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THE ASYMPTOTIC DISTRIBUTION THEORY OF BIVARIATE ORDER STATISTICS

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Abstract. In this paper the limit distribution function (d.f.) of general bivariate order statistics (o.s.) (extreme, intermediate and central) is studied by the notion of the exceedances of levels and characteristic function (c.f.) technique. The advantage of this approach is to give a simple and unified method to derive the limit d.f. of any bivariate o.s. The conditions under which the limit d.f. splits into the product of the limit marginals are obtained. Some illustrative examples are given.

Key words and phrases: Bivariate order statistics, exceedance of level, increasing rank.

1. Introduction

Let $\bar{X}_i = (X_{1i}, X_{2i})$, $i \geq 1$, denote i.i.d. random vectors, with a d.f. $F(\bar{x}) = F(x_1, x_2)$. Write $X_{t,1:n} \leq X_{t,2:n} \leq \cdots \leq X_{t,n:n}$, $t = 1, 2$, for the order marginal of the first n pairs (X_{1i}, X_{2i}). Let $\bar{v}_n = (v_{1n}, v_{2n})$ be a real vector which is regarded as a level (typically each of v_{1n} and v_{2n} becoming higher with n) and define $p_{1:n} = P(E_{1i}) = P(X_{1i} > v_{1n}, X_{2i} \leq v_{2n})$, $p_{2:n} = P(E_{2i}) = P(X_{1i} \leq v_{1n},\ X_{2i} > v_{2n})$, $p_{3:n} = P(E_{3i}) = P(X_{1i} > v_{1n},\ X_{2i} > v_{2n}) = G(\bar{v}_n)$ and $p_{4:n} = P(E_{4i}) = P(X_{1i} \leq v_{1n},\ X_{2i} \leq v_{2n}) = F(\bar{v}_n)$, representing the probabilities in the four quadrants defined by the point $\bar{v}_n$. Choosing $F_i(x_i)$ and $G_i(x_i) = 1 - F_i(x_i)$, $i = 1, 2$ denote the marginals of $F(\bar{x})$ and $G(\bar{x})$, respectively, we get $p_{i:n} = G_i(v_{in}) - G(\bar{v}_n)$, $i = 1, 2$, and $p_{4:n} = 1 - G_1(v_{1n}) - G_2(v_{2n}) + G(\bar{v}_n)$. Furthermore, let $\bar{S}_n = (S_{1n}, S_{2n})$ be the vector of the number of exceedances of $\bar{v}_n$ by $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$, i.e., $S_{jn} = \sum_{i=1}^{n} I_{\{X_{ji} > v_{jn}\}}$, $j = 1, 2$, where I_A denotes the indicator function of the set A. In this paper we investigate properties of $\bar{S}_n$, and as consequences, obtain limiting distributional results for the vector $\bar{Z}_{\bar{k}:n} = \bar{Z}_{k_1,k_2:n} = (X_{1,n-k_1+1:n},\ X_{2,n-k_2+1:n})$, for any integers k_1 and k_2 such that $1 \leq k_1, k_2 \leq n$. For fixed $k_i \geq 1 (i = 1, 2)$, as $n \to \infty$, k_i will be called fixed rank (or the case of extreme o.s.), while if $k_i \to \infty$, as $n \to \infty$, k_i is called increasing rank. Two particular rates of increase are of special interest. The first is called the case of central rank (or the case of central o.s.), in which we have $k_i/n \to \lambda_i \in (0, 1)$, $i = 1, 2$. The second case is called the intermediate rank (or the case of intermediate o.s.), in which we have $k_i \to \infty$, but $k_i/n \to 0$. We shall also use the term k_i-th lower o.s. for $X_{i,k_i:n}$ and k_i-th upper o.s. for $X_{i,n-k_i+1:n}$, $i = 1, 2$. In speaking of a general distribution theory for bivariate o.s. however, the following distinct cases must be considered

(1) (U.e.-U.e.);	(2) (U.e.-L.e.);	(3) (U.i.-U.i.);
(4) (U.i.-L.i.);	(5) (C.-C.);	(6) (U.e.-U.i.);

(7) (U.e.-L.i.); (8) (U.e.-C.); (9) (U.i.-C.),

where the words, upper, lower, extreme, intermediate and central are abbreviated, respectively to U., L., e., i., and C.

In a long and important series of paper over a period of twenty years or so, Finkelstein (1953), Sibuya (1960) and Tiago de Oliveira (1959, 1962, 1965, 1970, 1975 and 1977) have obtained a number of more or less equivalent results on the forms of the limit d.f.'s of the vector $\bar{Z}_{\bar{1}:n} = (X_{1,n:n}, X_{2,n:n})$ (i.e., the case (1)). The necessary and sufficient conditions for the weak convergence of the d.f. of $\bar{Z}_{\bar{1}:n}$ are derived by Galambos (1975), while the domains of attraction of these limit d.f.'s are discussed extensively by Marshall and Olkin (1983) and Takahashi (1994). The asymptotic independence of the components of $\bar{Z}_{\bar{1}:n}$ is studied by Mikhailov (1974) and Galambos (1975). The tail equivalence in the case (1) is discussed by Takahashi (1987). The weak convergence of the d.f.'s of the vectors $\bar{Z}_{\bar{1}:n}$ and $\bar{Z}_{n-k_1,k_2:n}$ (i.e., the cases (1) and (2)) and the vectors $\bar{Z}_{\bar{k}:\nu_n}$, $\bar{Z}_{\nu_n-k_1,k_2:\nu_n}$ are investigated by Barakat (1990, 1997), where ν_n is a discrete positive random variable (r.v.). More survey of recent developments can be found in Galambos (1978, 1987). The work for remaining cases under general conditions is limited. However, the asymptotic normality of the sample quantiles (case (5), when $k_i = \lambda_i n + o(n^{1/2})$, $\lambda_i \in (0,1)$ and $F_i(\xi_i) = \lambda_i$) is proved by Goel and Hall (1994), under the assumptions that the first partial derivatives of the functions $G_i(x_i) - G(\bar{x})$, $i = 1,2; G(\bar{x})$ and $F(\bar{x})$ exist in neighborhoods of (ξ_1, ξ_2) and are continuous at (ξ_1, ξ_2) and that $f_1(\xi_1)$, $f_2(\xi_2)$ are nonzero, where $f_i(x_i) = dF_i(x_i)/dx_i$. This result was proved previously under stronger assumption by Babu and Rao (1988), by using Bahadur's representation of sample quantiles. In this paper the limit d.f. of the bivariate o.s. in each of the cases (1-9) is obtained by using the notion of the exceedances of levels and the c.f. technique. The advantage of this approach is that one can handle all cases (1-9) in one theorem.

Throughout this paper, basic arithmetical operations are always meant componentwise (see Galambos (1978), Chapter 5). We shall also use the notations $\Psi_{\bar{k}:n}(\bar{x}) = P(\bar{Z}_{\bar{k}:n} \le \bar{x})$, $\bar{0} = (0,0)$, $\bar{1} = (1,1)$, $\bar{\infty} = (\infty, \infty)$, $\max(a,b) = a \vee b$ and $\min(a,b) = a \wedge b$. Finally, let $\mathcal{N}_\rho(.)$ denote the bivariate normal standard d.f. with correlation ρ.

In order to obtain the d.f. of $\bar{S}_n$ consider the random vector (W_1, W_2, W_3, W_4), where W_j denotes the number of $E_{j1}, E_{j2}, \ldots, E_{jn}$, $j = 1,2,3,4$, which occur. Clearly the random vector $\bar{W} = (W_1, W_2, W_3)$ has the probability mass function

$$(1.1) \qquad f_{\bar{W}:n}(\bar{w}) = \frac{n!}{w_1! w_2! w_3! (n - w_1 - w_2 - w_3)!} p_{1:n}^{w_1} p_{2:n}^{w_2} p_{3:n}^{w_3} p_{4:n}^{n-w_1-w_2-w_3},$$

where $p_{4:n} = 1 - (p_{1:n} + p_{2:n} + p_{3:n}) > 0$. Moreover, $\bar{S}_n = (W_1 + W_3, W_2 + W_3)$. It can be shown that

$$(1.2) \qquad f_{\bar{S}_n:n}(\bar{s}) = \sum_{r=0\vee(s_1+s_2-n)}^{s_1 \wedge s_2} \frac{n!}{(s_1 - r)! r! (s_2 - r)! (n - s_1 - s_2 + r)!} \cdot p_{1:n}^{s_1-r} p_{3:n}^{r} p_{2:n}^{s_2-r} p_{4:n}^{n-s_1-s_2+r}.$$

On the other hand, the obvious equivalence of the events $\{\bar{Z}_{\bar{k}:n} \le \bar{v}_n\}$ and $\{\bar{S}_n < \bar{k}\}$ leads directly to the relation

$$(1.3) \qquad \Psi_{\bar{k}:n}(\bar{v}_n) = F_{\bar{S}_n}(\bar{k} - \bar{1}),$$

where $F_{\bar{S}_n}(\bar{s}) = P(\bar{S}_n \le \bar{s})$.

Notice that $\bar{W}$ and $\bar{S}_n$ have a four-dimensional multinomial and trivariate multinomial distributions, respectively. This fact, in view of (1.3), reveals a strong relation between the limit behaviour of the general bivariate o.s. and the convergence theorems for the multinomial distribution. For example, in the next section, the central limit theorem for the multinomial distribution is used to obtain the limit results concerning the d.f.'s of the bivariate o.s. with variable ranks. On the other hand, it is worth to mention that many other asymptotic results are related to the limit results of this work. For example, if $\bar{S}_n^\star$ has the d.f. (1.2), with $p_{1:n} = \lambda_{10}/n$, $p_{2:n} = \lambda_{01}/n$, and $p_{3:n} = \lambda_{11}/n$, then by taking limits in (1.2), one can obtain a bivariate Poisson d.f., which corresponds to the d.f. (2.6) (in the next section). Moreover, it is easy to check that r.v.'s U_1, U_2 with this limit d.f. have a representation of the form $U_1 = U_1^\star + U$, $U_2 = U_2^\star + U$, where $U_1^\star$, $U_2^\star$, U are independent and have Poisson distributions with respective parameters λ_{10}, λ_{01}, and λ_{11}. Numerous applications are apparent from this representation. For example, counts of accidents over a period of time in two overlapping districts have bivariate Poisson distribution (see Marshall and Olkin (1985), and references therein).

2. Main Results

The following almost obvious theorem gives the necessary and sufficient conditions under which the components of the vector $\bar{S}_n$ ($\bar{Z}_{\bar{k}:n}$) are stochastically independent for any finite n.

THEOREM 2.1. *For any finite n, the components of the vector $\bar{S}_n$ ($\bar{Z}_{\bar{k}:n}$) are stochastically independent if and only if (iff) $p_{3:n} = p_{13:n}p_{23:n}(G(\bar{v}_n) = G_1(v_{1n})G_2(v_{2n}))$, where $p_{j3:n} = p_{j:n} + p_{3:n}$, $j = 1, 2$.*

Remark 2.1. Clearly, the independence of the r.v.'s X_1 and X_2, where $\bar{X} = (X_1, X_2)$ denotes a generic (X_{1i}, X_{2i}), $i \ge 1$, implies the condition $G(\bar{v}_n) = G_1(v_{1n})G_2(v_{2n})$. However, when n becomes large enough, the necessity of this condition will be relaxed, as we shall see later.

PROOF. The proof can be easily followed from the fact that

$$\bar{S}_n = \sum_{i=1}^{n}(I_{\{X_{1i}>v_{1n}\}}, I_{\{X_{2i}>v_{2n}\}}) = \sum_{i=1}^{n}(Y_{1i}, Y_{2i}), \qquad \text{say.}$$

Hence, the c.f.'s of $\bar{S}_n = (S_{1n}, S_{2n})$ and $\bar{Y} = (Y_1, Y_2)$, where $\bar{Y}$ denotes a generic (Y_{1i}, Y_{2i}), are related by $\phi_{\bar{S}_n}(\bar{t}) = \phi_{\bar{Y}}^n(\bar{t})$. Therefore, S_{1n} and S_{2n} are independent iff Y_1 and Y_2 are independent, which means that the events $\{X_1 > v_{1n}\}$ and $\{X_2 > v_{2n}\}$ are independent. □

The following theorem gives conditions for the convergence of $\Psi_{\bar{k}:n}(\bar{v}_n)$, where $\{\bar{v}_n\}$ is any sequence of real constants, not necessarily of the form $(\bar{a}_n\bar{x} + \bar{b}_n)$, $\bar{a}_n > \bar{0}$ or even dependent on the parameter $\bar{x} = (x_1, x_2)$ and k_1, k_2 are any increasing ranks.

THEOREM 2.2. *Let $(n - k_i) \wedge k_i \to \infty$, $i = 1, 2$, as $n \to \infty$. Furthermore, let $\bar{k}/n \to \bar{\lambda} = (\lambda_1, \lambda_2)$, as $n \to \infty$ such that $\bar{0} \le \bar{\lambda} < \bar{1}$ or $\lambda_1 = 0, \lambda_2 = 1$. If*

$$(2.1) \quad r_{12:n} = \frac{G(\bar{v}_n) - G_1(v_{1n})G_2(v_{2n})}{\sqrt{G_1(v_{1n})G_2(v_{2n})F_1(v_{1n})F_2(v_{2n})}} = \frac{p_{3:n} - p_{13:n}p_{23:n}}{\sqrt{p_{13:n}p_{23:n}q_{13:n}q_{23:n}}} \to \rho_{12},$$

for fixed constant ρ_{12}, $|\rho_{12}| \leq 1$, where $q_{i3:n} = 1 - p_{i3:n}$, $i = 1, 2$, and

$$\frac{k_i - nG_i(v_{in})}{\sqrt{k_i\left(1 - \frac{k_i}{n}\right)}} \to \tau_i, \qquad i = 1, 2 \tag{2.2}$$

for some fixed constants τ_1 and τ_2, or equivalently (as we shall see)

$$\frac{k_i - nG_i(v_{in})}{\sqrt{nG_i(v_{in})F_i(v_{in})}} = \frac{k_i - np_{i3:n}}{\sqrt{np_{i3:n}q_{i3:n}}} \to \tau_i, \qquad i = 1, 2 \tag{2.3}$$

hold, then

$$\Psi_{\bar{k}:n}(\bar{v}_n) \to \mathcal{N}_{\rho_{12}}(\bar{\tau}), \qquad as \quad n \to \infty. \tag{2.4}$$

Conversely, with $\bar{v}_n = \bar{v}_n(\bar{x}) = \bar{a}_n\bar{x} + \bar{b}_n, \bar{\tau}(\bar{x}) = (\tau_1(x_1), \tau_2(x_2))$, where $\bar{a}_n = (a_{1n}, a_{2n}) > \bar{0}$, $\bar{b}_n = (b_{1n}, b_{2n})$ are some vectors of real sequences and τ_1, τ_2 are continuous functions, if (2.4) holds so do (2.1), (2.2) and (2.3).

PROOF. First observe that $E(S_{in}) = np_{i3:n}$, $\text{var}(S_{in}) = np_{i3:n}q_{i3:n}$, $i = 1, 2$, and $\text{cor}(S_{1n}, S_{2n}) = \text{cov}(S_{1n}, S_{2n})/\sqrt{\text{var}(S_{1n})\text{var}(S_{2n})} = (p_{3:n} - p_{13:n}p_{23:n})/\sqrt{p_{13:n}p_{23:n}q_{13:n}q_{23:n}}$. Thus, $r_{12:n}$ is just the correlation of S_{1n} and S_{2n} i.e., $|r_{12:n}| \leq 1$, which in turn yields that $|\rho_{12}| \leq 1$, provided that the limit in (2.1) exists. Next, that (2.2) implies (2.3), for each $i = 1, 2$, follows by writing

$$k_i = nG_i(v_{in}) + \tau_i\sqrt{k_i\left(1 - \frac{k_i}{n}\right)}\,(1 + o(1)), \qquad i = 1, 2$$

and noting that this implies $k_i \sim nG_i(v_{in})$ and $(n - k_i) \sim nF_i(v_{in})$. Similarly, (2.3) implies (2.2). Now, consider $Z_{jn} = (S_{jn} - E(S_{jn}))/\sqrt{\text{var}(S_{jn})}$, $j = 1, 2$. The c.f. of the vector $\bar{Z}_n = (Z_{1n}, Z_{2n})$ is given by

$$\begin{aligned}
\phi_{\bar{Z}_n:n}(\bar{t}) &= E(e^{it_1Z_{1n} + it_2Z_{2n}}) = e^{-it_1\sqrt{np_{13:n}/q_{13:n}} - it_2\sqrt{np_{23:n}/q_{23:n}}} \\
&\quad \cdot (p_{1:n}e^{it_1/\sqrt{np_{13:n}q_{13:n}}} + p_{3:n}e^{it_1/\sqrt{np_{13:n}q_{13:n}} + it_2/\sqrt{np_{23:n}q_{23:n}}} \\
&\qquad + p_{2:n}e^{it_2/\sqrt{np_{23:n}q_{23:n}}} + p_{4:n})^n \\
&= (A_nB_np_{1:n} + A_nC_np_{3:n} + D_nC_np_{2:n} + D_nB_np_{4:n})^n,
\end{aligned}$$

where $\ln A_n = it_1\sqrt{q_{13:n}/np_{13:n}}$, $\ln B_n = -it_2\sqrt{p_{23:n}/nq_{23:n}}$, $\ln C_n = it_2\sqrt{q_{23:n}/np_{23:n}}$ and $\ln D_n = -it_1\sqrt{p_{13:n}/nq_{13:n}}$. Noting that $\phi_{\bar{Z}_n:n}(\bar{t})$ can be written in the form

$$\begin{aligned}
\phi_{\bar{Z}_n:n}(\bar{t}) = (&B_n(A_n - D_n)p_{13:n} + (A_n - D_n)(C_n - B_n)p_{3:n} \\
&+ D_n(C_n - B_n)p_{23:n} + D_nB_n)^n,
\end{aligned}$$

when, $\bar{0} \leq \bar{\lambda} < \bar{1}$, i.e., $p_{i3:n} \to \lambda_i \in [0, 1)$, $q_{i3:n} \to 1 - \lambda_i \in (0, 1]$, $i = 1, 2$, as $n \to \infty$. On the other hand $\phi_{\bar{Z}_n:n}(\bar{t})$ can also be written in the form

$$\begin{aligned}
\phi_{\bar{Z}_n:n}(\bar{t}) = (&B_n(A_n - D_n)p_{13:n} + (A_n - D_n)(C_n - B_n)p_{3:n} \\
&- D_n(C_n - B_n)q_{23:n} + D_nC_n)^n,
\end{aligned}$$

when, $\lambda_1 = 0$ and $\lambda_2 = 1$, i.e., $p_{13:n} \to 0 (q_{13:n} \to 1)$ and $p_{23:n} \to 1 (q_{23:n} \to 0)$, as $n \to \infty$. Therefore, in view of the condition $n(p_{13:n} \wedge p_{23:n}) \to \infty$, as $n \to \infty$, one can easily show that the proof of (2.4) is just an imitation of the proof of the classical central limit theorem by c.f.'s (see e.g., Billingsley (1995), Chapter 5, Section 27). Indeed, after simple calculations, we can show that

$$\phi^{1/n}_{\bar{Z}_{n:n}}(\bar{t}) = 1 - \frac{1}{2n}\left(t_1^2 + t_2^2 + 2t_1t_2r_{12:n}\right) + o\left(\frac{1}{n}\right), \tag{2.5}$$

which in turn, in view of (2.1), (2.3), (1.3) and by virtue of the converse limit theorem of c.f.'s yields (2.4). Hence the first results.

To prove the converse, notice that, the limit in (2.4) will be in the sense of weak convergence. Moreover, by the continuity of the limit $\mathcal{N}_{\rho_{12}}(\bar{\tau}(\bar{x}))$ and on account of Lemma 5.2.1 in Galambos (1978), the marginals of $\Psi_{\bar{k}:n}(\bar{a}_n\bar{x}+\bar{b}_n)$ converge to the corresponding marginals of $\mathcal{N}_{\rho_{12}}(\bar{\tau}(\bar{x}))$. Thus, an application of Theorem 2.5.2, in Leadbetter *et al.* (1983), leads immediately to (2.2) as well as (2.3). On the other hand, if (2.1) does not hold, we can select a subsequence $\{n'\}$ of $\{n\}$ on which (2.3) holds (remember that $|\, r_{12:n} \,| \leq 1$), where the limit ρ_{12} may depend on the actual subsequence $\{n'\}$. Observing, however, that $\mathcal{N}_{\rho'_{12}}(\bar{\tau}(\bar{x})) = \mathcal{N}_{\rho''_{12}}(\bar{\tau}(\bar{x}))$, leads to $\rho'_{12} = \rho''_{12}$, i.e., the limit ρ_{12}, in view of (2.4), cannot depend on $\{n'\}$. □

The tools are now available for developing a general theory for the limit distributions of bivariate o.s. This will be done in the following theorem.

THEOREM 2.3.

Part 1. (U.e.- U.e.) *Let $\bar{k}$ be a vector of fixed ranks. If, as $n \to \infty$, $nG_i(v_{in}) \to \tau_i$, $i = 1, 2$, and $nG(\bar{v}_n) \to \tau_3$ hold for some sequence $\{\bar{v}_n\} = \{(v_{1n}, v_{2n})\}$ and fixed constants τ_1, τ_2 and τ_3 such that $\tau_1 \wedge \tau_2 > 0$, then*

$$\Psi_{\bar{k}:n}(\bar{v}_n) \to \sum_{i=0}^{k_1-1}\sum_{j=0}^{k_2-1}\sum_{r=0}^{i\wedge j}\frac{1}{(i-r)!r!(j-r)!}(\tau_1-\tau_3)^{i-r}\tau_3^r(\tau_2-\tau_3)^{j-r}e^{-\tau_1-\tau_2+\tau_3} \tag{2.6}$$

and $0 \leq \tau_3 < \tau_1 \wedge \tau_2$. Moreover,

$$r_{12:n} = \frac{G(\bar{v}_n) - G_1(v_{1n})G_2(v_{2n})}{\sqrt{G_1(v_{1n})G_2(v_{2n})F_1(v_{1n})F_2(v_{2n})}} \to \rho_{12} = \frac{\tau_3}{\sqrt{\tau_1\tau_2}}, \quad \text{as} \quad n \to \infty.$$

Thus, the limit of $\Psi_{\bar{k}:n}$ splits into the product of the limit marginals iff $\tau_3 = 0$ (i.e., $nG(\bar{v}_n) \to 0$). (Note that $\rho_{12} \geq 0$.)

Part 2. (L.e.- U.e.) *(most of the results of this part are contained in Barakat, 1990) Let $\bar{k}$ be defined as in Part* 1. *If, as $n \to \infty$, $nF_1(u_{1n}) \to l_1$ and $nG_2(v_{2n}) \to \tau_2$ hold, and at least one of the limits $nG(u_{1n}, v_{2n}) \to \tau_3^\star$ and $nF(u_{1n}, v_{2n}) \to l_3^\star$ exists, for some sequence $\{(u_{1n}, v_{2n})\}$ and fixed $0 < l_1, \tau_2, \tau_3^\star, l_3^\star < \infty$, then*

$$\begin{aligned}\Psi_{n-k_1+1,k_2:n}(u_{1n}, v_{2n}) &= P\left(X_{1,k_1:n} \leq u_{1n}, X_{2,n-k_2+1} \leq v_{2n}\right) \\ &\to M_{.,k_2}(\tau_2) - M_{\bar{k}}\left(l_3^\star, l_1 - l_3^\star, \tau_3^\star, \tau_2 + l_3^\star\right) \\ &= M_{.,k_2}(\tau_2) - M_{\bar{k}}\left(l_3^\star, \tau_2 - \tau_3^\star, \tau_3^\star, l_1 + \tau_3^\star\right),\end{aligned} \tag{2.7}$$

where

$$M_{.,k_2}(\tau_2) = \sum_{j=0}^{k_2-1}\frac{1}{j!}\tau_2^j e^{-\tau_2}$$

and

$$M_{\bar{k}}(x_1,x_2,x_3,x_4)=\sum_{i=0}^{k_1-1}\sum_{j=0}^{k_2-1}\sum_{r=0}^{i\wedge j}\frac{1}{(i-r)!r!(j-r)!}x_1^{i-r}x_2^{r}x_3^{j-r}e^{-x_4}.$$

Moreover, $r_{12:n}\to\rho_{12}=-(\tau_2-\tau_3^\star)/\sqrt{l_1\tau_2}=-(l_1-l_3^\star)/\sqrt{l_1\tau_2}$. *Thus, the limit of* $\Psi_{n-k_1+1,k_2:n}$ *splits into the product of the limit marginals iff* $\tau_2=\tau_3^\star(G_2(v_{2n})\sim G(u_{1n},v_{2n}))$ *or equivalently* $l_1=l_3^\star(F_1(u_{1n})\sim F(u_{1n},v_{2n}))$. *(Note that* $\rho_{12}\le 0$.*)*

Part 3. (U.i.- U.i.) *Let* $\bar{k}$ *be such that* $k_1\wedge k_2\to\infty$, *but* $k_i=o(n)$, $i=1,2$, *as* $n\to\infty$. *If, as* $n\to\infty$, $(k_i-nG_i(v_{in}))/\sqrt{k_i}\to\tau_i$, $i=1,2$ *and* $G(\bar{v}_n)/\sqrt{G_1(v_{1n})G_2(v_{2n})}\to\rho_{12}$ *hold for some sequence* $\{\bar{v}_n\}$ *and fixed* ρ_{12}, τ_1, τ_2, *then* $\Psi_{\bar{k}:n}(\bar{v}_n)\to\mathcal{N}_{\rho_{12}}(\bar{\tau})$. *The asymptotic independence occurs iff* $G(\bar{v}_n)/\sqrt{G_1(v_{1n})G_2(v_{2n})}\to 0$ *(note that* $\rho_{12}\ge 0$*).*

Part 4. (L.i.- U.i.) *Let* $\bar{k}$ *be such that* $k_1\wedge(n-k_1)\wedge k_2\to\infty$, *but* $(n-k_1), k_2=o(n)$. *If, as* $n\to\infty$, $(nF_1(u_{1n})-(n-k_1))/\sqrt{n-k_1}\to\tau_1$, $(k_2-nG_2(v_{2n}))/\sqrt{k_2}\to\tau_2$ *and*

$$\frac{G(u_{1n},v_{2n})-G_2(v_{2n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}}=\frac{F(u_{1n},v_{2n})-F_1(u_{1n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}}\to\rho_{12}$$

hold for some sequence $\{(u_{1n},v_{2n})\}$ *and fixed* ρ_{12},τ_1, τ_2, *then* $\Psi_{n-k_1+1,k_2:n}(u_{1n},v_{2n})\to\mathcal{N}_{\rho_{12}}(\bar{\tau})$. *The asymptotic independence occurs iff*

$$\frac{G(u_{1n},v_{2n})-G_2(v_{2n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}}=\frac{F(u_{1n},v_{2n})-F_1(u_{1n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}}\to 0$$

(note that $\rho_{12}\le 0$*).*

Part 5. (C.- C.) *Let* $\bar{k}$ *be such that* $\sqrt{n}(k_i/n-\lambda_i)\to 0$, *as* $n\to\infty$, $i=1,2$, *where* $\lambda_i\in(0,1)$. *If, as* $n\to\infty$, $\sqrt{n}((\lambda_i-G_i(v_{in}))/\sqrt{\lambda_i(1-\lambda_i)})\to\tau_i$, $i=1,2$, *and* $G(\bar{v}_n)\to\lambda_{12}$ *hold for some sequence* $\{\bar{v}_n\}$ *and fixed* λ_{12}, τ_1,τ_2, *then* $\Psi_{\bar{k}:n}(\bar{v}_n)\to\mathcal{N}_{\rho_{12}}(\bar{\tau})$, *where* $\rho_{12}=(\lambda_{12}-\lambda_1\lambda_2)/\sqrt{\lambda_1\lambda_2(1-\lambda_1)(1-\lambda_2)}$. *The asymptotic independence occurs iff* $\lambda_{12}=\lambda_1\lambda_2$,*i.e.,* $G(\bar{v}_n)\sim G_1(v_{1n})G_2(v_{2n})$.

Part 6. (U.e.- U.i.) *Let* k_1 *be fixed, while* $k_2\to\infty$ *and* $k_2=o(n)$, *as* $n\to\infty$. *If,* $nG_1(v_{1n})\to\tau_1$ *and* $(k_2-nG_2(v_{2n}))/\sqrt{k_2}\to\tau_2$, *as* $n\to\infty$, *hold for some sequence* $\{\bar{v}_n\}$ *and fixed* τ_1,τ_2, *then* $\Psi_{\bar{k}:n}(\bar{v}_n)\to M_{k_1,.}(\tau_1)\mathcal{N}(\tau_2)$, *as* $n\to\infty$, *where* $M_{k_1,.}(\tau_1)=\sum_{i=0}^{k_1-1}\frac{1}{i!}\tau_1^ie^{-\tau_1}$ *and* $\mathcal{N}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt$ *is the standard normal d.f.*

Part 7. (U.e.- L.i.) *Let* k_1 *be fixed, while* $k_2\to\infty$ *and* $n-k_2=o(n)$, *as* $n\to\infty$. *If,* $nG_1(v_{1n})\to\tau_1$ *and* $(nF_2(u_{2n})-(n-k_2))/\sqrt{n-k_2}\to\tau_2$, *as* $n\to\infty$, *hold for some sequence* $\{(v_{1n},u_{2n})\}$ *and fixed* τ_1,τ_2, *then* $\Psi_{k_1,n-k_2+1:n}(v_{1n},u_{2n})\to M_{k_1,.}(\tau_1)\mathcal{N}(\tau_2)$, *as* $n\to\infty$.

Part 8. (U.e.- C.) *Let* k_1 *be fixed, while* $\sqrt{n}(k_2/n-\lambda_2)\to 0$, $\lambda_2\in(0,1)$, *as* $n\to\infty$. *If,* $nG_1(v_{1n})\to\tau_1$ *and* $\sqrt{n}(\lambda_2-G_2(v_{2n}))/\sqrt{\lambda_2(1-\lambda_2)}\to\tau_2$, *as* $n\to\infty$, *hold for some sequence* $\{\bar{v}_n\}$ *and fixed* τ_1,τ_2, *then* $\Psi_{\bar{k}:n}(\bar{v}_n)\to M_{k_1,.}(\tau_1)\mathcal{N}(\tau_2)$, *as* $n\to\infty$.

Part 9 (U.i.- C.) *Let* $k_1\to\infty, k_1=o(n)$ *and* $\sqrt{n}(k_2/n-\lambda_2)\to 0,\lambda_2\in(0,1)$, *as* $n\to\infty$. *If,* $(k_1-nG_1(v_{1n}))/\sqrt{k_1}\to\tau_1$ *and* $\sqrt{n}((\lambda_2-G_2(v_{2n}))/(\sqrt{\lambda_2(1-\lambda_2)})\to\tau_2$, *as* $n\to\infty$, *hold for some sequence* $\{\bar{v}_n\}$ *and fixed* τ_1,τ_2, *then* $\Psi_{\bar{k}:n}(\bar{v}_n)\to\mathcal{N}(\tau_1)\mathcal{N}(\tau_2)$, *as* $n\to\infty$.

PROOF. Parts 3–5 and 9 follow immediately from Theorem 2.2, in view of the following facts:

(1) In Part 3, we have $G_1(v_{1n}) \vee G_2(v_{2n}) \to 0$, as $n \to \infty$, then $r_{12:n} \sim G(\bar{v}_n)/\sqrt{G_1(v_{1n})G_2(v_{2n})}$.

(2) In Part 4, we have $F_1(u_{1n}) \vee G_2(v_{2n}) \to 0$, then

$$r_{12:n} \sim \frac{G(u_{1n}, v_{2n}) - G_1(u_{1n})G_2(v_{2n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}} = \frac{G(u_{1n}, v_{2n}) - G_2(v_{2n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}} + \sqrt{F_1(u_{1n})G_2(v_{2n})}$$

which leads to

$$\lim_{n\to\infty} r_{12:n} = \lim_{n\to\infty} \frac{G(u_{1n}, v_{2n}) - G_2(v_{2n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}} = \lim_{n\to\infty} \frac{F(u_{1n}, v_{2n}) - F_1(u_{2n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}}.$$

(3) In Part 5, we have $G_i(v_{in}) \sim \lambda_i \in (0,1), i = 1,2$.

(4) In Part 9, we have $G(\bar{v}_n) \leq G_1(v_{1n}) \to 0$, as $n \to \infty$ and $G_2(v_{2n}) \sim \lambda_2 \in (0,1)$, then

$$\begin{aligned}|r_{12:n}| &\sim \frac{|G(\bar{v}_n) - G_1(v_{1n})G_2(v_{2n})|}{\sqrt{G_1(v_{1n})\lambda_2(1-\lambda_2)}} \\ &\leq \frac{1}{\sqrt{\lambda_2(1-\lambda_2)}}(\sqrt{G(\bar{v}_n)} + G_2(v_{2n})\sqrt{G_1(v_{1n})}) \to 0.\end{aligned}$$

In order to prove Parts 6 and 8, notice that $G(\bar{v}_n) = p_{3:n} \leq p_{13:n} = G_1(v_{1n}) \sim \tau_1/n$ and $p_{23:n} = G_2(v_{2n}) \sim \lambda(np_{23:n} \sim k_2 \to \infty)$, where $\lambda = 0$ in Part 6 and $\lambda = \lambda_2 \in (0,1)$ in Part 8. On the other hand, we can write the c.f. of the vector (S_{1n}, Z_{2n}) in the form

$$\begin{aligned}\phi_{S_{1n},Z_{2n}}(\bar{t}) &= E(e^{it_1 S_{1n} + it_2 Z_{2n}}) \\ &= e^{-it_2\sqrt{np_{23:n}/q_{23:n}}}\phi_{W_1,W_2,W_3}\left(t_1, \frac{t_2}{\sqrt{np_{23:n}q_{23:n}}}, t_1 + \frac{t_2}{\sqrt{np_{23:n}q_{23:n}}}\right) \\ &= (B_n(e^{it_1}-1)p_{13:n} + (C_n - B_n)(e^{it_1}-1)p_{3:n} + (C_n - B_n)p_{23:n} + B_n)^n,\end{aligned}$$

where B_n and C_n are defined in Theorem 2.2. Now, we can easy show that, $B_n p_{13:n} \sim \tau_1/n; (C_n - B_n)p_{3:n} = it_2 p_{3:n}/\sqrt{np_{23:n}q_{23:n}} + o(\frac{1}{n}) = o(\frac{1}{n})$(since, $p_{3:n} \leq p_{13:n} \sim \tau_1/n$);

$$(C_n - B_n)p_{23:n} = \frac{it_2}{\sqrt{n}}\sqrt{\frac{p_{23:n}}{q_{23:n}}} + \frac{t_2^2}{2n}\frac{(p_{23:n} - q_{23:n})}{q_{23:n}} + o\left(\frac{1}{n}\right) \quad \textit{and}$$

$$B_n = 1 - \frac{it_2}{\sqrt{n}}\sqrt{\frac{p_{23:n}}{q_{23:n}}} - \frac{t_2^2}{2n}\frac{p_{23:n}}{q_{23:n}} + o\left(\frac{1}{n}\right).$$

Therefore,

$$\begin{aligned}\phi_{S_{1n},Z_{2n}}(\bar{t}) &= \left(1 + (e^{it_1}-1)\frac{\tau_1}{n} - \frac{t_2^2}{2n} + o\left(\frac{1}{n}\right)\right)^n \\ &\to \exp\left((e^{it_2}-1)\tau_1 - \frac{t_2^2}{2}\right) = \exp\left(-\frac{t_2^2}{2}\right)\exp\left(\tau_1(e^{it_1}-1)\right).\end{aligned}$$

This completes the proof of Parts 6 and 8. The proof of Part 7 is similar to the proof of Parts 6 and 8 (with only the obvious changes) and for brevity it will be omitted.

Turning to the proof of Parts 1 and 2. In Part 1 we can write the c.f. of $\bar{S}_n$ as

$$\begin{aligned}\phi_{\bar{S}_n}(\bar{t}) &= E(e^{it_1 S_{1n}+it_2 S_{2n}})\\ &= (1+(e^{it_1}-1)p_{13:n}+(e^{it_1}-1)(e^{it_2}-1)p_{3:n}+(e^{it_2}-1)p_{23:n})^n.\end{aligned}$$

After noting that $p_{13:n}=G_1(v_{1n})\sim\tau_1/n, p_{23:n}=G_2(v_{2n})\sim\tau_2/n$, and $p_{3:n}=G(\bar{v}_n)\sim\tau_3/n$, we get, as $n\to\infty$

$$\phi_{\bar{S}_n}(\bar{t})\to\exp((e^{it_1}-1)\tau_1+(e^{it_2}-1)\tau_2+(e^{it_1}-1)(e^{it_2}-1)\tau_3), \tag{2.8}$$

which is the c.f. of the limit d.f. in (2.6). It is easy to show that the correlation coefficient of this limit d.f. is given by $\tau_3/\sqrt{\tau_1\tau_2}$. Since, Parts 1 and 2 are related by the simple transformation $(X_1,X_2)\to(-X_1,X_2)$, Part 2 can be easily proved as follows: Let $\Psi_{\bar{k}:n}(\bar{x})=H_{\bar{k}:n}(G_1(x_1),G_2(x_2),G(\bar{x}))$, say. Then $\Psi_{n-k_1+1,k_2:n}(\bar{x})=P(X_{1,k_1:n}\le x_1,X_{2,n-k_2+1:n}\le x_2)=P(-X^\star_{1,n-k_1+1:n}\le x_1,X_{2,n-k_2+1:n}\le x_2)=P(X_{2,n-k_2+1:n}\le x_2)-P(X^\star_{1,n-k_1+1:n}<-x_1,X_{2,n-k_2+1:n}\le x_2)$, where $X^\star_{1,n-k_1+1:n}=(n-k_1+1)$-th largest of $(-X_{11},-X_{12},\ldots,-X_{1n})$. Hence, $\Psi_{n-k_1+1,k_2:n}(\bar{x})=\Psi_{\cdot,k_2:n}(x_2)-H_{\bar{k}:n}(F_1(x_1),G_2(x_2),F_1(x_1)-F(\bar{x}))=\Psi_{\cdot,k_2:n}(x_2)-H_{\bar{k}:n}(F_1(x_1),G_2(x_2),G_2(x_2)-G(\bar{x}))$, where $\Psi_{\cdot,k_2:n}(x_2)$ as usual is the marginal d.f. of $\Psi_{\bar{k}:n}(\bar{x})$. □

Remark 2.2. By using a linear parametrization in Theorem 2.3, by using identifications $\bar{v}_n=\bar{a}_n\bar{x}+\bar{b}_n$, $\bar{u}_n=\bar{\alpha}_n\bar{x}+\bar{\beta}_n$, $\bar{a}_n$, $\bar{\alpha}_n>\bar{0}$; $\tau_i=\tau_i(x_i)$, $l_i=l_i(x_i)$, $i=1,2$; $\tau_3^\star=\tau_3^\star(\bar{x})$ and $l_3^\star=l_3^\star(\bar{x})$ we see that the limit distributional types of the U.e. (L.e.) and U.i. (L.i.), in contrast to central o.s., are all continuous (see Leadbetter *et al.* (1983)). However, the asymptotic d.f.'s of the central o.s. are normal given weak conditions. Therefore, by proceeding as we did in the second part of Theorem 2.2, we can show that all sufficient conditions for the convergence of the d.f. of bivariate o.s. which are given in Parts 1–9 of Theorem 2.3 are also necessary, when we use the preceding linear parametrization and the limit distributional types are continuous.

Remark 2.3. By virtue of Theorems 2.1 and 2.3, we conclude that for any n (finite or infinite) the marginal d.f's of any bivariate o.s. are independent iff they are uncorrelated.

3. Examples

For all the following examples, we assume that the marginals of the d.f. of considered bivariate o.s. converge.

Example 3.1. (The Morgenstern distribution) For this d.f. we have $G(\bar{x})=G_1(x_1)G_2(x_2)(1+\alpha F_1(x_1)F_2(x_2))$, where $-1\le\alpha\le1$. Thus, we can show that, $nG(\overline{v_n})\to0$, in Part 1; $nG(u_{1n},v_{2n})\to\tau_2$, in Part 2; $\frac{G(\overline{v_n})}{\sqrt{G_1(v_{1n})G_2(v_{2n})}}=\sqrt{G_1(v_{1n})G_2(v_{2n})}(1+\alpha F_1(v_{1n})F_2(v_{2n}))\to0$, in Part 3; $(G(u_{1n},v_{2n})-G_2(v_{2n}))/\sqrt{F_1(u_{1n})G_2(v_{2n})}=-\sqrt{F_1(u_{1n})G_2(v_{2n})}(1-\alpha G_1(u_{1n})F_2(v_{2n}))\to0$, in Part 4 and $G(\overline{v_n})\to\lambda_1\lambda_2(1+\alpha(1-\lambda_1)(1-\lambda_2))$, i.e., $\rho_{12}=\alpha\sqrt{\lambda_1\lambda_2(1-\lambda_1)(1-\lambda_2)}$, in Part 5. Therefore, in all Parts 1–9 of Theorem 2.3 the d.f. of bivariate o.s. converges, whenever its marginals converge. Moreover, if $\alpha=0$, in Part 5, and in all remaining parts, the limit d.f. splits into the product of the limit marginals.

Example 3.2. (Mardia's distribution) For this d.f. we have $G(\bar{x}) = G_1(x_1)G_2(x_2)/(G_1(x_1) + G_2(x_2) - G_1(x_1)G_2(x_2))$. Thus, we can show that, $nG(\overline{v_n}) \to \tau_1\tau_2/(\tau_1+\tau_2)$, i.e., $\rho_{12} = \sqrt{\tau_1\tau_2}/(\tau_1+\tau_2)$, in Part 1; $nG(u_{1n}, v_{2n}) \to \tau_2$, in Part 2;

$$\frac{G(\overline{v_n})}{\sqrt{G_1(v_{1n})G_2(v_{2n})}} = \left(\sqrt{\frac{G_2(v_{2n})}{G_1(v_{1n})}} + \sqrt{\frac{G_1(v_{1n})}{G_2(v_{2n})}} - \sqrt{G_1(v_{1n})G_2(v_{2n})}\right)^{-1}$$
$$\to \begin{cases} \dfrac{\sqrt{c}}{1+c}, & \text{if } \dfrac{k_1}{k_2} \to c \in (0,\infty), \\ 0, & \text{if } \frac{k_1}{k_2} \to c = 0 \quad \text{or} \quad \infty; \end{cases}$$

in Part 3;

$$\frac{G(u_{1n}, v_{2n}) - G_2(v_{2n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}} = -\frac{G_2(v_{2n})\sqrt{F_1(u_{1n})G_2(v_{2n})}}{G_1(u_{1n}) + G_2(v_{2n}) - G_1(u_{1n})G_2(v_{2n})} \to 0,$$

in Part 4 and $G(\overline{v_n}) \to \lambda_1\lambda_2/(\lambda_1+\lambda_2-\lambda_1\lambda_2)$, i.e., $\rho_{12} = \sqrt{\lambda_1\lambda_2(1-\lambda_1)(1-\lambda_2)}/(\lambda_1+\lambda_2-\lambda_1\lambda_2)$, in Part 5. Therefore, in Part 3, when $k_1/k_2 \to c \in [0,\infty]$ and in all remaining parts, the d.f. of bivariate o.s. converges if the marginals converge. Moreover, in Part 3, if $k_1/k_2 \to 0$ or ∞, and in all remaining parts except Parts 1 and 5, the limit d.f. splits into the product of the limit marginals.

Example 3.3 (Gumbel's type I exponential distribution) For this distribution we have $G(\bar{x}) = e^{-x_1-x_2-\theta x_1 x_2}$, $0 \le \theta < 1$, $x_1, x_2 > 0$, i.e., we have $G(\bar{x}) = G_1(x_1)G_2(x_2)\, e^{-\theta \ln G_1(x_1)\ln G_2(x_2)}$. Thus, $nG(\overline{v_n}) \sim \tau_1 G_2(v_{2n})e^{-\theta \ln G_1(v_{1n})\ln G_2(v_{2n})} \sim \tau_2 G_1(v_{1n})\, e^{-\theta \ln G_1(v_{1n})\ln G_2(v_{2n})} \to 0$, in Part 1; $(nG_2(v_{2n}))G_1(u_{1n})e^{-\theta(l_1/n)\ln n} \sim nG_2(v_{2n}) \to \tau_2$ (note that $G_1(u_{1n}) \sim 1 - l_1/n$, $G_2(v_{2n}) \sim \tau_2/n$), in Part 2; $G(\overline{v_n})/\sqrt{G_1(v_{1n})G_2(v_{2n})} = \sqrt{G_1(v_{1n})G_2(v_{2n})}\; e^{-\theta \ln G_1(v_{1n})\ln G_2(v_{2n})} \to 0$ (note that $G_i(v_{in}) \sim k_i/n \to 0, i = 1,2$), in Part 3;

$$e^{-\theta \ln G_1(u_{1n})\ln G_2(v_{2n})} = e^{-\theta c}(1+o(1)), \quad \text{if} \quad \left(1-\frac{k_1}{n}\right)\ln\frac{k_2}{n} \to -c \le 0,$$

i.e.,

$$\frac{G(u_{1n}, v_{2n}) - G_2(v_{2n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}} \sim \sqrt{F_1(u_{1n})G_2(v_{2n})} \to 0, \quad \text{if} \quad c = 0;$$

$$\frac{G(u_{1n}, v_{2n}) - G_2(v_{2n})}{\sqrt{F_1(u_{1n})G_2(v_{2n})}} \sim -\sqrt{\frac{k_2}{n}\left(1-\frac{k_1}{n}\right)^{-1}}\, e^{-\theta c} \to 0, \quad \text{if} \quad 0 < c < \infty,$$

(note that $(k_2/n)(1-k_1/n)^{-1} \sim (1-k_1/n)^{-1}\, e^{-c(1-k_1/n)^{-1}} \to 0$, if $0 < c < \infty$), in Part 4 and $G(\overline{v_n}) \to \lambda_1\lambda_2 e^{-\theta \ln\lambda_1 \ln\lambda_2}$, i.e., $\rho_{12} = -\sqrt{\lambda_1\lambda_2/((1-\lambda_1)(1-\lambda_2))}\,(1-e^{-\theta\ln\lambda_1\ln\lambda_2}) \le 0$, in Part 5. Therefore, in Part 4, when $(1-k_1/n)\ln k_2/n \to -c, 0 \le c < \infty$ and in all remaining parts, the d.f. of bivariate o.s. converges if the marginals converge. Moreover, in Parts 4 and 5 if $c = 0$ and $\theta = 0$, respectively and in all remaining parts the limit d.f. splits into the product of the limit marginals.

Example 3.4. (Standard bivariate normal distribution, with correlation ρ) In this example we need the following facts (for the first fact, see Example 5.3.1 of Galambos (1978) and for the second fact, see Esary *et al.* (1967) and Joag-Dev and Proschan

(1983)):

$$\frac{G(\bar{x})}{G_i(x_i)} \to 0, \quad i = 1, 2, \quad \text{as } \bar{x} \to \bar{\infty} \left(\frac{F(\bar{x})}{F_i(x_i)} \to 0, \text{ as } \bar{x} \to -\bar{\infty} \right)$$

and $F(\bar{x}) \geq F_1(x_1)F_2(x_2)(G(\bar{x}) \geq G_1(x_1)G_2(x_2))$ (i.e., positively quadrant dependent), if $\rho \geq 0$. Thus we can show that, $nG(\overline{v_n}) \sim \tau_1(G(\overline{v_n})/G_1(v_{1n})) \sim \tau_2(G(\overline{v_n})/G_2(v_{2n})) \to 0$, in Part 1; $1 \geq F(u_{1n}, v_{2n})/F_1(u_{1n}) \geq F_2(v_{2n}) \to 1$, i.e.,$nF(u_{1n}, v_{2n}) \sim nF_1(u_{1n}) \to l_1$, if $\rho \geq 0$, in Part 2; $G(\overline{v_n})/\sqrt{G_1(v_{1n})G_2(v_{2n})} = \sqrt{G(\overline{v_n})/G_1(v_{1n})}\sqrt{G(\overline{v_n})/G_2(v_{2n})} \to 0$, in Part 3; $0 \geq (G(u_{1n}, v_{2n}) - G_2(v_{2n}))/\sqrt{F_1(u_{1n})G_2(v_{2n})} \geq -\sqrt{F_1(u_{1n})G_2(v_{2n})} \to 0$, if $\rho \geq 0$, in Part 4 and $G(\overline{v_n}) \to \lambda_{12}$, where $G(\bar{\mu}) = \lambda_{12}$, $\bar{\mu} = (\mu_1, \mu_2)$ and $G_i(\mu_i) = \lambda_i(\lambda_{12} = \lambda_1\lambda_2$, if $\rho = 0)$, in Part 5. Therefore, in Parts 2 and 4, if $\rho \geq 0$ and in all remaining parts the d.f. of bivariate o.s. converges, whenever its marginals converge. Moreover, in Parts 2 and 4, if $\rho \geq 0$, in Part 5, if $\rho = 0$ and in all remaining parts the limit d.f. splits into the product of the limit marginals.

Example 3.5. (The correlation of the marginal ranges) Many authors have considered the correlation of the marginal ranges $R_{1:n}$, $R_{2:n}$, where $R_{i:n} = X_{i,n:n} - X_{i,1:n}$, $i = 1, 2$ for samples from a standardized bivariate normal distribution, e.g., Hartley (1950), Smith and Hartley (1968) and Barnett (1976). For any d.f. $F(\bar{x})$ and real sequences $\{\bar{u}_n\} = \{(u_{1n}, u_{2n})\}$, $\{\bar{v}_n\} = \{(v_{1n}, v_{2n})\}$, if $nG_i(v_{in}) \to \tau_i$, $nF_i(u_{in}) \to l_i, i = 1, 2$, $nG(\bar{v}_n) \to \tau_3, nF(\bar{u}_n) \to l_3, nF(u_{1n}, v_{2n}) \to l_3^\star$ and $nG(u_{1n}, v_{2n}) \to \tau_3^\star$, we can show that

$$\lim_{n\to\infty} \operatorname{cor}(R_{1:n}, R_{2:n}) = \frac{\tau_3}{\sqrt{\tau_1\tau_2}} + \frac{l_3}{\sqrt{l_1 l_2}} + \frac{\tau_2 - \tau_3^\star}{\sqrt{\tau_2 l_1}} + \frac{l_1 - l_3^\star}{\sqrt{l_2 \tau_1}}.$$

(Note that $\tau_2 - \tau_3^\star = l_1 - l_3^\star$.) Therefore, in Examples 3.1, 3.2, 3.3 and 3.4 we have, respectively $\lim_{n\to\infty} \operatorname{cor}(R_1, R_2) =$ zero, $\sqrt{\tau_1\tau_2}/(\tau_1 + \tau_2) + \sqrt{l_1 l_2}/(l_1 + l_2)$, zero and zero if $\rho \geq 0$.

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