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INFORMATION INEQUALITIES IN A FAMILY OF UNIFORM DISTRIBUTIONS

MASAFUMI AKAHIRA[1] AND KEI TAKEUCHI[2]

[1]*Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan*
[2]*Faculty of International Studies, Meiji-Gakuin University, Kamikurata-cho 1598, Totsuka-ku, Yokohama 244-0816, Japan*

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Abstract. For a family of uniform distributions, it is shown that for any small $\varepsilon > 0$ the average mean squared error (MSE) of any estimator in the interval of θ values of length ε and centered at θ_0 can not be smaller than that of the midrange up to the order $o(n^{-2})$ as the size n of sample tends to infinity. The asymptotic lower bound for the average MSE is also shown to be sharp.

Key words and phrases: Best location equivariant estimator, average mean squared error, sufficient statistic.

1. Introduction

Estimation of the mean θ of the uniform distribution with known range, which may be assumed to be equal to 1, is simple but a typical case of non-regular estimation. It is known that the variance of the locally best unbiased estimator at any $\theta = \theta_0$ is equal to zero even when the sample size is equal to one (see, e.g. Akahira and Takeuchi (1995)), while the best location equivariant estimator

$$\hat{\theta}^* = \frac{1}{2}\left(\min_{1\le i\le n} X_i + \max_{1\le i\le n} X_i\right)$$

has the variance equal to $1/\{2(n+1)(n+2)\}$ for all θ. There are several ways to construct unbiased estimators with zero variance at a specified value $\theta = \theta_0$, but they all have variance larger than that of $\hat{\theta}^*$ for other values of θ, and when the size of sample is large, even for values arbitrarily close to θ. The purpose of this paper is to show that for any small $\varepsilon > 0$ the average mean squared error of any estimator $\hat{\theta}$ in the interval of θ values of length ε and centered at θ_0 can not be smaller than that of $\hat{\theta}^*$. More precisely we shall prove that for any estimator $\hat{\theta}$ based on the sample of size n

$$\underline{\lim}_{\varepsilon\to 0}\ \underline{\lim}_{n\to\infty} \frac{1}{\varepsilon}\int_{\theta_0-\varepsilon/2}^{\theta_0+\varepsilon/2} n^2 E_\theta[(\hat{\theta}-\theta)^2]d\theta \ge \frac{1}{2}.$$

This means that in a sense asymptotically $\hat{\theta}^*$ can be regarded as uniformly best.

The result can be generalized to the case of estimation of the unknown location parameter θ with the density $f(x-\theta)$, where f has the following conditions:

(i) $f(x) > 0$ for $a < x < b$, $f(x) = 0$ otherwise.

(ii) $\lim_{x\to a+0} f(x) = \lim_{x\to b-0} f(x) = A > 0$.

(iii) f is continuously differentiable in the interval (a, b).

Indeed, it is derived from the fact that the estimator which minimizes

$$\int_{\theta_0-\varepsilon/2}^{\theta_0+\varepsilon/2} E_\theta[(\hat{\theta}-\theta)^2]d\theta$$

is the Bayes estimator with respect to the uniform prior over the interval $[\theta_0-(\varepsilon/2), \theta_0+(\varepsilon/2)]$, and that is asymptotically equivalent to the one with respect to the prior over the entire interval and asymptotically equal to the estimator $\hat{\theta}^* = (\min_{1\le i\le n} X_i + \max_{1\le i\le n} X_i)/2$. The logic of the proof is nearly the same as is shown, in this paper, from the rectangular distribution.

The related results to the above are found in Vincze (1979), Khatri (1980) and Móri (1983). In particular, the Móri type inequality is shown to be derived from the information inequality in this paper.

2. Information inequalities

Suppose that X be distributed uniformly over the interval $[\theta-(1/2), \theta+(1/2)]$. Let $\hat{\theta}(X)$ be an unbiased estimator of θ, i.e.

$$\int_{\theta-1/2}^{\theta+1/2} \hat{\theta}(x)dx = \theta \quad \text{for all} \quad \theta \in \mathbb{R}.$$

Denote the variance of $\hat{\theta}$ by

$$v(\theta) := V_\theta\big(\hat{\theta}(X)\big) = \int_{\theta-1/2}^{\theta+1/2} \{\hat{\theta}(x)-\theta\}^2 dx. \tag{2.1}$$

Now we have the following.

THEOREM 2.1. *For any* $\theta \in \mathbb{R}$

$$\int_{\theta-1/2}^{\theta+1/2} v(t)dt = \int_{-1/2}^{1/2} v(t)dt = \int_{-1/2}^{1/2} \{\hat{\theta}(x)-x\}^2 dx + \int_{-1/2}^{1/2} x^2 dx \ge V_\theta(X) = \frac{1}{12}.$$

PROOF. Denote

$$\psi(x) := \hat{\theta}(x) - x.$$

Then $\psi(x)$ is a periodic function with periodicity 1, i.e. $\psi(x+1) = \psi(x)$ for almost all x. Indeed, since

$$\int_{\theta-1/2}^{\theta+1/2} \psi(x)dx = 0 \quad \text{for all} \quad \theta \in \mathbb{R},$$

by differentiation we have

$$\psi\left(\theta+\frac{1}{2}\right) - \psi\left(\theta-\frac{1}{2}\right) = 0 \quad \text{a.e.}$$

This shows the periodicity of $\psi(x)$.

Now we have

$$v(\theta) = \int_{\theta-1/2}^{\theta+1/2} \{\psi(x) + x - \theta\}^2 dx \tag{2.2}$$
$$= \int_{\theta-1/2}^{\theta+1/2} \psi^2(x)dx + 2\int_{\theta-1/2}^{\theta+1/2} (x-\theta)\psi(x)dx + \int_{\theta-1/2}^{\theta+1/2} (x-\theta)^2 dx.$$

Since ψ is a periodic function, if we express θ by $n+p$ with an integer n and $0 \le p < 1$, we have

$$\int_{\theta-1/2}^{\theta+1/2} \psi^2(x)dx = \int_{p-1/2}^{p+1/2} \psi^2(x)dx = \int_{p-1/2}^{1/2} \psi^2(x)dx + \int_{1/2}^{p+1/2} \psi^2(x)dx \tag{2.3}$$
$$= \int_{p-1/2}^{1/2} \psi^2(x)dx + \int_{-1/2}^{p-1/2} \psi^2(x)dx = \int_{-1/2}^{1/2} \psi^2(x)dx$$

and

$$\int_{\theta-1/2}^{\theta+1/2} (x-\theta)\psi(x)dx = \int_{p-1/2}^{p+1/2} (x-p)\psi(x)dx = \int_{p-1/2}^{p+1/2} x\psi(x)dx \tag{2.4}$$
$$= \int_{p-1/2}^{1/2} x\psi(x)dx + \int_{1/2}^{p+1/2} x\psi(x)dx$$
$$= \int_{p-1/2}^{1/2} x\psi(x)dx + \int_{-1/2}^{p-1/2} (x+1)\psi(x)dx$$
$$= \int_{-1/2}^{1/2} x\psi(x)dx + \int_{-1/2}^{p-1/2} \psi(x)dx.$$

From (2.2), (2.3) and (2.4) we have

$$v(\theta) = v(p) = \int_{-1/2}^{1/2} \psi^2(x)dx + \int_{-1/2}^{1/2} x^2 dx + 2\left\{\int_{-1/2}^{1/2} x\psi(x)dx + \int_{-1/2}^{p-1/2} \psi(x)dx\right\}.$$

Therefore, if we can prove that

$$\int_0^1 \left\{\int_{-1/2}^{1/2} x\psi(x)dx + \int_{-1/2}^{p-1/2} \psi(x)dx\right\} dp = 0, \tag{2.5}$$

then the theorem is established. We denote

$$\Psi(p) := \int_{-1/2}^{p-1/2} \psi(x)dx. \tag{2.6}$$

Then we have $\Psi(0) = \Psi(1) = 0$, and it is shown that

$$\int_0^1 \left\{\int_{-1/2}^{1/2} x\psi(x)dx\right\} dp = \int_{-1/2}^{1/2} x\psi(x)dx \tag{2.7}$$
$$= \int_0^1 \left(p - \frac{1}{2}\right)\psi\left(p - \frac{1}{2}\right) dp = \int_0^1 (p-1)\Psi'(p)dp$$
$$= -\int_0^1 \Psi(p)dp.$$

From (2.6) and (2.7) we get (2.5). □

For estimators not necessarily unbiased, we have the following.

THEOREM 2.2. *Let $M(\theta)$ be the mean squared error (MSE) of an estimator $\hat{\theta}(X)$ of θ, i.e.*

$$M(\theta) := \int_{\theta-1/2}^{\theta+1/2} \{\hat{\theta}(x) - \theta\}^2 dx.$$

Then for any $\theta_0 \in \mathbb{R}$

$$\int_{\theta_0-\varepsilon/2}^{\theta_0+\varepsilon/2} M(\theta)d\theta \geq \begin{cases} \frac{1}{12}\left(\varepsilon - \frac{1}{2}\right) & \textit{for} \quad \varepsilon > 1, \\ \frac{\varepsilon^3}{12}\left(1 - \frac{\varepsilon}{2}\right) & \textit{for} \quad \varepsilon \leq 1. \end{cases} \tag{2.8}$$

PROOF. We assume $\theta_0 = 0$ without loss of generality. Then we have for $\varepsilon > 1$

$$\begin{aligned}
(2.9) \quad & \int_{-\varepsilon/2}^{\varepsilon/2} M(\theta)d\theta \\
&= \int_{-\varepsilon/2}^{\varepsilon/2} d\theta \int_{\theta-1/2}^{\theta+1/2} \{\hat{\theta}(x) - \theta\}^2 dx \\
&= \int_{-\varepsilon/2}^{\varepsilon/2} d\theta \int_{\theta-1/2}^{\theta+1/2} \{\hat{\theta}^2(x) - 2\theta\hat{\theta}(x) + \theta^2\} dx \\
&= \iint_{\{(x,y)||x-y|<1/2,\, 0<x<(\varepsilon/2)+(1/2)\}} \{\hat{\theta}^2(x) - 2y\hat{\theta}(x)\} dxdy \\
&\quad + \iint_{\{(x,y)||x-y|<1/2, -(\varepsilon/2)-(1/2)<x<0\}} \{\hat{\theta}^2(x) - 2y\hat{\theta}(x)\} dxdy + \frac{\varepsilon^3}{12} \\
&=: I_1 + I_2 + \frac{\varepsilon^3}{12},
\end{aligned}$$

where

$$I_1 = \iint_{\{(x,y)||x-y|<1/2, 0<x<(\varepsilon/2)+(1/2)\}} \{\hat{\theta}^2(x) - 2y\hat{\theta}(x)\} dydx,$$

$$I_2 = \iint_{\{(x,y)||x-y|<1/2, -(\varepsilon/2)-(1/2)<x<0\}} \{\hat{\theta}^2(x) - 2y\hat{\theta}(x)\} dydx.$$

Then we have

$$\begin{aligned}
(2.10) \quad I_1 &= \left(\int_0^{\varepsilon/2-1/2} \int_{x-1/2}^{x+1/2} + \int_{\varepsilon/2-1/2}^{\varepsilon/2+1/2} \int_{x-1/2}^{\varepsilon/2}\right) \{\hat{\theta}^2(x) - 2y\hat{\theta}(x)\} dydx \\
&= \int_0^{\varepsilon/2-1/2} \{\hat{\theta}^2(x) - 2x\hat{\theta}(x)\} dx \\
&\quad + \int_{\varepsilon/2-1/2}^{\varepsilon/2+1/2} \left\{\left(\frac{\varepsilon}{2} - x + \frac{1}{2}\right)\hat{\theta}^2(x) - \left(\frac{\varepsilon}{2} - x + \frac{1}{2}\right)\left(\frac{\varepsilon}{2} + x - \frac{1}{2}\right)\hat{\theta}(x)\right\} dx
\end{aligned}$$

$$\begin{aligned}
&\geq -\int_0^{\varepsilon/2-1/2} x^2 dx - \int_{\varepsilon/2-1/2}^{\varepsilon/2+1/2} \frac{1}{4}\left(\frac{\varepsilon}{2} - x + \frac{1}{2}\right)\left(\frac{\varepsilon}{2} + x - \frac{1}{2}\right)^2 dx \\
&= -\frac{1}{24}(\varepsilon-1)^3 - \frac{1}{48}(6\varepsilon^2 - 8\varepsilon + 3) \\
&= \frac{1}{48}(-2\varepsilon^3 + 2\varepsilon - 1),
\end{aligned}$$

where the equality holds for

$$\hat{\theta}(x) = \begin{cases} x & \text{for } \ 0 < x \leq \dfrac{\varepsilon}{2} - \dfrac{1}{2}, \\ \dfrac{1}{2}\left(\dfrac{\varepsilon}{2} + x - \dfrac{1}{2}\right) & \text{for } \ \dfrac{\varepsilon}{2} - \dfrac{1}{2} < x \leq \dfrac{\varepsilon}{2} + \dfrac{1}{2}. \end{cases}$$

Similarly we obtain

$$I_2 \geq \frac{1}{48}(-2\varepsilon^3 + 2\varepsilon - 1). \tag{2.11}$$

From (2.9), (2.10) and (2.11) we have the inequality (2.8) for $\varepsilon > 1$. For $\varepsilon \leq 1$, we have

$$\begin{aligned}
\int_{-\varepsilon/2}^{\varepsilon/2} M(\theta) d\theta &= \iint_{\{(x,\theta) \mid |x-\theta|<1/2, |\theta|<\varepsilon/2\}} \{\hat{\theta}^2(x) - 2\theta\hat{\theta}(x)\} dx d\theta \\
&= I_1' + I_2' + \frac{\varepsilon^3}{12},
\end{aligned} \tag{2.12}$$

where I_1' and I_2' denote I_1 and I_2, respectively. Then we obtain

$$\begin{aligned}
I_1' &= \int_0^{\varepsilon/2} \int_{\theta-1/2}^{\theta+1/2} \{\hat{\theta}^2(x) - 2\theta\hat{\theta}(x)\} dx d\theta \\
&= \left(\int_0^{-\varepsilon/2+1/2} \int_{-\varepsilon/2}^{\varepsilon/2} + \int_{-\varepsilon/2+1/2}^{\varepsilon/2+1/2} \int_{x-1/2}^{\varepsilon/2} \right) \{\hat{\theta}^2(x) - 2y\hat{\theta}(x)\} dy dx \\
&= \int_0^{-\varepsilon/2+1/2} \varepsilon\hat{\theta}^2(x) dx - \int_{-\varepsilon/2+1/2}^{\varepsilon/2+1/2} \left(\frac{\varepsilon}{2} - x + \frac{1}{2}\right) \\
&\quad \cdot \left\{ \hat{\theta}^2(x) - \left(\frac{\varepsilon}{2} + x - \frac{1}{2}\right)\hat{\theta}(x) \right\} dx \\
&\geq -\frac{1}{4} \int_{-\varepsilon/2+1/2}^{\varepsilon/2+1/2} \left(\frac{\varepsilon}{2} - x + \frac{1}{2}\right)\left(\frac{\varepsilon}{2} + x - \frac{1}{2}\right)^2 dx \\
&= -\frac{\varepsilon^3}{48}.
\end{aligned} \tag{2.13}$$

Similarly we have

$$I_2' \geq -\frac{\varepsilon^3}{48}. \tag{2.14}$$

From (2.12), (2.13) and (2.14) we get the inequality (2.8) for $\varepsilon \leq 1$. □

COROLLARY 2.1. *For the case when the range is equal to ℓ instead of 1, let $M_\ell(\theta)$ be the MSE of an estimator $\hat{\theta}(X)$ of θ, i.e.*

$$M_\ell(\theta) := \int_{\theta-\ell/2}^{\theta+\ell/2} \{\hat{\theta}(x) - \theta\}^2 dx.$$

Then for any $\theta_0 \in \mathbb{R}$

$$\int_{\theta_0-\varepsilon/2}^{\theta_0+\varepsilon/2} M_\ell(\theta)d\theta \geq \begin{cases} \dfrac{1}{12}\left(\ell^2\varepsilon - \dfrac{\ell^3}{2}\right) & \text{for} \quad \varepsilon > \ell, \\ \dfrac{\varepsilon^3}{12}\left(1 - \dfrac{\varepsilon}{2\ell}\right) & \text{for} \quad \varepsilon \leq \ell. \end{cases}$$

OUTLINE OF THE PROOF. Let $Y := X/\ell$ and $\theta' := \theta/\ell$. Then it follows that Y is uniformly distributed on the interval $[\theta' - (1/2), \theta' + (1/2)]$. Letting $\theta_0' := \theta_0/\ell$ and $\varepsilon' := \varepsilon/\ell$, from Theorem 2.2 we have

$$\begin{aligned} \int_{\theta_0-\varepsilon/2}^{\theta_0+\varepsilon/2} M_\ell(\theta)d\theta &\geq \begin{cases} \dfrac{\ell^3}{12}\left(\dfrac{\varepsilon}{\ell} - \dfrac{1}{2}\right) & \text{for} \quad \varepsilon > \ell, \\ \dfrac{\ell^3}{12}\left(\dfrac{\varepsilon}{\ell}\right)^3\left(1 - \dfrac{\varepsilon}{2\ell}\right) & \text{for} \quad \varepsilon \leq \ell \end{cases} \\ &= \begin{cases} \dfrac{1}{12}\left(\ell^2\varepsilon - \dfrac{\ell^3}{2}\right) & \text{for} \quad \varepsilon > \ell, \\ \dfrac{\varepsilon^3}{12}\left(1 - \dfrac{\varepsilon}{2\ell}\right) & \text{for} \quad \varepsilon \leq \ell. \end{cases} \end{aligned}$$

□

3. Asymptotic lower bound for the average mean squared error

Now suppose that $X_1, \ldots, X_n$ are independently, identically and uniformly distributed over the interval $[\theta - (1/2), \theta + (1/2)]$. Let $Y := (X_{(1)} + X_{(n)})/2$ and $R = X_{(n)} - X_{(1)}$, where $X_{(1)} := \min_{1\leq i\leq n} X_i$ and $X_{(n)} := \max_{1\leq i\leq n} X_i$. Then it is shown that the pair $(X_{(1)}, X_{(n)})$ is a sufficient statistic, and given R, Y is uniformly distributed over the interval $[\theta - \{(1-R)/2\}, \theta + \{(1-R)/2\}]$. Let $\hat{\theta} := \hat{\theta}(X_{(1)}, X_{(n)})$ be an estimator of θ. Define

$$\begin{aligned} J_\varepsilon : &= \int_{-\varepsilon/2}^{\varepsilon/2} E_\theta\left[(\hat{\theta} - \theta)^2\right] d\theta \\ &= E^R\left[\int_{-\varepsilon/2}^{\varepsilon/2} E_\theta\left[(\hat{\theta} - \theta)^2 \mid R\right] d\theta\right]. \end{aligned}$$

From Corollary 2.1, we have

$$\text{(3.1)} \quad \int_{-\varepsilon/2}^{\varepsilon/2} E_\theta\left[(\hat{\theta} - \theta)^2 \mid R\right] d\theta \geq \begin{cases} \dfrac{1}{12}\left\{(1-R)^2\varepsilon - \dfrac{1}{2}(1-R)^3\right\} & \text{for} \quad \varepsilon > 1 - R, \\ \dfrac{\varepsilon^3}{12}\left\{1 - \dfrac{\varepsilon}{2(1-R)}\right\} & \text{for} \quad \varepsilon \leq 1 - R. \end{cases}$$

The density of R is given by

$$\text{(3.2)} \quad f(R) = \begin{cases} n(n-1)R^{n-2}(1-R) & \text{for} \quad 0 < R < 1, \\ 0 & \text{otherwise,} \end{cases}$$

hence it follows that for any $\varepsilon \le 1$

$$
\begin{aligned}
(3.3) \quad J_\varepsilon \ge\ & n(n-1)\int_{1-\varepsilon}^{1} \frac{1}{12}\left\{\varepsilon(1-R)^2 - \frac{1}{2}(1-R)^3\right\} R^{n-2}(1-R)dR \\
&+n(n-1)\int_0^{1-\varepsilon} \frac{\varepsilon^3}{12}\left\{1-\frac{\varepsilon}{2(1-R)}\right\} R^{n-2}(1-R)dR \\
=\ & \frac{\varepsilon}{2(n+1)(n+2)} - \frac{n}{12}\varepsilon(1-\varepsilon)^{n-1} + \frac{1}{4}(n-1)\varepsilon(1-\varepsilon)^n \\
& -\frac{n(n-1)}{4(n+1)}\varepsilon(1-\varepsilon)^{n+1} + \frac{n(n-1)}{12(n+2)}\varepsilon(1-\varepsilon)^{n+2} \\
& -\frac{1}{(n+1)(n+2)(n+3)} + \frac{n}{24}(1-\varepsilon)^{n-1} - \frac{1}{6}(n-1)(1-\varepsilon)^n \\
& +\frac{n(n-1)}{4(n+1)}(1-\varepsilon)^{n+1} - \frac{n(n-1)}{6(n+2)}(1-\varepsilon)^{n+2} + \frac{n(n-1)}{24(n+3)}(1-\varepsilon)^{n+3} \\
& +\frac{n}{12}\varepsilon^3(1-\varepsilon)^{n-1} - \frac{1}{12}(n-1)\varepsilon^3(1-\varepsilon)^n - \frac{n}{24}\varepsilon^4(1-\varepsilon)^{n-1}.
\end{aligned}
$$

Then we have the following.

THEOREM 3.1. *For any estimator* $\hat\theta = \hat\theta(X_{(1)}, X_{(n)})$

$$
(3.4) \qquad \varliminf_{\varepsilon\to 0}\varliminf_{n\to\infty} n^2\frac{J_\varepsilon}{\varepsilon} = \varliminf_{\varepsilon\to 0}\varliminf_{n\to\infty} \frac{n^2}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} E_\theta\left[(\hat\theta-\theta)^2\right]d\theta \ge \frac{1}{2}.
$$

The proof is straightforwardly derived from (3.3). We also have somewhat weaker result.

COROLLARY 3.1. For any estimator $\hat\theta = \hat\theta(X_{(1)}, X_{(n)})$

$$
(3.5) \qquad \lim_{\varepsilon\to 0}\varliminf_{n\to\infty}\sup_{|\theta|<\varepsilon/2}\frac{n^2}{\varepsilon}E_\theta\left[(\hat\theta-\theta)^2\right] \ge \frac{1}{2}.
$$

The proof is omitted since (3.5) is easily derived from Theorem 3.1. Note that the equalities in (3.4) and (3.5) are attained by

$$
\hat\theta^* = \frac{1}{2}(X_{(1)} + X_{(n)}).
$$

Indeed, since the probability density function of $\hat\theta^*$ is given by

$$
f_{\hat\theta^*}(y) = \begin{cases} n(1-2|y-\theta|)^{n-1} & \text{for } \theta - \frac{1}{2} \le y \le \theta + \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}
$$

it follows that

$$
(3.6) \qquad E_\theta\left[(\hat\theta^*-\theta)^2\right] = \frac{1}{2(n+1)(n+2)}.
$$

Since $J_\varepsilon = \varepsilon/\{2(n+1)(n+2)\}$ by (3.6), it is easily seen that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} n^2\frac{J_\varepsilon}{\varepsilon} = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{n^2}{2(n+1)(n+2)} = \frac{1}{2},$$

which implies that the equality in (3.4) is attained by $\hat{\theta}^*$. Since by (3.6)

$$\lim_{n\to\infty} n^2 \sup_{|\theta|<\varepsilon/2} E_\theta\left[(\hat{\theta}^* - \theta)^2\right] = \lim_{n\to\infty}\frac{n^2}{2(n+1)(n+2)} = \frac{1}{2},$$

the equality in (3.5) is seen to be attained by $\hat{\theta}^*$.

Finally we shall show that the Móri type inequality is easily derived from the inequality (3.1). Letting $c = \varepsilon/2$, we have from (3.1)

$$\int_{-c}^{c} E_\theta\left[(\hat{\theta}-\theta)^2 \mid R\right]d\theta \geq \frac{1}{12}\left\{2c(1-R)^2 - \frac{1}{2}(1-R)^3\right\} \quad \text{for} \quad c > (1-R)/2,$$

hence

$$\text{(3.7)} \qquad \begin{aligned} \int_{-c}^{c} E_\theta\left[(\hat{\theta}-\theta)^2\right]d\theta &= E^R\left[\int_{-c}^{c} E_\theta\left[(\hat{\theta}-\theta)^2 \mid R\right]d\theta\right] \\ &\geq \frac{1}{12}E^R\left[2c(1-R)^2 - \frac{1}{2}(1-R)^3\right] \end{aligned}$$

for large c. Since by (3.2)

$$E^R[(1-R)^2] = \frac{6}{(n+1)(n+2)},$$
$$E^R[(1-R)^3] = \frac{24}{(n+1)(n+2)(n+3)},$$

it follows from (3.7) that

$$\frac{1}{2c}\int_{-c}^{c} E_\theta[(\hat{\theta}-\theta)^2]d\theta \geq \frac{1}{2(n+1)(n+2)} - \frac{1}{2c(n+1)(n+2)(n+3)}$$

for large c, which implies

$$\lim_{c\to\infty}\frac{1}{2c}\int_{-c}^{c} E_\theta\left[(\hat{\theta}-\theta)^2\right]d\theta \geq \frac{1}{2(n+1)(n+2)}.$$

This type inequality is given by Móri (1983).

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