1}x^{k-1}$ by $(1 - qx)/p$ (this is feasible because $V(x) = 0$) and perform elementary algebra in the outcome.

It is worth mentioning that x can be satisfactorily approximated by $x^* = 1 + (k-1)p^2q^{k-2}$. To justify this, write first $V(x) = 0$ in the alternative form

$$x = \frac{1}{q}(1 - pq^{k-1}x^{k-1}) = f(x)$$

and observe that the dominant root x can be numerically calculated by successive approximations setting $x_0 = 1$ and $x_{v+1} = f(x_v)$ (see e.g. Feller (1968)). The first two iterations yield

$$x_1 = f(x_0) = \frac{1 - pq^{k-1}}{q}$$
$$x_2 = f(x_1) = \frac{1}{q}[1 - p(1 - pq^{k-1})^{k-1}] \cong \frac{1}{q}\{1 - p[1 - (k-1)pq^{k-1}]\} = x^*$$

and since the higher-step approximations become rather cumbersome, one could terminate the process here (at the price of a cruder approximation as compared to subsequent terms of the iteration scheme).

Let us now proceed to the investigation of the asymptotic behavior of the random variable $Y_r^{(0)}$. As indicated after Corollary 3.1, for $k = 2$, the distribution of $Y^{(0)}$ is geometric with parameter p^2; consequently, the distribution of $Y_{2,r}^{(0)}$ is negative binomial with probability generating function

$$p^{2r}(1 - (1 - p^2)z)^{-r} = \left(\frac{p^2}{1 - qz - pqz}\right)^r = G_{2,r}^{(0)}(z).$$

It is well known that, if $rq \to \lambda > 0$ as $r \to \infty$, the negative binomial distribution with parameters r and p converges in law to the Poisson distribution with parameter λ. In the next theorem, we prove that, under the same assumption (i.e. $\lim_{r\to\infty} rq = \lambda$), the asymptotic distribution of $Y_{k,r}^{(0)}$ is Poisson for all $k \geq 2$.

THEOREM 5.2. *If* $\lim_{r\to\infty} rq = \lambda > 0$ *then the random variable* $Y_{k,r}^{(0)}$, $k \geq 2$ *converges in law to the Poisson distribution with parameter* 2λ, *i.e.*

$$\lim_{r\to\infty} \Pr(Y_{k,r}^{(0)} = x) = e^{-2\lambda}\frac{(2\lambda)^x}{x!}, \qquad x = 0, 1, 2, \ldots$$

PROOF. It is immediate that

$$\lim_{r\to\infty} p^{2r} = \lim_{r\to\infty} (1-q)^{2r} = e^{-2\lambda},$$

$$\lim_{r\to\infty} [1 - A(z)] = \lim_{r\to\infty} r[(qz)^{k-1} - qz] = \begin{cases} 0, & \text{if } k = 2 \\ -\lambda z, & \text{if } k \geq 3, \end{cases}$$

$$\lim_{r\to\infty} r(qz + pq^{k-1}z^{k-1}) = \begin{cases} 2\lambda z, & \text{if } k = 2 \\ \lambda z, & \text{if } k \geq 3. \end{cases}$$

The last two formulae ascertain that

$$\lim_{r\to\infty} A^r(z) = \begin{cases} 1, & \text{if } k = 2 \\ \exp(\lambda z), & \text{if } k \geq 3 \end{cases}$$

$$\lim_{r\to\infty} (1 - qz - pq^{k-1}z^{k-1})^r = \begin{cases} \exp(-2\lambda z), & \text{if } k = 2 \\ \exp(-\lambda z), & \text{if } k \geq 3 \end{cases}$$

and recalling Theorem 4.1 we get

$$\lim_{r\to\infty} G_{k,r}^{(0)}(z) = \exp(-2\lambda + 2\lambda z)$$

which proves the asymptotic result we are interested in.

Since $Y_r^{(0)}$ can be decomposed in a sum of r iid random variables (see (4.1)), the following additional asymptotic result will hold true (by virtue of the Central Limit Theorem).

THEOREM 5.3. *If* $r \to \infty$ *and* p *is fixed, then the standardized random variable*

$$\frac{1}{\sqrt{r\,\mathrm{Var}[Y^{(0)}]}}\left\{Y_r^{(0)} - \frac{rq(2-q^{k-1})}{p(1-q^{k-1})}\right\}$$

converges to the standard Normal distribution $N(0,1)$. ($\mathrm{Var}[Y^{(0)}]$ *is given explicitly in Corollary* 3.4).

6. Applications

The random variables studied in the previous sections can be fruitfully exploited in several areas, including actuarial science, reliability engineering, quality control etc. A few specific examples will now be discussed in some detail.

An automobile insurance company is considering to offer their customers the following plan. Every "policy year" i , the company will pay the claims only if the aggregate claims C_i throughout the year exceed a certain threshold c. In the opposite case, i.e. if $C_i < c$, the policyholder will pay the damages by himself and the year will be declared as "claim-free". For every pair of claim-free years, or a pair of claim-free years separated by at most $k-2$ non claim-free years the customer is awarded a bonus point and once he manages to collect r bonus points, the company applies a special discount to the premium of the insurance contract. Apparently, in order to specify the parameters involved in the plan (c, k discount rate), the company has to study carefully the total claim W of the policyholder at the time he gains the premium reduction. The mathematical model for the plan can be summarized as follows: Let C_i denote the total claim for year i and define

$$X_i = \begin{cases} 1, & \text{if } C_i < c \\ 0, & \text{if } C_i \geq \text{c.} \end{cases}$$

Then $T_{k,r}$ is the year in which the company is liable for the discount whereas $Y_{k,r}^{(0)}$ enumerates the number of non claim-free years. Therefore, W is a sum of a random number $Y_{k,r}^{(0)}$ of iid random variables (selected out of $C_1, C_2, \ldots$) and the probability generating function of W is computed as

$$E(z^W) = G^{(0)}(G_c(z));$$

$G_c(z) = E[z^C]$ is the probability generating function of C_i , and $G^0(z)$ is given by Corollary 3.1. The distribution of W can now be easily studied if the distribution law of the annual aggregate claims C is specified. It is worth mentioning that a similar analysis to the one conducted above could be performed for a penalized insurance plan which would impose a penalty (e.g. increased premium) to the policyholder whenever annual aggregate claims exceeding a threshold c, show up very frequently (for example in two consecutive years or in policy periods which are very close to each other).

Another interesting application, emanating from engineering models is the following version of moving (sliding) window detection problem (compare to Glaz (1983)). A radar sweep with a quantizer transmits to a detector the digit 1 or 0 according to whether the signal-plus-noise waveform exceeds a predetermined threshold. The detector's memory is keeping track of the last k transmitted digits and generates a pulse when two 1's are present. Should this occur, the detector's memory is cleared and the next transmitted digit is the first to be registered. The r-th pulse initiates an alarm. Clearly, the random variable $Y_{k,r}^{(0)}$ enumerates the number of times the signal-plus-noise waveform was below the threshold level, till the time the alarm was triggered.

In the same flavor, one could construct a sampling inspection plan which rejects a lot of equipment (such as power generators or lawn mowers) whenever r pairs of defective items are spotted, with each pair being separated by at most k-2 non-defective items. If the term "defective" indicates that the equipment subject to testing broke down before a (pre- specified) test period, then $Y_{k,r}^{(0)}$ gives the number of items that have survived in a rejected lot.

REFERENCES

Aki, S. (1985). Discrete distributions of order k on a binary sequence, *Ann. Inst. Statist. Math.*, **37**, 205–224.

Aki, S. and Hirano, K. (1994). Distributions of numbers of failures and successes until the first consecutive k successes, *Ann. Inst. Statist. Math.*, **46**,193–202.

Antzoulakos, D. L. and Philippou, A. N. (1996). Distributions of the numbers of successes and failures until the occurrence of the r-th overlapping and non-overlapping success run of length k, *J. Indian Soc. Statist. Oper. Res.*, **17**, 19–30.

Balakrishnan, N, (1997). Joint distributions of number of success-runs and failures until the first consecutive k successes in a binary sequence, *Ann. Inst. Statist. Math.*, **49**, 519–529.

Balakrishnan, N. and Koutras, M. V. (2001). *Runs, and Scans with Applications*, Wiley, New York.

Chen, J. and Glaz, J. (1997). Approximations and inequalities for the distribution of a scan statistic for 0-1 Bernoulli trials, *Advances in Combinatorial Methods and Applications to Probability and Statistics* (ed. N. Balakrishnan), 285–298, Birkhäuser, Boston.

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd ed. Wiley, New York.

Glaz, J. (1983). Moving window detection for discrete-data, *IEEE Trans. Inform. Theory*, **29**, 457–462.

Glaz, J. (1989). Approximations and bounds for the distribution of the scan statistic, *J. Amer. Statist. Assoc.*, **84**, 560–566.

Glaz, J. and Naus, J. (1991). Tight bounds and approximations for scan statistic probabilities for discrete data, *Ann. Appl. Probab.*, **1**, 306–318.

Greenberg, I. (1970). The first occurrence of n successes in N trials, *Technometrics*, **12**, 627–634.

Johnson, N. L., Kotz, S. and Kemp, A. W. (1992). *Univariate Discrete Distributions*, Wiley, New York.

Keilson, J. and Gerber, H. (1971). Some results on discrete unimodality, *J. Amer. Statist. Assoc.*, **66**, 386–389.

Koutras, M. V. (1996). On a waiting time distribution in a sequence of Bernoulli trials, *Ann. Inst. Statist. Math.*, **48**, 789–806.

Koutras, M. V. and Balakrishnan, N. (1999). A start-up demonstration test using a simple scan-based statistic, *Scan Statistics and Applications* (eds. J. Glaz and N. Balakrishnan), 251–267, Birkhäuser, Boston.

Leslie, R. T. (1967). Recurrent composite events, *J. Appl. Probab.*, **4**, 34–61.

Saperstein, B. (1973). On the occurrence of n successes within N Bernoulli trials, *Technometrics*, **15**, 809–818.