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TESTING FOR UNIFORMITY OF THE RESIDUAL LIFE TIME BASED ON DYNAMIC KULLBACK-LEIBLER INFORMATION

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Abstract. In this article a goodness of fit test for distributional assumptions regarding the residual lifetime is proposed. The test is based on a Vasicek type sum log-spacings estimators of a dynamic version of Kullback-Leibler information. The specific distributional hypothesis considered is of the unformity over [0,1]. However, the test can be used for testing any simple goodness of fit hypothesis. The asymptotic distribution of the test statistic together with a tabulation of the critical points for different sample sizes are given. Finally, the power function of the test is empirically studied in comparison with some competitors, and the test appears to be meritorious.

Key words and phrases: Discrimination information, uniformity tests, goodness-of-fit tests, consistent tests.

1. Introduction

Let T be a random variable representing time to failure of a system. For example, it might be the time to failure of a bio-system or the time to failure of an engineering system. Let $F(t) = P(T \leq t)$ be the lifetime distribution of T with the survival function $\bar{F}(t) = 1 - F(t)$. We assume that F is differentiable with density function $f(t)$ concentrated on the interval [0,1]. Consider the problem of testing the null hypothesis H_0 that the residual lifetime distribution of a system given that it has survived until time say t_0 is the uniform over $(t_0, 1)$. That is, given that the age of a system is t_0 we want to test

$$H_0 : \frac{\bar{F}(x + t_0)}{\bar{F}(t_0)} = \frac{1 - x - t_0}{1 - t_0} \quad \text{for all} \quad 0 \leq x \leq 1 - t_0. \tag{1.1}$$

The alternative to H_0 is

$$H_a : \frac{\bar{F}(x + t_0)}{\bar{F}(t_0)} \neq \frac{1 - x - t_0}{1 - t_0} \quad \text{for at least one } x \text{ in} \quad [0, 1 - t_0]. \tag{1.2}$$

If H_0^* states that $\frac{\bar{F}(x+t_0)}{\bar{F}(t_0)} = \frac{\bar{F}_0(x+t_0)}{\bar{F}_0(t_0)}$ for all $x > 0$, where $\bar{F}_0$ is continuous and fully specified distribution, then our test of uniformity also allows one to test H_0^*; see Lemma 1. The main purpose of this paper is to propose a method for testing (1.1) against (1.2).

In Section 2 the test statistic based on the dynamic version of Kullback-Leibler information is formulated and its main properties are stated. An advantage of our test statistic is that it incorporates information about the age and therefore it can be used for testing of a certain probability model for the residual lifetime distribution. In Section 3, the percentage points of our test statistic were estimated for various sample sizes and levels, discussed how to implement the proposed test and gave an illustrative example. Section 3 also compares the powers of the proposed test with other competing tests. Finally in Section 4, we derive the asymptotic behaviors of our test statistic.

2. Test statistic

To discriminate between the two hypotheses (1.1) and (1.2), we use the dynamic version of Kullback-Leibler discrimination information function between two residual lifetime distributions given by

$$(2.1) \qquad K(F,F_0;t_0) = \int_{t_0}^{1} \frac{f(x)}{\bar{F}(t_0)} \log \left\{ \frac{\frac{f(x)}{\bar{F}(t_0)}}{\frac{f_0(x)}{\bar{F}_0(t_0)}} \right\} dx$$
$$= \log \bar{F}(t_0) + H(F;t_0) - \int_{t_0}^{1} \frac{f(x)}{\bar{F}(t)} \log f_0(x) dx,$$

where

$$(2.2) \qquad H(F;t_0) = \int_{t_0}^{1} \frac{f(x)}{\bar{F}(t_0)} \log \frac{f(x)}{\bar{F}(t_0)} dx$$
$$= \frac{1}{\bar{F}(t_0)} \left[\int_{t_0}^{1} f(x) \log f(x) dx \right] - \log \bar{F}(t_0).$$

It is well known that $K(F,F_0;t_0) \geq 0$ and the equality holds if and only if $\frac{f(x)}{\bar{F}(t_0)} = \frac{f_0(x)}{\bar{F}_0(t_0)}$ for all $x \geq t_0$; Ebrahimi (1996) and Ebrahimi and Kirmani (1996*a*,1996*b*).

In our situation, discriminating between $\frac{\bar{F}(x+t_0)}{\bar{F}(t_0)}$ the true residual survival distribution at age t_0 and the corresponding $\frac{\bar{F}_0(x+t_0)}{\bar{F}_0(t_0)} = \frac{1-x-t_0}{1-t_0}$ the residual survival distribution for the uniform (0,1), the equation (2.1) reduces to

$$(2.3) \qquad K(F,F_0;t_0) = \log\,(1-t_0) + H(F;t_0).$$

The discrimination function (2.3) is a measure of disparity between residual lifetime distribution at age t_0 and corresponding residual lifetime distribution for the uniform (0,1). Under the null hypothesis H_0 in (1.1) $K(F,F_0;t_0) = 0$, and large values of $K(F,F_0;t_0)$ favor H_a.

Since evaluation of (2.3) requires complete knowledge of F, then $K(F,F_0;t_0)$ is not operational. We operationalize $K(F,F_0;t_0)$ by developing the discrimination information statistic as follows.

Given a random sample $T_1,\ldots,T_n$ from F, let $T(1),\ldots,T(n)$ be the n ordered observations. To estimate $H(F;t_0)$ in (2.3) we write

$$(2.4) \qquad H(F;t_0) = -\log \bar{F}(t_0) - \frac{1}{\bar{F}(t_0)} \int_{F(t_0)}^{1} \log \left(\frac{d}{dp} F^{-1}(p) \right) dp.$$

Now a natural estimate of (2.4) can be constructed by replacing $F(t)$ and $\bar{F}(t)$ by $F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(T_i \le t)$ and $\bar{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(T_i > t)$. Also using a difference operator in place of differential operator, the derivative of $F^{-1}(p)$ can be estimated by

$$(2.5)\quad u_{m,n}(p) = \begin{cases} \frac{n}{2m}(T(i+m) - T(i-m)), & \frac{i-1}{n} < p \le \frac{i}{n}, \\ i = m+1, \ldots, n-m & \\ \frac{n}{2m}(T(i+m) - T(1)), & p \le \frac{m}{n}, \quad \frac{i-1}{n} < p \le \frac{i}{n}, \quad i = 1, \ldots, m \\ \frac{n}{2m}(T(n) - T(i-m)), & p > \frac{n-m}{n}, \quad \frac{i-1}{n} < p \le \frac{i}{n}, \\ i = n-m+1, \ldots, n. & \end{cases}$$

It should be mentioned that we can adjust (2.5) for the end points using methods described by Ebrahimi *et al.* (1994). However, our study shows that tests based on adjusted and unadjusted estimators perform the same.

From (2.5), an estimate $H_{m,n}(F_n; t_0)$ of $H(F; t_0)$ is

$$\begin{aligned} (2.6)\quad H_{m,n}(F_n; t_0) &= -\log \bar{F}_n(t_0) - \frac{1}{\bar{F}_n(t_0)} \int_{F_n(t_0)}^{1} \log(u_{m,n}(p))dp \\ &= -\log \frac{n-j+1}{n} - \frac{1}{n-j+1} \sum_{i=j}^{n} \log \frac{n}{2m}(T(i+m) - T(i-m)), \\ &= \frac{-1}{n-j+1} \sum_{i=j}^{n} \log \frac{n-j+1}{2m}(T(i+m) - T(i-m)), \end{aligned}$$

for $T(j-1) \le t_0 < T(j)$, $j = 2, \ldots, n$, where $T(i) = T(1)$ if $i < 1$, $T(i) = T(n)$, $i > n$, and m is a positive integer smaller than $\frac{n}{2}$ and is called window size. If $0 \le t_0 < T(1)$, then

$$(2.7)\qquad H_{m,n}(F_n; t_0) = -\frac{1}{n} \sum_{i=1}^{n} \log \frac{n}{2m}(T(i+m) - T(i-m)).$$

It should be mentioned that if $t_0 = 0$, then $H_{m,n}(F_n; 0)$ coincides with the estimator proposed by Vasicek (1976).

We estimate the right hand side of the equation (2.3) by

$$(2.8)\qquad K_{m,n}(t_0) = \log(1 - t_0) + H_{m,n}(F_n; t_0).$$

Large values of $K_{m,n}(t_0)$ indicate that the residual lifetime distribution at age t_0 is not uniform. Therefore, for given m and n we reject H_0 in favor of H_a at the significance level α if $K_{m,n}(t_0) \ge C_{m,n}(t_0, \alpha)$, where the critical point $C_{m,n}(t_0, \alpha)$ is determined by the $(1-\alpha)$-quantile of the distribution $K_{m,n}(t_0)$ under the null hypothesis H_0. In Equation (2.8) if $t_0 = 0$, then $K_{m,n}(0)$ coincides with the statistic proposed by Dudewicz and van der Meulen (1981) for testing uniformity over (0,1).

The proposed test statistic in (2.8) for the uniform goodness-of-fit test under discussion has wider applicability than has been indicated previously. The following lemma extends the applicability in one direction.

LEMMA 1. (Ebrahimi and Kirmani (1996*b*)). *Let X and Y be two non-negative random variables with distribution functions G_1 and G_2 respectively. For any monotone function ϕ,*

$$K(X,Y;t_0) = K(G_1,G_2;t_0) = K(\phi(X),\phi(Y);\phi(t_0)).$$

Lemma 1 simply says that testing for H_0^* is equivalent to testing for H_0, where H_0 is the true residual survival function of $Y = F_0(T)$ at age $F_0(t_0)$.

3. Implementation of the test

The asymptotic distribution of $K_{m,n}(t_0)$ can be obtained and is given in Section 4. However, the sampling distribution for small sample sizes is interactable. We determine the critical points $C_{m,n}(t_0,\alpha)$ of the statistics $K_{m,n}(t_0)$ by the means of Monte Carlo simulations for α equal to .01, .05 and .10.

For selected values of the sample size $n(n = 5,10,12,14,16,18,20,25,30,35,40,45,50,60,70,80,90,100,110,120,130,140,150,170,190,200)$ we generated $N = 10000$ samples of size n from the uniform distribution on (0,1). The random number generator used in the study is due to Ahrens and Dieter (1974). This random number generator uses a multiplicative congruential method with modulus $(2)^{32}$ and sets $x_{i+1} = ax_i$ mod (2^{32}), where $a = 663608941$. For our purpose we use $x_0 = 1$. It should be mentioned that this generator has period 1073741824. The TESTRAND test was performed to this random number generator and it did pass the test, see Karian and Dudewicz (1999). The TESTRAND tests are a set of 19 chi-squared tests developed by Dudewicz and Ralley (1981) that can be used to test quality of random number generators. For each sample, the statistic $K_{m,n}(t_0)$ was calculated for each m such that $m < \frac{n}{2}$ and for different values of t_0. For each m, t_0 and n, the upper α-quantile of the distribution $K_{m,n}(t_0)$, $C_{m,n}(t_0)$ under the null hypothesis was estimated from the 10000 sample values of $K_{m,n}(t_0)$ generated by this Monte Carlo experiment.

REMARK 1. It should be mentioned that we first used the IMSL uniform random number generator with $a = 397204094$ to determine critical points $C_{m,n}(t_0,\alpha)$ of the statistics $K_{m,n}(t_0)$. It was noticed that this random number did not pass the TESTRAND test. Furthermore for the small sample sizes the critical values obtained under this random number generator were substantially different from the ones given in Tables 1–4. In fact, the only example in the literature of this phenomenon that I am aware of is by Chen *et al.* (1999).

To assess the accuracy of our estimates of $C_{m,n}(t_0,\alpha)$ one can use the methods described by Dudewicz and van der Meulen (1984), see also Karian and Dudewicz (1999). For example, when $n = 10$, $t_0 = .1$, $m = 3$ and $\alpha = .01$, we estimated $C_{m,n}(t_0,\alpha)$ by $Z(9900) = .7839$, where $Z(1), Z(2), \ldots, Z(10000)$ denote the order statistics of the sample of $N = 10000$ values of $K_{3,10}(.1)$. In order to assess the precision of our estimate, we construct 95% confidence interval for $C_{3,10}(.1,.01)$ using the methods described by Dudewicz and van der Meulen (1984). The 95% confidence interval goes from .7485 to .8053 which is accurate to at least $\pm.0284$.

For each n and t_0, the m that gives the maximum $C_{m,n}(t_0,\alpha)$ produces the most conservative test, and the test that gives the minimum $C_{m,n}(t_0,\alpha)$ produces the least

Table 1. Values of m corresponding to minimum critical points of $K_{m,n}(t_0)$.

	t_0								
n	.1	.2	.3	.4	.5	.6	.7	.8	.9
5	2	2	2	2	2	2	2	2	2
10	3	3	3	2	2	2	2	4	4
12	3	3	3	2	2	2	2	4	4
14	3	3	3	2	2	2	2	4	4
16	3	3	3	2	2	2	2	4	4
18	4	4	4	3	3	3	3	5	5
20	4	4	4	3	3	3	3	5	5
25	4	4	4	3	3	3	3	5	5
30	5	5	5	4	4	4	4	6	6
35	5	5	5	4	4	4	4	6	6
40	6	6	6	5	5	5	5	7	7
45	6	6	6	5	5	5	5	7	7
50	7	7	7	6	6	6	6	8	8
60	7	7	7	6	6	6	6	8	8
70	7	7	7	6	6	6	6	8	8
80	7	7	7	6	6	6	6	8	8
90	8	8	8	7	7	7	7	9	9
100	8	8	8	7	7	7	7	9	9
110	8	8	8	7	7	7	7	9	9
120	8	8	8	7	7	7	7	9	9
130	9	9	9	8	8	8	8	10	10
140	9	9	9	8	8	8	8	10	10
150	9	9	9	8	8	8	8	10	10
170	9	9	9	8	8	8	8	10	10
190	11	11	11	10	10	10	10	12	12
200	11	11	11	10	10	10	10	12	12

conservative test. However, the simulation reveals that for large $n (n > 100)$ the maximum and the minimum are very close. That is the most conservative test and the least conservative test are almost identical. Furthermore, the least conservative tests showed the highest power, in comparison with other statistics, against several alternatives considered in the power study reported in the next section.

Table 1 shows the least conservative window sizes corresponding to various sample sizes and different values of t_0. Observe that the least conservative m increases as n increases. Tables 2–4 give the minimum $C_{m,n}(t_0, \alpha)$ for several sample sizes and several values of t_0.

To implement the test statistic $K_{m,n}(t_0)$, we must first fix a window size m. We recommend the least conservative m. Therefore,

(a) Use Table 1 to find the window size corresponding to sample size n and t_0;

(b) Use one of the Tables 2–4 to find critical point $C_{m,n}(t_0, \alpha)$ and reject H_0 if $K_{m,n}(t_0) > C_{m,n}(t_0, \alpha)$. It should be mentioned that if $n > 200$ then use the asymptotic distribution of $K_{m,n}(t_0)$ under H_0 from Section 4 to compute the approximate critical

Table 2. Critical values of $K_{m,n}(t_0)$-statistics ($C_{m,n}(t_0, \alpha)$).

($\alpha = .01$)

	t_0								
n	.1	.2	.3	.4	.5	.6	.7	.8	.9
5	2.568	2.639	3.241	3.623	4.123	4.938	6.123	8.156	9.232
10	.7839	1.165	1.282	1.298	1.635	1.815	1.936	2.565	3.135
12	.7368	.8938	.9653	1.128	1.573	1.713	1.916	2.463	2.918
14	.6438	.7436	.8636	.9142	.9754	1.173	1.865	2.130	2.716
16	.5831	.6158	.6306	.7945	.8125	1.065	1.785	1.988	2.703
18	.5639	.5973	.5928	.7381	.7926	1.063	1.712	1.937	2.628
20	.4312	.5123	.5695	.6507	.7160	.9063	1.669	1.888	2.338
25	.4015	.4168	.4836	.5156	.7032	.8359	1.438	1.858	2.325
30	.3352	.3558	.4045	.4636	.5573	.7150	1.012	1.801	2.323
35	.2895	.3156	.3692	.4056	.5989	.6162	.8162	1.786	2.320
40	.2598	.2835	.3188	.3575	.4212	.5336	.7406	1.577	2.296
45	.2369	.2756	.2698	.3109	.4026	.4563	.6321	1.331	2.193
50	.2187	.2308	.2592	.2982	.3548	.4424	.5930	1.235	2.106
60	.1963	.2131	.2398	.2638	.3158	.4163	.5063	.9632	2.036
70	.1892	.1992	.2063	.2346	.2853	.3589	.4921	.7763	1.783
80	.1697	.1783	.1835	.1965	.2406	.2685	.3821	.6187	1.629
90	.1415	.1489	.1678	.1893	.2107	.2561	.3468	.5680	1.404
100	.1315	.1386	.1554	.1718	.2009	.2381	.3093	.5187	1.268
110	.1298	.1318	.1495	.1665	.1863	.2275	.2862	.2965	1.162
120	.1267	.1287	.1365	.1532	.1795	.1938	.2659	.3789	.9190
130	.1213	.1197	.1293	.1483	.1603	.1873	.2413	.3656	.8910
140	.1132	.1163	.1206	.1386	.1568	.1803	.2308	.3632	.8491
150	.0974	.1047	.1179	.1325	.1486	.1750	.2219	.3478	.8283
170	.0895	.0965	.1126	.1227	.1363	.1613	.2087	.2936	.6875
190	.0863	.0892	.1102	.1213	.1258	.1477	.1993	.2797	.6293
200	.0816	.0864	.0992	.1069	.1216	.1449	.1893	.2759	.6174

point.

The following example illustrates the calculation procedures.

3.1 *Example*

Grubbs (1971) has given the data on mileages for 19 military personnel carriers that failed in service. The mileages are

162 200 271 320 393 508 539 629 706 778

884 1003 1101 1182 1463 1603 1984 2355 2880

An important question is whether the failure time for a personnel carrier with zero mileage on it is exponentially distributed; this is discussed by Ebrahimi *et al.* (1992). In fact, they were unable to reject that the failure time is exponentially distributed. Here, we examine whether the remaining lifetime of a personnel carrier with t_0 miles on it, could conceivably still have arisen from an exponential distribution. That is, we test the hypothesis that the remaining lifetime has the distribution function of the form

Table 3. Critical values of the $K_{m,n}(t_0)$ statistics ($C_{m,n}(t_0, \alpha)$).

($\alpha = .05$)

	t_0								
n	.1	.2	.3	.4	.5	.6	.7	.8	.9
5	2.268	2.366	2.881	3.168	3.468	4.138	5.268	6.333	7.015
10	.5923	.6098	.7320	.7994	.8953	1.051	1.460	1.582	2.133
12	.5836	.6178	.6938	.7663	.8568	.9926	1.326	1.401	2.028
14	.5653	.5853	.6821	.7635	.7835	.9805	1.321	1.389	1.993
16	.4616	.4672	.5416	.5955	.6831	.8138	1.138	1.386	1.976
18	.4165	.4533	.4831	.5367	.5912	.7863	.9683	1.381	1.957
20	.3255	.3726	.4087	.4749	.5687	.7056	.9132	1.380	1.921
25	.2873	.3013	.3495	.4058	.4428	.6162	.8381	1.376	1.903
30	.2666	.2692	.3001	.3449	.3926	.4952	.6591	1.322	1.835
35	.2456	.2632	.2866	.3069	.3685	.4136	.6053	1.203	1.754
40	.2109	.2212	.2454	.2709	.3192	.3980	.5291	1.036	1.723
45	.1903	.2056	.2169	.2473	.2932	.3787	.4635	.9805	1.719
50	.1750	.1849	.2061	.2250	.2676	.3308	.4335	.8327	1.620
60	.1632	.1803	.1835	.1916	.2033	.3106	.4106	.7835	1.439
70	.1498	.1595	.1658	.1623	.1841	.2738	.3693	.6012	1.108
80	.1305	.1413	.1363	.1493	.1663	.2103	.3005	.4658	.9865
90	.1186	.1323	.1295	.1402	.1605	.2016	.2813	.4069	.9391
100	.1060	.1118	.1254	.1385	.1585	.1892	.2421	.3907	.9101
110	.0938	.1167	.1173	.1268	.1403	.1605	.2268	.3615	.8267
120	.0865	.1023	.1039	.1205	.1328	.1576	.2039	.3181	.7532
130	.0814	.0965	.1013	.1137	.1265	.1438	.1865	.2973	.6315
140	.0813	.0903	.1003	.1093	.1195	.1403	.1805	.2813	.6121
150	.0812	.0887	.0954	.1064	.1182	.1399	.1794	.2749	.5902
170	.0736	.0834	.0913	.0963	.1116	.1376	.1636	.2638	.5093
190	.0693	.0732	.0876	.0907	.1065	.1206	.1596	.2392	.4832
200	.0682	.0716	.0789	.0880	.0989	.1183	.1478	.2264	.4677

$\bar{F}(x \mid t_0) = \frac{\bar{F}(x+t_0)}{\bar{F}(t_0)} = \exp(-\lambda(t_0)x)$.

For illustration we use $t_0 = 350$. Using the Lemma 1, we transfer the data to $y_i = 1 - \exp(-\frac{1}{997}t_i)$, $i = 1, \ldots, 19$. The question now is whether the conditional distribution of $Y - (1 - \exp(-\frac{350}{997}))$ given that $Y > (1 - \exp(-\frac{350}{997}))$ is the uniform over (.3,1). To answer this first we find the window size using Table 1, $m = 4$. Then, we use the equation (2.8) for the transferred data $y_1, \ldots, y_{19}$ to compute $K_{m,n}(.3) = .107$. For $n = 19$, $m = 4$ and $\alpha = .1$ Table 4 gives the critical value $.3885 < C_{4,19}(.3, .1) < .5163$. Thus we cannot reject the null hypothesis that the remaining lifetime of a carrier with 350 miles on it has an exponential distribution. That is the exponential distribution with mean 997 provides an adequate fit for the remaining lifetime of a personnel carrier with 350 miles on it.

Table 4. Critical values of $K_{m,n}(t_0)$ statistics ($C_{m,n}(t_0, \alpha)$).

($\alpha = .1$)

	t_0								
n	.1	.2	.3	.4	.5	.6	.7	.8	.9
5	2.031	2.268	2.563	2.956	3.013	3.923	4.987	5.328	6.921
10	.5666	.5752	.6712	.7385	.8588	.9990	1.392	1.485	1.997
12	.5428	.5763	.6328	.6883	.8313	.9516	1.312	1.389	1.988
14	.5027	.5415	.5962	.6365	.7712	.9235	1.128	1.328	1.926
16	.4638	.4938	.5621	.5701	.6532	.7758	.9912	1.316	1.903
18	.3987	.4286	.5163	.5628	.6382	.7033	.8738	1.304	1.876
20	.3032	.3454	.3885	.4396	.5154	.6368	.8718	1.294	1.808
25	.2651	.3153	.3338	.3965	.4123	.5963	.7895	1.286	1.719
30	.2416	.2936	.3087	.3168	.3835	.4317	.6155	1.138	1.693
35	.2189	.2598	.2938	.2958	.3316	.3852	.5585	1.038	1.613
40	.1896	.2012	.2221	.2414	.2972	.3444	.4575	.9132	1.593
45	.1728	.1835	.2059	.2153	.2493	.3242	.4263	.8835	1.438
50	.1563	.1728	.1939	.1976	.2396	.3151	.3968	.7136	1.380
60	.1439	.1648	.1798	.1886	.2103	.2838	.3732	.6510	1.213
70	.1328	.1416	.1537	.1656	.1815	.2668	.3217	.5883	1.010
80	.1256	.1329	.1278	.1531	.1575	.2238	.2832	.4557	.8526
90	.1073	.1138	.1203	.1368	.1501	.1938	.2398	.3938	.7932
100	.0951	.1001	.1119	.1237	.1403	.1658	.2097	.3354	.7486
110	.0895	.0987	.1039	.1302	.1356	.1532	.1983	.3031	.7052
120	.0813	.0905	.0963	.1196	.1205	.1483	.1825	.2828	.6315
130	.0773	.0867	.0985	.1103	.1178	.1368	.1703	.2531	.5085
140	.0731	.0803	.0903	.1063	.1102	.1305	.1603	.2457	.4982
150	.0720	.0789	.0858	.0946	.1061	.1246	.1566	.2399	.4942
170	.0695	.0712	.0813	.0879	.1008	.1103	.1403	.2185	.4621
190	.0638	.0698	.0756	.0813	.0938	.1098	.1308	.2095	.4057
200	.0627	.0650	.0716	.0791	.0882	.1043	.1297	.2009	.4027

3.2 *Power determination under seven alternatives*

To provide information on the power of $K_{m,n}(t_0)$, the Monte Carlo study of the power of our test was carried out under seven alternative distributions. For each sample size n ($n = 20, 50, 100$), $N = 10000$ samples of size n were generated from the uniform (0,1) and for each alternative these samples were transformed to the alternative distribution by the application of the appropriate inverse function.

We consider the following alternatives:

$$A_k : F(x) = 1 - (1-x)^k \quad \text{if} \quad 0 \le x \le 1 \ (\text{for } k = 1.5, 2)$$

$$B_k : F(x) = 2^{k-1}x^k, \quad 0 \le x \le \frac{1}{2} = 1 - 2^{k-1}(1-x)^k, \quad \frac{1}{2} \le x \le 1 \ (\text{for } k = 1.5, 2, 3)$$

$$C_k : F(x) = .5 - 2^{k-1}(.5-x)^k, \quad 0 \le x \le \frac{1}{2}$$

$$= .5 + 2^{k-1}(x-.5)^k, \quad \frac{1}{2} \le x \le 1 \ (\text{for } k = 1.5, 2)$$

Table 5. Powers of .05 tests against some alternatives based on 10000 replications.

n	Alternative	$t_0 = .3$	$t_0 = .5$	$t_0 = .7$
20	$A_{1.5}$	.43	.30	.20
50		.69	.51	.33
100		.91	.80	.52
20	A_2	.54	.49	.39
50		.95	.81	.71
100		.98	.88	.80
20	$B_{1.5}$	.60	.51	.39
50		.79	.68	.52
100		.92	.82	.70
20	B_2	.92	.86	.71
50		.97	.91	.83
100		.99	.94	.90
20	B_3	.98	.9	.79
50		.99	.96	.93
100		1	.92	.86
20	$C_{1.5}$	.18	.11	.08
50		.39	.21	.12
100		.75	.51	.39
20	C_2	.51	.20	.16
50		.85	.74	.67
100		.99	.87	.80

Alternatives A_k, B_k and C_k were also used by Stephens (1974) and Dudewicz and van der Meulen (1981) in their studies for comparisons of several tests for uniformity. First we computed $K(F, F_0; t_0)$ using equations (2.1) and (2.2) for these alternatives. Under the alternative A_k, $K(F, F_0; t_0) = \log k + \frac{1-k}{k}$ which is free from t_0. (This is always true as long as F and F_0 are proportional; see Ebrahimi and Kirmani (1996a)). We observe that as k increases the discrimination between the uniform and the alternative A_k gets easier and easier for any t_0. Under the alternative B_k,

$$K(F, F_0; t_0) = \begin{cases} \log(1-t_0) - \log(1-2^{k-1}t_0^k) + \log(k2^{k-1}) \\ \qquad + \frac{2^{k-1}(k-1)}{1-2^{k-1}t_0^k}[2((\frac{1}{2})^k \log(\frac{1}{2}) - \frac{(\frac{1}{2})^k}{k}) - t_0^k \log t_0 + \frac{t_0^k}{k}] \text{ if } t_0 \leq \frac{1}{2} \\ \log k + \frac{1-k}{k} \text{ if } t_0 \geq \frac{1}{2}. \end{cases}$$

We note that as k increases, for any t_0, the discrimination between the uniform and the alternative B_k gets easier. Finally under the alternative C_k,

$$K(F, F_0; t_0) = \begin{cases} \log(1-t_0) - \log(\frac{1}{2} + 2^{k-1}(.5-t_0)^k) + \log(k2^{k-1}) \\ \qquad + \frac{2^{k-1}(k-1)}{\frac{1}{2}+2^{k-1}(.5-t_0)^k}[(.5-t_0)^k \log(.5-t_0) - (\frac{1}{2})^k \log 2 - \frac{1}{k2^k} - \frac{(.5-t_0)^k}{k}] \text{ if } t_0 \leq \frac{1}{2} \\ \log(1-t_0) - \log(.5 - 2^{k-1}(t_0-.5)^k) + \log(k2^{k-1}) \\ \qquad + \frac{2^{k-1}(k-1)}{\frac{1}{2}-2^{k-1}(t_0-.5)^k}[\frac{(t_0-.5)^k}{k} - (t_0-.5)^k \log(t_0-.5) - (\frac{1}{2})^k \log 2 - \frac{1}{k2^k}] \text{ if } t_0 \geq \frac{1}{2}. \end{cases}$$

Table 6. Power comparison of $K_{m,n}(t_0)$ versus the test with highest power.

n		$t_0 = .3$	$t_0 = .5$	$t_0 = .7$
20	$A_{1.5}$	$Q = .52, K = .43$	$Q = .45, K = .30$	$Q = .19, K = .20$
40		$Q = .75, K = .50$	$Q = .68, K = .47$	$Q = .47, K = .33$
20	A_2	$W^2 = .71, K = .55$	$W^2 = .60, K = .49$	$W^2 = .2, K = .38$
40		$Q = .99, K = .93$	$Q = .97, K = .84$	$Q = .45, K = .6$
20	$B_{1.5}$	$ENT = .51, K = .59$	$ENT = .33, K = .51$	$ENT = .12, K = .39$
40		$ENT = .63, K = .72$	$ENT = .56, K = .58$	$ENT = .38, K = .46$
20	B_2	$ENT = .82, K = .92$	$ENT = .62, K = .86$	$ENT = .42, K = .71$
40		$ENT = .93, K = .93$	$ENT = .91, K = .91$	$ENT = .68, K = .72$
20	B_3	$ENT = .95, K = .97$	$ENT = .93, K = .90$	$ENT = .78, K = .79$
40		$ENT = 1, K = 1$	$ENT = 1, K = .95$	$ENT = .94, K = .91$
20	$C_{1.5}$	$U^2 = .33, K = .19$	$U^2 = .16, K = .12$	$U^2 = .08, K = .08$
40		$U^2 = .48, K = .36$	$U^2 = .37, K = .22$	$U^2 = .23, K = .12$
20	C_2	$U^2 = .60, K = .51$	$U^2 = .25, K = .20$	$U^2 = .18, K = .16$
40		$U^2 = .86, K = .79$	$U^2 = .77, K = .69$	$U^2 = .51, K = .62$

Q: log statistic, W^2: Cramer-Von Mises statistic
ENT: Dudewicz and van der Meulen statistic, U^2: Watson statistic
K: Our proposed statistic

Again we observe that as k increases the discrimination between the uniform and the alternative C_k becomes easier.

We estimated also the power of our test $K_{m,n}(t_0)$ at $t_0 = .3, .5, .7$. Table 5 shows the estimated powers at significance level $\alpha = .05$. The powers for the $K_{m,n}(t_0)$ statistic are based on the window sizes reported in Table 1. These choices of m gave the maximum power for our statistic. We observe that $K_{m,n}(t_0)$ performs very well. However, the simulation reveals that the proposed test is generally more powerful for smaller t_0 than larger t_0. We also observe that our test has higher power for larger k than smaller k which is consistent with the actual values of $K(F, F_0; t_0)$.

3.3 *Power comparisons with other tests for uniformity*

If L is the number of T_i's larger than t_0 then L has a binomial distribution with parameters n and $\bar{F}(t_0)$. Given $L = \ell$ the set of T_i exceeding t_0 is a random sample of size ℓ from the conditional survival function of $\frac{\bar{F}(x)}{\bar{F}(t_0)}$. Now, on the null hypothesis of conditional uniformity one can produce an independent identically distributed uniform [0,1] sample by computing $\frac{T_i - t}{1-t}$ and test the null hypothesis H_0 by applying any test of uniformity to the ℓ resulting values.

Tests of goodness of fit of a uniform distribution have been proposed by many authors, see Dudewicz and van der Meulen (1981). To compare the power of our proposed test, we consider the Kolmogorov-Smirnov D, Cramer-Von Mises W^2, Kuiper V, Watson U^2, Anderson Darling A^2, log statistic Q, χ^2 and the entropy based test of Dudewicz and van der Meulen (1981), ENT against the alternatives A_k, B_k, C_k and for sample sizes 20 and 40.

Given the findings of Dudewicz and van der Meulen (1981) Table 3, from their table we take the statistic with the highest power and compare it with our proposed test $K_{m,n}(t_0)$. In Table 6 we report the estimated powers at significance level $\alpha = .05$ at $t_0 = .3$, .5 and .7.

We observe that for the alternative B our proposed test K has the higher power than the ENT test. Under alternatives A and C our test does not perform as well as the test with the highest power. However, for larger t_0 our test is competitive with the best test.

4. Asymptotic properties of $H_{m,n}(F_n;t_0)$ and $K_{m,n}(t_0)$

Our goal here is to obtain the asymptotic properties of $H_{m,n}(F_n;t_0)$ and $K_{m,n}(t_0)$. In this section, the limits are taken as $m, n \to \infty$ and $\frac{m}{n} \to 0$ unless otherwise stated.

The following theorem states the consistency of $H_{m,n}(F_n;t_0)$ and consequently $K_{m,n}(t_0)$.

THEOREM 1. *Let $t_1, \ldots, t_n$ be a sample from F with a density function f and a finite variance. Then*

$$H_{m,n}(F_n;t_0) \xrightarrow{P} H(F;t_0),$$

for any $t_0 \epsilon [0,b], b < 1$.

PROOF. First observe that given $\ell_{t_0} = \sum_{i=1}^{n} I(t_i - t_0)$, where $I(a) = 0$ or 1 according to $a > 0$ or $a \leq 0$, there is a sample of size ℓ_{t_0} from $\frac{f(y)}{\bar{F}(t_0)} I(y - t_0)$ and ℓ_{t_0} is Binomial random variable. Now using Theorem 1 of Vasicek (1976) the result follows. This completes the proof.

From Theorem 1 it is clear that our proposed test $K_{m,n}(t_0)$ in (2.8) is a consistent test.

To investigate the asymptotic distribution of $K_{m,n}(t_0)$ first notice that for a fixed $t_0 \epsilon [0,b]$, $b < 1$ if we define

$$L(m,n) = \sum_{i=0}^{n-m+1} \log(Y(i+m) - Y(i)) - \log(1 - t_0), \tag{4.1}$$

where $Y(0) = t_0$ and $Y(n+1) = 1$ with $Y(1), Y(2), \ldots, Y(n)$ are the order statistics of the random sample $Y_1, \ldots, Y_n$, and m is a positive integer, then using Cressie (1976) one can show that if $m = o(n^{1/3})$ and $Y_1, \ldots, Y_n$ is a random sample from the uniform distribution over $(t_0, 1)$,

$$\left(\frac{n}{3m}\right)^{-1/2} [L(m,n) + (n + 2 - m)\{\log\,(n+1) + \gamma - R(1, m-1)\}], \tag{4.2}$$

is asymptotically standard normal as $n \to \infty$, where γ is Euler constant .5772 and $R(1, m-1) = \frac{1}{m-1} + \cdots + \frac{1}{2} + 1$ if $m \geq 2$ and $R(1,0) = 0$. (This result comes from the fact that if Y is the uniform over $(t,1)$, then $\frac{Y-t}{1-t}$ is the uniform over (0,1).) Thus,

$$\left(\frac{6m}{n}\right)^{1/2} [L(2m,n) + (n + 2 - 2m)\{\log\,(n+1) + \gamma - R(1, 2m-1)\}] \tag{4.3}$$

is asymptotically standard normal. Finally, following similar arguments used by Dudewicz and van der Meulen (1981) one can prove that

$$(4.4) \qquad (\sqrt{6mn})[\Delta(m,n) + \log 2m + \gamma - R(1, 2m-1)],$$

is asymptotically standard normal, where $m = o(n^{1/3-\delta})$, $\delta > 0$ and $\Delta(m,n) = \frac{1}{n}\sum_{i=1}^{n} \log \frac{n}{2m}(Y(i+m) - Y(i-m)) - \log(1-t_0)$.

The following theorem gives the asymptotic distribution of $K_{m,n}(t_0)$ which is the finite mixture of normal distributions.

THEOREM 2. *Under H_0, as $n \to \infty$ and $m = o(n^{1/3-\delta})$, $\delta > 0$,*

$$(4.5) \quad P(K_{m,n}(t_0) \le y) = \sum_{k=0}^{n} \Phi(\sqrt{6mk}(y + \log 2m + \gamma - R(1, 2m-1))) \binom{n}{k} t_0^{n-k}(1-t_0)^k,$$

where $R(1, m-1) = \frac{1}{m-1} + \cdots + \frac{1}{2} + 1$ if $m \ge 2$, $R(1,0) = 0$ and Φ is the distribution function of the standard normal distribution.

Proof. The proof relies heavily on the fact that given $\ell_{t_0} = \sum_{i=1}^{n} I(t_i - t_0)$, there is a sample of size ℓ_{t_0} from the uniform distribution over the interval $(t_0, 1)$ and ℓ_{t_0} is Binomial random variable with parameters n and $1 - t_0$.

Now, with some reorganization, $H_{m,n}(F_n; t_0)$ in (2.6) can be written as

$$(4.6) \qquad H_{m,n}(F_n; t_0) = \frac{-1}{n-j+1} \sum_{i=j}^{n} \log \frac{n-j+1}{2m}(T(i+m) - T(i-m)).$$

When t_0 belongs to an interval $[0, b]$, $b < 1$, given ℓ_{t_0}, from (4.3) and (4.4) it follows that $K_{m,n}(t_0)$ is asymptotically normal. Furthermore, using the fact that ℓ_{t_0} is Binomial,

$$\begin{aligned} P(K_{m,n}(t_0) < y) &= \sum_{k=0}^{n} P(K_{m,n}(t_0) < y \mid \ell_{t_0} = k) \binom{n}{k} (1-t_0)^k t_0^{n-k} \\ &\approx \sum_{k=0}^{n} \Phi(\sqrt{6mk}(y + \log 2m + \gamma - R(1, 2m-1))) \binom{n}{k} (1-t_0)^k t_0^{n-k}. \end{aligned}$$

This completes the proof.

Based on the asymptotic distribution of $K_{m,n}(t_0)$ under H_0 (Theorem 2), approximate percentiles can be obtained by the equation (4.6). A simple approximation to (4.6) can also be obtained simply by replacing the average over k by the single normal cumulative distribution function with $k = nt_0$. That is,

$$(4.7) \qquad P(K_{m,n}(t_0) < y) \approx \Phi(\sqrt{6mnt_0}(y + \log 2m + \gamma - R(1, 2m-1))).$$

We have compared the percentiles of $K_{m,n}(t_0)$ for different values of n and t_0 using the approximation given by (4.7) and estimated percentiles from Tables 2–4. It seems that only for $n \ge 200$ our asymptotic result becomes reasonable. For this reason, the Monte Carlo values of the percentages given in Tables 2–4 will be needed for most practical applications.

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