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ESTIMATION WITH SEQUENTIAL ORDER STATISTICS FROM EXPONENTIAL DISTRIBUTIONS

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Abstract. The lifetime of an ordinary k-out-of-n system is described by the $(n-k+1)$-st order statistic from an iid sample. This set-up is based on the assumption that the failure of any component does not affect the remaining ones. Since this is possibly not fulfilled in technical systems, sequential order statistics have been proposed to model a change of the residual lifetime distribution after the breakdown of some component. We investigate such sequential k-out-of-n systems where the corresponding sequential order statistics, which describe the lifetimes of these systems, are based on one- and two-parameter exponential distributions. Given differently structured systems, we focus on three estimation concepts for the distribution parameters. MLEs, UMVUEs and BLUEs of the location and scale parameters are presented. Several properties of these estimators, such as distributions and consistency, are established. Moreover, we illustrate how two sequential k-out-of-n systems based on exponential distributions can be compared by means of the probability $P(X < Y)$. Since other models of ordered random variables, such as ordinary order statistics, record values and progressive type II censored order statistics can be viewed as sequential order statistics, all the results can be applied to these situations as well.

Key words and phrases: Sequential k-out-of-n system, sequential order statistics, order statistics, record values, progressive type II censoring, maximum likelihood estimation, best linear unbiased estimation, uniformly minimum variance unbiased estimation, exponential distribution, Weinman multivariate exponential distribution.

1. Introduction

k-out-of-n systems are important technical structures which are often considered in the literature. Such systems consist of n components of the same kind with independent and identically distributed (iid) lifelengths. All components start working simultaneously, and the system will work as long as k components function. Parallel and series systems are particular cases of k-out-of-n systems corresponding to $k = 1$ and $k = n$, respectively. Other examples of systems with k-out-of-n structure are presented in, e.g., Meeker and Escobar (1998). For instance, an aircraft with three engines will not crash if at least two of them are functioning, or a satellite will have enough power to send signals if not more than four out of its ten batteries are discharged. In the conventional modeling of these structures it is supposed that the failure of any component does not affect the remaining ones. Hence, the $(n - k + 1)$-st order statistic from an iid sample describes the lifetime of some k-out-of-n system. Due to this connection, the theory of order statistics is utilized in the probabilistic analysis of these models and in the related statistical inference. For detailed expositions on order statistics we refer to David (1981)

$$
\begin{array}{ccl|ccccc l}
X_*^{(1)} & \leftarrow & & Z_1^{(1)} & Z_2^{(1)} & \cdots & Z_{n-1}^{(1)} & Z_n^{(1)} & \sim F_1(\cdot) \\
X_*^{(2)} & \leftarrow & & Z_1^{(2)} & Z_2^{(2)} & \cdots & Z_{n-1}^{(2)} & & \sim \dfrac{F_2(\cdot) - F_2(z_{1,n}^{(1)})}{1 - F_2(z_{1,n}^{(1)})} \\
\vdots & & & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & & & \qquad \vdots \\
X_*^{(n-1)} & \leftarrow & & Z_1^{(n-1)} & Z_2^{(n-1)} & & & & \\
X_*^{(n)} & \leftarrow & & Z_1^{(n)} & & & & & \sim \dfrac{F_n(\cdot) - F_n(z_{1,2}^{(n-1)})}{1 - F_n(z_{1,2}^{(n-1)})} \\
\uparrow & & & & & & & &
\end{array}
$$

line minima $X_*^{(1)} \leq \cdots \leq X_*^{(n)}$

Fig. 1. Triangular scheme for sequential order statistics.

and Arnold *et al.* (1992).

The assumption that the breakdown of some component does not influence the components at work will generally not be fulfilled in real life. For example, the breakdown of an aircraft's engine will increase the load put on the remaining engines, such that their lifetime should tend to be shorter.

Sequential order statistics have been introduced as an extension of (ordinary) order statistics to model 'sequential k-out-of-n systems', where the failures of components possibly affect the remaining ones. This can be thought of as a damage caused by failures or, as mentioned before, as an increased stress put on the active components. The model of sequential order statistics is flexible in the sense that, after the failure of some component, the distribution of the residual lifetime of the components may change. For a more detailed discussion we refer to Kamps ((1995), Chapter I.1) and to Cramer and Kamps (1996).

The sequential $(n-r+1)$-out-of-n system, the lifetime of which is described by the r-th sequential order statistic, can be illustrated as follows:

Let $F_1, \ldots, F_n$ be continuous distribution functions, and let $z_{1,n}^{(1)} \leq z_{1,n-1}^{(2)} \leq \cdots \leq z_{1,2}^{(n-1)}$ be real numbers. Consider a triangular scheme (see Fig. 1) $(Z_j^{(i)})_{1\leq i\leq n, 1\leq j\leq n-i+1}$ of random variables. The random variables $(Z_j^{(i)})_{1\leq j\leq n-i+1}$ are supposed to be iid according to the distribution function

$$\frac{F_i(\cdot) - F_i(z_{1,n-i+2}^{(i-1)})}{1 - F_i(z_{1,n-i+2}^{(i-1)})}, \quad 1 \leq i \leq n, \quad z_{1,n+1}^{(0)} = -\infty,$$

which is F_i truncated on the left at the occurrence time $z_{1,n-i+2}^{(i-1)}$ of the $(i-1)$-st failure in the system. Supposing $z_{1,n-i+2}^{(i-1)}$ to be the realization of the sample minimum in line $i-1$ (cf. Fig. 1), the next failure time $X_*^{(i)}$ is modeled as the minimum in the sample $Z_1^{(i)}, \ldots, Z_{n-i+1}^{(i)}$ of iid random variables with survival function $(1 - F_i(\cdot))/(1 - F_i(z_{1,n-i+2}^{(i-1)}))$. These random variables represent the lifetimes of the remaining $n-i+1$ components after the $(i-1)$-st failure. A precise description in the general set-up as well as the joint density function of the first r sequential order statistics based on F_1, $F_2, \ldots$

Table 1. Parameter choices for models of ordered random variables (cf. Kamps, 1999).

model	α_j, $1 \le j \le n$
ordinary order statistics	1
generalized order statistics	$\gamma_j/(n-j+1)$
k-th record values	$k/(n-j+1)$
progressive type II censoring	$(N+1-j-\sum_{v=1}^{j-1} R_v)/(n-j+1)$

are shown in Kamps (1995).

In a previous paper (Cramer and Kamps (1996)), the authors were concerned with maximum likelihood estimators (MLE) of the model parameters in a sequential k-out-of-n system based on the distribution functions $F_i = 1-(1-F)^{\alpha_i}$, $1 \le i \le n$, where F is an absolutely continuous distribution function and $\alpha_1, \dots, \alpha_n$ are positive real numbers. This choice of the distribution functions leads to the hazard function $\alpha_{i+1} f/(1-F)$ of each component at work on level $i+1$, i.e., after the i-th failure, such that the parameters α_1, $\alpha_2, \dots$ model the influence of a failure on the remaining components. The MLEs $\alpha_1^*, \dots, \alpha_r^*$ of $\alpha_1, \dots, \alpha_r$ based on independent observations of some sequential $(n-r+1)$-out-of-n system turned out to be independent, inverted gamma distributed random variables (see Cramer and Kamps (1996)). Moreover, it has been shown that the estimators are sufficient, strongly consistent and asymptotically normal with respect to an increasing number of independent observations. We also considered simultaneous maximum likelihood estimation of both the model parameters and the parameters of specific distributions such as Pareto and Weibull distributions.

As mentioned above, the model discussed in Cramer and Kamps (1996) is restricted to a particular choice of the underlying distribution functions, i.e., $F_i = 1-(1-F)^{\alpha_i}$, $1 \le i \le n$. Aside from the interpretation in terms of hazard rates, this limitation is reasonable in order to reduce the uncertainty in the model to the parameters α_1, $\alpha_2, \dots$ and the distribution function F. Although this setting seems to be very restrictive, many models of ordered random variables are included in the distribution theoretical sense. Table 1 gives some examples of well known models along with the respective choices of the parameters α_j, $1 \le j \le n$.
For more details and further models we refer to Kamps (1995, 1999). Since, in the following, the model of progressive type II censoring becomes an important particular case, we describe it in some detail. This sampling scheme proceeds as follows: N objects are subjected to a life test. The random variables representing their lifetimes are assumed to be independent and identically distributed. At the time of the first breakdown, R_1 objects of the remaining $N-1$ working components are withdrawn at random from the experiment, such that $N-R_1-1$ items remain in the test. At the time of the second failure, R_2 items are removed randomly, etc. This procedure is continued until n objects have failed and the others have been withdrawn from the experiment. The sampling scheme is represented by the censoring scheme $(R_1, \dots, R_n)$ and the number of items $N = n + \sum_{j=1}^{n} R_j$ at the beginning of the experiment. Recent results for progressive type II censored samples are provided by, e.g., Viveros and Balakrishnan (1994), Balakrishnan and Sandhu (1995, 1996) and Cohen (1995).

In this paper, we consider one- and two-parameter exponential distributions and focus on three estimation concepts for the distribution parameters. In Section 2 we introduce the model and derive an interesting interpretation of sequential order statistics in the case of underlying exponential distributions. Following (Section 3.1), we derive the MLE of the scale parameter assuming that the location parameter is known. Some properties such as consistency and asymptotic normality are presented. By calculating the Cramér-Rao lower bound, the MLE of the scale parameter is seen to coincide with the uniformly minimum variance unbiased estimator (UMVUE). Moreover, since the MLE is linear, it is the best linear unbiased estimator (BLUE) as well. Section 3.2 is concerned with the two-parameter exponential distribution. We calculate the MLEs, the UMVUEs and the BLUEs of the distribution parameters. In contrast to the case of a known location parameter, the estimators of the scale parameter turn out to be different. Furthermore, we present several properties of the estimators. Obviously, estimation results for the particular models given in Table 1 are included. In Section 4 we summarize some results on the UMVUE of $P(X < Y)$ based on order statistics from a sample, whose joint distribution is a Weinman multivariate exponential distribution (for details see Cramer and Kamps (1997*b*)). We illustrate how these results can be applied to compare two different sequential k-out-of-n systems represented by sequential order statistics $X_*^{(1)}, \dots, X_*^{(r_1)}$ and $Y_*^{(1)}, \dots, Y_*^{(r_2)}$, respectively. Moreover, we point out that these results apply to estimation procedures based on record observations as well as on those based on progressive type II censored data. In these situations the resulting estimators of $P(X < Y)$ simplify considerably, if the location parameter is assumed to be unknown.

2. Sequential order statistics from exponential distributions

In the present paper, we consider exponential distributions as underlying lifetime distributions. Given this assumption, the definition of sequential order statistics simplifies considerably. Let the two-parameter exponential distribution be defined by its survival function

$$(2.1) \qquad 1 - F(t) = \exp\left\{-\frac{t-\mu}{\vartheta}\right\}, \qquad t \geq \mu,$$

with location parameter $\mu \in \mathbb{R}$ and scale parameter $\vartheta > 0$. We denote this distribution by $\mathrm{Exp}(\mu, \vartheta)$. For an excellent survey on the exponential distribution we refer to the collection of Balakrishnan and Basu (1995).

The general definition of sequential order statistics based on continuous distribution functions $F_1, \dots, F_n$ is given by

$$X_*^{(i)} = \min_{1 \leq j \leq n-i+1} F_i^{-1}[F_i(Y_j^{(i)})(1 - F_i(X_*^{(i-1)})) + F_i(X_*^{(i-1)})], \qquad 1 \leq i \leq n,$$

where $X_*^{(0)} = -\infty$ and $(Y_j^{(i)})_{1 \leq i \leq n, 1 \leq j \leq n-i+1}$ is a sequence of independent random variables with $Y_j^{(i)} \sim F_i$ (cf. Kamps (1995), p. 27). In the case of exponential distributions (2.1), the inverse of $F_i(t) = 1 - \exp\{-(\alpha_i/\vartheta)(t-\mu)\}$, $t \geq \mu$, is given by $F_i^{-1}(x) = -(\vartheta/\alpha_i)\ln(1-x) + \mu$, $x \in [0, 1)$. Applying this distributional assumption, we obtain:

LEMMA 2.1. *Let* $(Y_j^{(i)})_{1 \leq i \leq n, 1 \leq j \leq n-i+1}$ *be a triangular scheme of independent random variables, where the random variables* $(Y_j^{(i)})_{1 \leq j \leq n-i+1}$ *are iid according to* $\mathrm{Exp}(\mu, \vartheta/\alpha_i)$, $1 \leq i \leq n$, *with* $\mu \in \mathbb{R}$ *and positive real numbers* $\alpha_1, \dots, \alpha_n$ *and* ϑ.

Sequential order statistics $X_*^{(1)}, \dots, X_*^{(n)}$ based on these random variables are then defined by

$$(2.2) \qquad X_*^{(1)} = Y_{1,n}^{(1)}, \qquad X_*^{(j)} = X_*^{(j-1)} + Y_{1,n-j+1}^{(j)} - \mu = \mu + \sum_{k=1}^{j} (Y_{1,n-k+1}^{(k)} - \mu), \quad 2 \le j \le n,$$

where $Y_{1,n-j+1}^{(j)}$ denotes the minimum of $Y_1^{(j)}, \dots, Y_{n-j+1}^{(j)}$, $1 \le j \le n$.

Hence, the distribution of $X_*^{(j)} - \mu$ is given by the distribution of the convolution of independent minima from exponential distributions with possibly different scale parameters and common location parameter zero.

The situation in Lemma 2.1 could be paraphrased as follows: If $r-1$ components have failed, the residual lifetimes of the remaining components are supposed to be exponentially distributed with scale parameters $\vartheta/\alpha_r > 0$, $2 \le r \le n$.

The joint distribution of $X_*^{(1)}, \dots, X_*^{(r)}$ is given by the Lebesgue density

$$(2.3) \quad f^{X_*^{(1)}, \dots, X_*^{(r)}}(x_1, \dots, x_r) = \frac{n!}{(n-r)!} \frac{1}{\vartheta^r} \prod_{i=1}^{r} \left[\alpha_i \exp\left\{ -\frac{\alpha_i}{\vartheta}(n-i+1)(x_i - x_{i-1}) \right\} \right],$$

where $\mu = x_0 \le x_1 \le \cdots \le x_r$. This distribution coincides with the distribution of the first r (out of n) order statistics based on a Weinman multivariate exponential distribution (cf. Weinman (1966), Johnson and Kotz (1972), p. 268–269, Block (1975), p. 303), which in Cramer and Kamps (1997*b*) is referred to as $\mathrm{WME}_n(\mu, \vartheta, \tilde{\alpha})$, $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n)$. Hence, our results contribute to the analysis of Weinman's multivariate exponential distribution. Regarding related works in estimation theory, we are only aware of point estimation results for the parameters α_i given by Weinman (1966) and for the entropy of the distribution (2.3) (cf. Ahmed and Gokhale (1989)).

Putting $\alpha_1 = \cdots = \alpha_r = 1$, we obtain in (2.3) the joint density function of ordinary order statistics $X_{1,n}, \dots, X_{r,n}$ based on iid random variables $X_1, \dots, X_n$ with distribution $\mathrm{Exp}(\mu, \vartheta)$.

Remark 2.1. In Lemma 2.1, the situation of sequential k-out-of-n systems is described in a distribution theoretical way, which means that we do not assign failures to components. This may be done by analogy with Heinrich and Jensen (1995), who present a general, bivariate set-up in the sense of Freund (1961).

Starting with a scheme $(Y_j^{(i)})_{1 \le i,j \le n}$ of independent random variables, where the $(Y_j^{(i)})_{1 \le j \le n}$ are iid according to $\mathrm{Exp}(\mu, \vartheta/\alpha_i)$, $1 \le i \le n$, let $X_*^{(1)} = Y_{1,n}^{(1)}$, $X_*^{(2)} = X_{2,n}^{(2)}$, which denotes the second order statistic from the sample $X_j^{(2)} = X_*^{(1)} + (Y_j^{(2)} - \mu)\mathbb{1}_{\{Y_j^{(1)} \ne Y_{1,n}^{(1)}\}}$, $1 \le j \le n$. Obviously, we have $X_{2,n}^{(2)} \sim X_*^{(1)} + Y_{1,n-1}^{(2)} - \mu$ (cf. Lemma 2.1). This description becomes complicated in the next steps, and thus we proceed as in Lemma 2.1, instead.

3. Estimation of location and scale parameters

Throughout this section, we are concerned with the following situation. We take a sample of size s of possibly differently structured $(n_i - r_i + 1)$-out-of-n_i systems, $1 \le r_i \le$

n_i, $1 \le i \le s$, with r_i (dependent) observations each, and with known model parameters $(\alpha_{ij})_{1\le i\le s, 1\le j\le r_i}$. This leads to the set of data $(x_{ij})_{1\le i\le s, 1\le j\le r_i}$ with $x_{i1} \le \cdots \le x_{ir_i}$, $1 \le i \le s$.

The corresponding sequential order statistics are denoted by $(X_{*i}^{(j)})_{1\le i\le s, 1\le j\le r_i}$, which, by assumption, are independent with respect to the index i.

3.1 *Known location parameter*

First, we consider maximum likelihood estimation of the scale parameter ϑ.

THEOREM 3.1. *The MLE of ϑ is given by*

$$\vartheta^* = \frac{1}{R}\sum_{i=1}^{s}\sum_{j=1}^{r_i}(n_i - j + 1)\alpha_{ij}(X_{*i}^{(j)} - X_{*i}^{(j-1)})$$

with $X_{*i}^{(0)} = \mu$, $1 \le i \le s$, $R = \sum_{i=1}^{s} r_i$.

PROOF. Considering the log-likelihood function

$$\begin{aligned} l(\vartheta, \mu; (\alpha_{ij})_{i,j}, (x_{ij})_{i,j}) = & \sum_{i=1}^{s}\log\frac{n_i!}{(n_i - r_i)!} + \sum_{i=1}^{s}\sum_{j=1}^{r_i}\log\alpha_{ij} \\ & -\frac{1}{\vartheta}\sum_{i=1}^{s}\sum_{j=1}^{r_i}(n_i - j + 1)\alpha_{ij}(x_{ij} - x_{i,j-1}) - R\log\vartheta, \end{aligned}$$

the assertion directly follows. □

Remark 3.1. (i) In the situation of ordinary type II censoring described by ordinary order statistics, i.e., $\alpha_{ij} = 1$, $r_i = r$, $n_i = n$, $1 \le i \le s$, $1 \le j \le r$, the results of Theorem 3.1 can be found in Lawless ((1982), p. 102) and in Johnson *et al.* ((1994), p. 514) in the case of one sample ($s = 1$).

(ii) In the particular case $\alpha_{ij} = \alpha_i$ for all j, the estimator ϑ^* can be written as

$$\begin{aligned} \vartheta^* &= \frac{1}{R}\sum_{i=1}^{s}\alpha_i\left[(n_i - r_i + 1)X_{*i}^{(r_i)} + \sum_{j=1}^{r_i-1}X_{*i}^{(j)} - n_i\mu\right] \\ &= \frac{1}{R}\sum_{i=1}^{s}\alpha_i\left[(n_i - r_i + 1)(X_{*i}^{(r_i)} - \mu) + \sum_{j=1}^{r_i-1}(X_{*i}^{(j)} - \mu)\right], \end{aligned}$$

which, for $s = 1$ and $\alpha_i = 1$ for all i, leads to the representation usually found in the literature (see, e.g., Epstein (1957), eq. (3), $\mu = 0$).

(iii) The result in terms of progressive type II censoring with $s = 1$ is given in Cohen (1995) ($\mu = 0$).

The MLE ϑ^* possesses some interesting properties, which we summarize in the following theorem.

THEOREM 3.2. *In the above situation, with $\vartheta^* = \vartheta^*(R)$, we find that*

(i) $\vartheta^* \sim \Gamma(R, \vartheta/R)$, *i.e.*, ϑ^* *is a gamma distributed random variable with parameters* R *and* ϑ/R. *Its density function is given by*

$$f_{\vartheta^*}(t) = \frac{(R/\vartheta)^R}{(R-1)!} t^{R-1} e^{-Rt/\vartheta}, \quad t \geq 0.$$

(ii) $E(\vartheta^*)^k = \frac{(R+k-1)!}{(R-1)!}(\frac{\vartheta}{R})^k$, $k \in \mathbb{N}$; *in particular*, $E\vartheta^* = \vartheta$ *and* $\mathrm{Var}(\vartheta^*) = \frac{\vartheta^2}{R}$. *Hence*, ϑ^* *is an unbiased estimator of* ϑ.

(iii) ϑ^* *is sufficient for* ϑ.

(iv) $(\vartheta^*(R))_R$ *is strongly consistent for* ϑ, *i.e.*, $\vartheta^*(R) \to \vartheta$ *a.e. w.r.t.* $R \to \infty$.

(v) $(\vartheta^*(R))_R$ *is asymptotically normal, i.e.*, $\sqrt{R}(\vartheta^*(R)/\vartheta - 1) \to^d \mathfrak{N}(0,1)$ *w.r.t.* $R \to \infty$.

PROOF. (i), (ii), (iv) and (v) are based on the fact that normalized spacings of sequential order statistics with $F \equiv \mathrm{Exp}(0,1)$ are independent and again standard exponentially distributed (cf. Kamps (1995), p. 81; see Viveros and Balakrishnan (1994) for the particular case of progressive type II censoring). Hence,

$$\frac{1}{\vartheta}(n_i - j + 1)\alpha_{ij}(X_{*i}^{(j)} - X_{*i}^{(j-1)}) \sim \mathrm{Exp}(0,1), \quad 1 \leq i \leq s, \quad 1 \leq j \leq r_i,$$

and all these random variables are independent ($X_{*i}^{(0)} = \mu$, $1 \leq i \leq s$).

(iii) follows applying the Fisher-Neyman factorization criterion for sufficient statistics (cf. Lehmann and Casella (1998), p. 35). □

THEOREM 3.3. *The MLE* ϑ^* *attains the Cramér-Rao lower bound. Since* ϑ^* *is unbiased and linear, it coincides with the UMVUE and the BLUE of* ϑ.

PROOF. The log-likelihood function is shown in the Proof of Theorem 3.1. Differentiating twice with respect to ϑ yields

$$\frac{\partial^2}{\partial \vartheta^2} l(\vartheta; \mu, (\alpha_{ij})_{i,j}, (X_{*i}^{(j)})_{i,j}) = -\frac{2}{\vartheta^3} \sum_{i=1}^{s} \sum_{j=1}^{r_i} (n_i - j + 1)\alpha_{ij}(X_{*i}^{(j)} - X_{*i}^{(j-1)}) + \frac{R}{\vartheta^2}.$$

Noticing that

$$\frac{1}{\vartheta}(n_i - j + 1)\alpha_{ij}(X_{*i}^{(j)} - X_{*i}^{(j-1)}) \sim \mathrm{Exp}(0,1), \quad 1 \leq j \leq r_i, \quad 1 \leq i \leq s,$$

(cf. the Proof of Theorem 3.2), we obtain for the Fisher information

$$-E\left(\frac{\partial^2}{\partial \vartheta^2} l(\vartheta; \mu, (\alpha_{ij})_{i,j}, (X_{*i}^{(j)})_{i,j})\right) = \frac{R}{\vartheta^2}.$$

The Cramér-Rao lower bound is thus ϑ^2/R (cf. Lehmann and Casella (1998), p. 116–120).

Since ϑ^* is linear, unbiased and $\mathrm{Var}(\vartheta^*) = \vartheta^2/R$ (cf. Theorem 3.2 (ii)), the MLE ϑ^* coincides with the BLUE and the UMVUE. □

It is possible to calculate the BLUE of ϑ by the classical Gauß-Markov theorem as well. This derivation is based on the covariance matrix of the sequential order statistics.

Although we have already calculated the BLUE we compute the covariances, since we need them in the next section.

In the case of an underlying exponential distribution ($F \equiv \text{Exp}(\mu, \vartheta)$), the sequential order statistics $X_{*i}^{(1)}, \ldots, X_{*i}^{(n_i)}$ follow the distribution of order statistics from a Weinman multivariate exponential distribution $\text{WME}_{n_i}(\mu, \vartheta, \tilde{\alpha}_i)$, $\tilde{\alpha}_i = (\alpha_{i1}, \ldots, \alpha_{in_i})$, $1 \leq i \leq s$. We utilize this fact to calculate the covariance matrix Σ of the sequential order statistics $X_{*i}^{(1)}, \ldots, X_{*i}^{(r_i)}$, $1 \leq i \leq s$.

From the independence assumption we conclude that the covariance matrix Σ is a block diagonal matrix $\Sigma = \text{diag}(\Sigma_1, \ldots, \Sigma_s)$, where $\Sigma_i \in \mathbb{R}^{r_i \times r_i}$ is the covariance matrix of $(X_{*i}^{(1)}, \ldots, X_{*i}^{(r_i)})'$, $1 \leq i \leq s$. We evaluate this expression for one block Σ_i applying the moment generating function of the Weinman multivariate exponential distribution, i.e.,

$$E\left[\exp\left(\sum_{j=1}^{n_i} t_j X_{*i}^{(j)}\right)\right] = \prod_{j=0}^{n_i-1}\left[1 - \frac{\vartheta}{(n_i - j)\alpha_{i,j+1}} \sum_{k=j+1}^{n_i} t_k\right]^{-1}$$

(cf. Johnson and Kotz (1972), p. 269). This leads to the covariances ($1 \leq j \leq k \leq r_i$)

$$\text{Cov}(X_{*i}^{(j)}, X_{*i}^{(k)}) = \text{Var}(X_{*i}^{(j)}) = \vartheta^2 \sum_{\nu=1}^{j} [(n_i - \nu + 1)\alpha_{i\nu}]^{-2} = \vartheta^2 a_i^{(j)}, \quad \text{say.}$$

Hence, the covariance matrix of $(X_{*i}^{(1)}, \ldots, X_{*i}^{(r_i)})'$ is given by

$$\Sigma_i = \vartheta^2 \begin{bmatrix} a_i^{(1)} & a_i^{(1)} & a_i^{(1)} & \cdots & a_i^{(1)} \\ a_i^{(1)} & a_i^{(2)} & a_i^{(2)} & \cdots & a_i^{(2)} \\ a_i^{(1)} & a_i^{(2)} & a_i^{(3)} & \cdots & a_i^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_i^{(1)} & a_i^{(2)} & a_i^{(3)} & \cdots & a_i^{(r_i)} \end{bmatrix} = \vartheta^2 \Delta_i, \quad \text{say,}$$

with $a_i^{(1)} < a_i^{(2)} < \cdots < a_i^{(r_i)}$. Choosing $\alpha_{ij} = 1$ for all $1 \leq j \leq r_i$, Σ_i reduces to the covariance matrix in the case of a sample of ordinary order statistics from an exponential distribution (cf. Sarhan (1954), Balakrishnan and Cohen (1991)). The inverse of Δ_i is given by

$$\Delta_i^{-1} = \begin{bmatrix} b_i^{(1)} + b_i^{(2)} & -b_i^{(2)} & 0 & \cdots & \cdots & 0 \\ -b_i^{(2)} & b_i^{(2)} + b_i^{(3)} & -b_i^{(3)} & 0 & & \\ 0 & -b_i^{(3)} & b_i^{(3)} + b_i^{(4)} & -b_i^{(4)} & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & & & -b_i^{(r_i)} \\ & & & & -b_i^{(r_i)} & b_i^{(r_i)} \end{bmatrix}$$

with $b_i^{(j)} = [(n_i - j + 1)\alpha_{ij}]^2$, $1 \leq j \leq r_i$, (cf. Roy and Sarhan (1956), Graybill (1983), p. 187).

By this, we may complete the straightforward calculation to obtain the BLUE of ϑ (cf. Theorem 3.3). The Gauß-Markov theorem leads to the representation

$$(\beta'\Delta^{-1}\beta)^{-1}\beta'\Delta^{-1}\vec{X} = \left(\sum_{i=1}^{s} \beta_i \Delta_i^{-1} \beta_i'\right)^{-1} \sum_{i=1}^{s} \beta_i \Delta_i^{-1} (X_{*i}^{(1)}, \ldots, X_{*i}^{(r_i)})'$$

of ϑ^*, where $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_s)$, $\vec{X} = (X_{*1}^{(1)}, \ldots, X_{*1}^{(r_1)}, X_{*2}^{(1)}, \ldots, X_{*s}^{(r_s)})'$, $\beta = (\beta_1, \ldots, \beta_s)'$, and $\beta_i = E(X_{*i}^{(1)}, \ldots, X_{*i}^{(r_i)})/\vartheta$, $1 \leq i \leq s$.

Remark 3.2. Choosing particular values for the parameters α_{ij}, the preceding calculations lead to results for the particular models as pointed out in the introduction. Previous results deal with only one sample, i.e., $s = 1$. For ordinary order statistics we refer to Epstein (1957) and Engelhardt (1995). Progressive type II censoring is considered in Cohen (1995) and Balakrishnan and Sandhu (1995, 1996). In the model of record values, i.e., $\alpha_{ij} = 1/(n_i - j + 1)$, the estimator ϑ^* simplifies as follows:

$$\vartheta^* = \frac{1}{R}\sum_{i=1}^{s} X_{*i}^{(r_i)} \stackrel{s=1}{=} \frac{1}{r_1} X_{*1}^{(r_1)}.$$

The latter expression can be found in the dissertation of Houchens ((1984), p. 26) (see also Arnold *et al.* (1998), p. 122).

3.2 *Unknown location parameter*

As in the preceding section we start with the MLEs of the distribution parameters. We omit the proof since it proceeds similar to the case of a known location parameter.

THEOREM 3.4. *The simultaneous MLEs of* μ *and* ϑ *are given by*

$$\tilde{\mu} = \min\{X_{*1}^{(1)}, \ldots, X_{*s}^{(1)}\} \quad \textit{and}$$
$$\tilde{\vartheta} = \frac{1}{R}\sum_{i=1}^{s}\sum_{j=1}^{r_i}(n_i - j + 1)\alpha_{ij}(X_{*i}^{(j)} - X_{*i}^{(j-1)}), \quad \textit{respectively,}$$

with $X_{*i}^{(0)} = \tilde{\mu}$, $1 \leq i \leq s$, $R = \sum_{i=1}^{s} r_i$.

Now, we focus on some properties of the stated MLEs. First of all, it is shown that the MLE $\tilde{\mu}$ of the location parameter and the MLE $\tilde{\vartheta}$ of the scale parameter are independent. We make use of the following auxiliary result:

Let $(V_j)_{j\in\mathbb{N}}$ be a sequence of iid random variables with $V_j \sim \text{Exp}(0,1)$, let $(a_j)_{j\in\mathbb{N}}$ be a sequence of nonnegative real numbers and $m \in \mathbb{N}$. Then the following equations hold for $x, t \geq 0$:

$$\begin{aligned}(3.1)\quad & P\left(\sum_{j=1}^{m} V_j \leq x + \left(\sum_{j=1}^{m} a_j\right) t, V_j > a_j t, j = 1, \ldots, m\right)\\ &= \int_{a_m t}^{\infty} P\left(\sum_{j=1}^{m-1} V_j \leq x + \left(\sum_{j=1}^{m} a_j\right) t - z, V_j > a_j t,\right.\\ &\qquad\qquad \left. j = 1, \ldots, m-1\right) dP^{V_m}(z)\\ &= \exp\left\{-t\sum_{j=1}^{m} a_j\right\} P\left(\sum_{j=1}^{m} V_j \leq x\right) = \exp\left\{-t\sum_{j=1}^{m} a_j\right\}\left(1 - e^{-x}\sum_{\nu=0}^{m-1}\frac{x^\nu}{\nu!}\right).\end{aligned}$$

The second equation in (3.1) follows by induction on m.

Equation (3.1) enables us to establish the mentioned result.

THEOREM 3.5. (i) *The MLEs $\tilde{\vartheta}$ and $\tilde{\mu}$ are stochastically independent.*
(ii) $\tilde{\vartheta} \sim \Gamma(R-1, \vartheta/R)$, $\tilde{\mu} \sim \mathrm{Exp}(\mu, \vartheta/\sum_{i=1}^{s}(n_i\alpha_{i1}))$.
(iii) *$(\tilde{\mu}, \tilde{\vartheta})$ is a complete sufficient statistic for (μ, ϑ).*

PROOF. In the case $s = 1$, assertions (i) and (ii) are obvious (cf. Engelhardt (1995), p. 85). Hence, suppose $s \geq 2$. From Theorem 3.4 we deduce the following additive decomposition of $\tilde{\vartheta}$:

$$\begin{aligned}\tilde{\vartheta} &= \frac{1}{R}\sum_{i=1}^{s}\sum_{j=2}^{r_i}(n_i - j + 1)\alpha_{ij}(X_{*i}^{(j)} - X_{*i}^{(j-1)}) \\ &\quad + \frac{1}{R}\left(\sum_{i=1}^{s} n_i\alpha_{i1}(X_{*i}^{(1)} - \mu) - \left(\sum_{i=1}^{s} n_i\alpha_{i1}\right)(\tilde{\mu} - \mu)\right) \\ &= \vartheta_1^* + \vartheta_2^*, \quad \text{say.}\end{aligned}$$

The random variables ϑ_1^* and ϑ_2^* are independent, because ϑ_1^* is a function of the spacings $X_{*i}^{(j)} - X_{*i}^{(j-1)}$, $j \geq 2$, and ϑ_2^* depends only on the minima $X_{*i}^{(1)}$, $1 \leq i \leq s$. Moreover, since $\tilde{\mu}$ is a function of $X_{*i}^{(1)}$, $1 \leq i \leq s$, ϑ_1^* and $\tilde{\mu}$ are independent, too. Hence, it is sufficient to prove that ϑ_2^* and $\tilde{\mu}$ are independent. We establish this assertion by calculating the joint cumulative distribution function of $\tilde{\mu}$ and ϑ_2^*. Putting $a_i = n_i\alpha_{i1}$ and $Y_i = (X_{*i}^{(1)} - \mu) \sim \mathrm{Exp}(0, \vartheta/a_i)$, $1 \leq i \leq s$, we rewrite ϑ_2^* as follows

$$\vartheta_2^* = \frac{1}{R}\left(\sum_{i=1}^{s} a_i Y_i - \left(\sum_{i=1}^{s} a_i\right) Y_{1,s}\right), \qquad \tilde{\mu} = Y_{1,s} + \mu, \quad Y_{1,s} = \min_{i=1,\ldots,s} Y_i.$$

Hence, the calculation proceeds as follows ($x \geq 0, y \geq \mu$):

$$\begin{aligned}&P(\vartheta_2^* \leq x, \tilde{\mu} \leq y) \\ &= P\left(\sum_{i=1}^{s} a_i Y_i - \left(\sum_{i=1}^{s} a_i\right) Y_{1,s} \leq Rx, Y_{1,s} \leq y - \mu\right) \\ &= \sum_{j=1}^{s} P\left(\sum_{i=1}^{s} a_i Y_i - \left(\sum_{i=1}^{s} a_i\right) Y_{1,s} \leq Rx, Y_{1,s} \leq y - \mu, Y_j = Y_{1,s}\right) \\ &= \sum_{j=1}^{s}\int_0^{y-\mu} P\left(\sum_{i=1, i\neq j}^{s} a_i Y_i \leq Rx + \left(\sum_{i=1, i\neq j}^{s} a_i\right) z, z < Y_i, i \neq j\right) dP^{Y_j}(z) \\ &= \sum_{j=1}^{s}\int_0^{y-\mu} \exp\left\{-\frac{z}{\vartheta}\sum_{i=1, i\neq j}^{s} a_i\right\}\left(1 - \mathrm{e}^{-Rx/\vartheta}\sum_{\nu=0}^{s-2}\frac{(Rx/\vartheta)^\nu}{\nu!}\right) dP^{Y_j}(z) \\ &= \left(1 - \exp\left\{-\frac{y-\mu}{\vartheta}\sum_{i=1}^{s} a_i\right\}\right)\left(1 - \mathrm{e}^{-Rx/\vartheta}\sum_{\nu=0}^{s-2}\frac{(Rx/\vartheta)^\nu}{\nu!}\right),\end{aligned}$$

where we have used (3.1) and that $a_i Y_i/\vartheta \sim \mathrm{Exp}(0, 1)$. Having the preceding remarks in mind, this proves the independence of $\tilde{\vartheta}$ and $\tilde{\mu}$.

Obviously, ϑ_2^* follows a gamma distribution with parameters $s-1$ and ϑ/R. Since $\vartheta_1^* \sim \Gamma(R-s, \vartheta/R)$ (cf. the Proof of Theorem 3.2) and ϑ_1^* and ϑ_2^* are independent, we conclude that $\tilde{\vartheta} \sim \Gamma(R-1, \vartheta/R)$. The distribution of $\tilde{\mu}$ immediately follows from the representation of the joint cumulative distribution function.

The sufficiency of $(\tilde{\mu}, \tilde{\vartheta})$ is deduced from the Fisher-Neyman factorization of the likelihood function. The completeness is proved as in Chiou and Cohen (1984). □

Remark 3.3. Suppose that $X_1, \dots, X_s$ form an iid sample from a normal population with parameters μ and σ^2. It is well known that the maximum likelihood estimators of μ and σ^2, i.e., $\bar{X} = \frac{1}{s}\sum_{i=1}^s X_i$ and $s^2 = \frac{1}{s}\sum_{i=1}^s (X_i - \bar{X})^2$, are independent (cf. Bickel and Doksum (1977), p. 20). Moreover, their distributions are given by $\mathfrak{N}(\mu, \sigma^2/s)$ and $(s/\sigma^2)\chi^2_{s-1}$. Considering statements (i) and (ii) of Theorem 3.5, we observe an analogy in our situation, i.e., the estimators $\tilde{\mu}$ and $\tilde{\vartheta}$ are independent with distributions $\mathrm{Exp}(\mu, \vartheta/\sum_{i=1}^s n_i\alpha_{i1})$ and $(R/\vartheta)\chi^2_{2(R-1)}$. For the case of one type II censored sample see Epstein and Sobel (1954) and David ((1981), p. 153). If we consider μ in both situations as location parameter and ϑ and σ^2 as scale parameter, respectively, the analogy is striking.

From Theorem 3.5 we find the following results similar to those given in Theorem 3.2 for a known location parameter.

COROLLARY 3.1. (i) $E(\tilde{\vartheta})^k = \frac{(R+k-2)!}{(R-2)!}(\frac{\vartheta}{R})^k$, $k \in \mathbb{N}$; *in particular*, $E\tilde{\vartheta} = \frac{R-1}{R}\vartheta$ *and* $\mathrm{Var}(\tilde{\vartheta}) = \frac{R-1}{R^2}\vartheta^2$. *Hence*, $\hat{\vartheta} = \frac{R}{R-1}\tilde{\vartheta}$ *is an unbiased estimator of* ϑ.

(ii) $(\tilde{\vartheta}(R))_R$ *is strongly consistent for* ϑ, *i.e.*, $\tilde{\vartheta}(R) \to \vartheta$ *a.e. w.r.t.* $R \to \infty$.

(iii) $(\tilde{\vartheta}(R))_R$ *is asymptotically normal, i.e.*, $\sqrt{R}(\tilde{\vartheta}(R)/\vartheta - 1) \to^d \mathfrak{N}(0,1)$ *w.r.t.* $R \to \infty$.

PROOF. Assertion (i) is obvious. Statement (ii) is seen as follows: Writing

$$\tilde{\vartheta}(R) = \vartheta^*(R) - \frac{\sum_{i=1}^s n_i\alpha_{i1}}{R}(\tilde{\mu}(R) - \mu) \tag{3.2}$$

we conclude from Theorem 3.2 that $\vartheta^*(R) \to \vartheta$ a.e. w.r.t. $R \to \infty$. Moreover, we have from Theorem 3.5 that $Y_s = (\sum_{i=1}^s n_i\alpha_{i1})(\tilde{\mu}(R) - \mu) \sim \mathrm{Exp}(0, \vartheta)$. Hence, we find for $\varepsilon > 0$

$$\begin{aligned}\sum_{s=1}^\infty P\left(\left|\frac{\sum_{i=1}^s n_i\alpha_{i1}}{R}(\tilde{\mu}(R) - \mu)\right| > \varepsilon\right) &= \sum_{s=1}^\infty P(Y_s/R > \varepsilon) = \sum_{s=1}^\infty \exp\left\{-\frac{R\varepsilon}{\vartheta}\right\} \\ &\le \sum_{s=0}^\infty \exp\left\{-\frac{s\varepsilon}{\vartheta}\right\} = \frac{1}{1 - \exp(-\varepsilon/\vartheta)} < \infty.\end{aligned}$$

Applying a result given in Serfling ((1980), 1.3.4., p. 10) we conclude that $Y_s/R \to 0$ a.e. w.r.t. $R \to \infty$. This establishes the assertion.

(iii) From Theorem 3.5 we know that $\tilde{\vartheta}(R) \sim \Gamma(R-1, \vartheta/R)$. Hence,

$$\frac{R}{\sqrt{R-1}}\left(\frac{\tilde{\vartheta}(R)}{\vartheta} - 1 + \frac{1}{R}\right) \xrightarrow{d} \mathfrak{N}(0,1)$$

(cf. Johnson *et al.* (1994), p. 340). An application of Slutsky's Lemma (cf. Serfling (1980), p. 19) proves the result. □

COROLLARY 3.2. (i) *$E(\tilde{\mu})^k = \sum_{j=0}^{k} \frac{k!}{(k-j)!}(\frac{\vartheta}{\sum_{i=1}^{s} n_i\alpha_{i1}})^j \mu^{k-j}$; in particular, $E\tilde{\mu} = \mu + \frac{\vartheta}{\sum_{i=1}^{s} n_i\alpha_{i1}}$ and $\mathrm{Var}(\tilde{\mu}) = (\frac{\vartheta}{\sum_{i=1}^{s} n_i\alpha_{i1}})^2$. Hence, the mean squared error (MSE) is given by $\mathrm{MSE}(\tilde{\mu}) = 2(\frac{\vartheta}{\sum_{i=1}^{s} n_i\alpha_{i1}})^2$.*

(ii) *$(\tilde{\mu}(R))_R$ is asymptotically unbiased provided $\sum_{i=1}^{\infty} n_i\alpha_{i1} = \infty$, i.e., $E\tilde{\mu}(R) \to \mu$ w.r.t. $R \to \infty$. If $\sum_{i=1}^{\infty} n_i\alpha_{i1} < \infty$, the asymptotic bias is given by $\vartheta / \sum_{i=1}^{\infty} n_i\alpha_{i1}$.*

(iii) *$(\tilde{\mu}(R))_R$ is strongly consistent iff $\sum_{i=1}^{\infty} n_i\alpha_{i1} = \infty$.*

PROOF. Assertions (i) and (ii) are deduced immediately from the exponential distribution of $\tilde{\mu}$. The consistency is derived from the result that convergence of a sequence of distributions $(P^{Y_s})_s$ to a degenerate distribution ε_t is equivalent to convergence in probability of the sequence $(Y_s)_s$ to the point t (cf. Serfling (1980), p. 19). It is easily seen that convergence in distribution to a degenerate distribution, i.e., ε_0, holds iff $\sum_{i=1}^{\infty} n_i\alpha_{i1} = \infty$. Additionally, the sequence $(\tilde{\mu}(R) - \mu)_R$ is decreasing and nonnegative. Hence, a well known result yields that convergence in probability and convergence almost everywhere coincide provided these assumptions. □

Since $\hat{\vartheta} = \frac{R}{R-1}\tilde{\vartheta}$ is an unbiased estimator of ϑ, we obtain an unbiased estimator $\hat{\mu}$ of the location parameter μ via a bias correction, i.e.,

$$\hat{\mu} = \tilde{\mu} - \left(\sum_{i=1}^{s} n_i\alpha_{i1}\right)^{-1} \hat{\vartheta}.$$

Theorem 3.6 gives the variances of $\hat{\mu}$ and $\hat{\vartheta}$ as well as their covariance. It extends a result quoted in Cohen (1995) for $s = 1$ and ordinary order statistics.

THEOREM 3.6. (i) *The UMVUE of ϑ is given by $\hat{\vartheta}$. The variance of $\hat{\vartheta}$ is*

$$\mathrm{Var}(\hat{\vartheta}) = \frac{\vartheta^2}{R-1}.$$

(ii) *The UMVUE of μ is given by $\hat{\mu}$. The variance of $\hat{\mu}$ is*

$$\mathrm{Var}(\hat{\mu}) = \frac{R}{(R-1)(\sum_{i=1}^{s} n_i\alpha_{i1})^2}\vartheta^2.$$

(iii) $\mathrm{Cov}(\hat{\mu}, \hat{\vartheta}) = -\frac{1}{(R-1)\sum_{i=1}^{s} n_i\alpha_{i1}}\vartheta^2.$

PROOF. The theorems of Rao-Blackwell and Lehmann-Scheffé yield the optimality of $\hat{\vartheta}$ and $\hat{\mu}$, respectively. The values of the variances follow from the preceding corollaries. In case of $\hat{\mu}$ we have to take into account that $\tilde{\mu}$ and $\tilde{\vartheta}$ are stochastically independent. □

Asymptotic properties of the UMVUEs are obvious and can be derived directly from Corollaries 3.1 and 3.2 applying the results for the MLEs. Thus, we omit the details.

In the case of a known location parameter we have seen that the MLE, the UMVUE and the BLUE of ϑ coincide. This property does not generally hold true in the case of an unknown location parameter, because the MLE $\tilde{\mu}$ of μ is nonlinear if $s \geq 2$. In the following, the BLUEs of μ and ϑ are deduced from the Gauß-Markov theorem, which yields the following matrix representation:

$$\begin{pmatrix} \vartheta_{\text{BLUE}} \\ \mu_{\text{BLUE}} \end{pmatrix} = ((\beta, \mathbb{1}_R)'\Delta^{-1}(\beta, \mathbb{1}_R))^{-1}(\beta, \mathbb{1}_R)'\Delta^{-1}\vec{X}, \tag{3.3}$$

where $\beta = (\beta_1, \ldots, \beta_s)'$ and $\beta_i = ([n_i\alpha_{i1}]^{-1}, \sum_{j=1}^{2}[(n_i - j + 1)\alpha_{ij}]^{-1}, \ldots, \sum_{j=1}^{r_i}[(n_i - j + 1)\alpha_{ij}]^{-1})$, $1 \leq i \leq s$. $\mathbb{1}_R$ denotes the vector $(1, \ldots, 1)'$ of dimension R. The covariance matrix of the BLUEs is given by

$$\operatorname{Cov}\begin{pmatrix} \vartheta_{\text{BLUE}} \\ \mu_{\text{BLUE}} \end{pmatrix} = ((\beta, \mathbb{1}_R)'\Delta^{-1}(\beta, \mathbb{1}_R))^{-1}\vartheta^2.$$

Evaluating these matrix expressions we obtain the following theorem. The proof follows from equation (3.3) by some lengthy calculations, and it is therefore omitted.

THEOREM 3.7. *Let $R = \sum_{i=1}^{s} r_i$, $c^{-1} = R\sum_{i=1}^{s}(n_i\alpha_{i1})^2 - (\sum_{i=1}^{s} n_i\alpha_{i1})^2 > 0$ and $X_{*i}^{(0)} = 0$, $1 \leq i \leq s$.*

The BLUEs of μ and ϑ are given by

$$\vartheta_{\text{BLUE}} = c\left[\left(\sum_{i=1}^{s}(n_i\alpha_{i1})^2\right)\sum_{i=1}^{s}\sum_{j=1}^{r_i}(n_i - j + 1)\alpha_{ij}(X_{*i}^{(j)} - X_{*i}^{(j-1)}) - \left(\sum_{i=1}^{s} n_i\alpha_{i1}\right)\sum_{i=1}^{s}(n_i\alpha_{i1})^2 X_{*i}^{(1)}\right],$$

$$\mu_{\text{BLUE}} = c\left[R\sum_{i=1}^{s}(n_i\alpha_{i1})^2 X_{*i}^{(1)} - \left(\sum_{i=1}^{s} n_i\alpha_{i1}\right)\sum_{i=1}^{s}\sum_{j=1}^{r_i}(n_i - j + 1)\alpha_{ij}(X_{*i}^{(j)} - X_{*i}^{(j-1)})\right].$$

Remark 3.4. (i) The condition $c^{-1} > 0$ in Theorem 3.7 is fulfilled but for the case of record values with only one observation in each sample, i.e., $\alpha_{i1} = 1/n_i$ and $r_i = 1$, $i = 1 \ldots, s$. Applying the Cauchy-Schwarz inequality this is directly seen from the inequality $c^{-1} \geq (R - s)\sum_{i=1}^{s} n_i\alpha_{i1} \geq 0$ with equality in the first inequality iff $\alpha_{i1} = 1/n_i$, $i = 1, \ldots, s$. If we consider this particular model, we have $R = s$ and the matrix $(\beta, \mathbb{1}_R)'\Delta^{-1}(\beta, \mathbb{1}_R)$ given in (3.3) is $A = s\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$. Observe that A is singular. In this situation a linear unbiased estimator of (μ, ϑ) does not exist. It is only possible to deal with so-called estimable linear combinations of μ and ϑ. Here, the BLUE of $\mu + \vartheta$ is given by $\frac{1}{s}\sum_{i=1}^{s} X_{*i}^{(1)}$.

(ii) From equation (3.3) we derive the alternative representation

$$\vartheta_{\text{BLUE}} = \frac{1}{R}\sum_{i=1}^{s}\left(\sum_{j=1}^{r_i-1}((n_i - j + 1)\alpha_{ij} - (n_i - j)\alpha_{i,j+1})(X_{*i}^{(j)} - \mu_{\text{BLUE}}) + (n_i - r_i + 1)\alpha_{i,r_i}(X_{*i}^{(r_i)} - \mu_{\text{BLUE}})\right).$$

In the particular case of an ordinary k-out-of-n system and $s = 1$ this yields the result of Epstein (1957).

(iii) In the case $s = 1$ the estimators given in Theorem 3.7 simplify considerably. We obtain the representations:

$$\vartheta_{\text{BLUE}} = \frac{1}{r_1 - 1} \sum_{j=2}^{r_1} (n_1 - j + 1)\alpha_{1j}(X_{*1}^{(j)} - X_{*1}^{(j-1)}), \qquad \mu_{\text{BLUE}} = X_{*1}^{(1)} - \vartheta_{\text{BLUE}}.$$

For completeness we give the covariance matrix of the BLUEs in an explicit form, i.e.,

$$\text{Cov}\begin{pmatrix} \vartheta_{\text{BLUE}} \\ \mu_{\text{BLUE}} \end{pmatrix} = \vartheta^2 \frac{1}{R\sum_{i=1}^{s}(n_i\alpha_{i1})^2 - (\sum_{i=1}^{s} n_i\alpha_{i1})^2} \begin{pmatrix} \sum_{i=1}^{s}(n_i\alpha_{i1})^2 & -\sum_{i=1}^{s} n_i\alpha_{i1} \\ -\sum_{i=1}^{s} n_i\alpha_{i1} & R \end{pmatrix}.$$

Remark 3.5. As in the model of a known location parameter, particular cases are found in the literature. For references see Remark 3.2. Considering the model of record values, we have a similar simplification of the proposed estimators (for the BLUEs let $R \geq s + 1$ and for $s = 1$ let $r_1 \geq 2$):

$$\begin{array}{lll}
\text{MLE:} & \tilde{\mu} = \min_i X_{*i}^{(1)} & \overset{s=1}{=} X_{*1}^{(1)} \\
& \tilde{\vartheta} = \frac{1}{R}\sum_{i=1}^{s} X_{*i}^{(r_i)} - \frac{s}{R}\tilde{\mu} & \overset{s=1}{=} \frac{1}{r_1}(X_{*1}^{(r_1)} - X_{*1}^{(1)}) \\
\text{UMVUE:} & \hat{\mu} = \left(1 + \frac{s^2}{R-1}\right)\min_i X_{*i}^{(1)} - \frac{s}{R-1}\sum_{i=1}^{s} X_{*i}^{(r_i)} & \overset{s=1}{=} \frac{1}{r_1 - 1}(r_1 X_{*1}^{(1)} - X_{*1}^{(r_1)}) \\
& \hat{\vartheta} = \frac{1}{R-1}\sum_{i=1}^{s} X_{*i}^{(r_i)} - \frac{s}{R-1}\tilde{\mu} & \overset{s=1}{=} \frac{1}{r_1 - 1}(X_{*1}^{(r_1)} - X_{*1}^{(1)}) \\
\text{BLUE:} & \mu_{\text{BLUE}} = \frac{1}{(R-s)s}\left[R\sum_{i=1}^{s} X_{*i}^{(1)} - s\sum_{i=1}^{s} X_{*i}^{(r_i)}\right] & \overset{s=1}{=} \frac{1}{r_1 - 1}(r_1 X_{*1}^{(1)} - X_{*1}^{(r_1)}) \\
& \vartheta_{\text{BLUE}} = \frac{1}{R-s}\sum_{i=1}^{s}(X_{*i}^{(r_i)} - X_{*i}^{(1)}) & \overset{s=1}{=} \frac{1}{r_1 - 1}(X_{*1}^{(r_1)} - X_{*1}^{(1)})
\end{array}$$

The expressions for the BLUEs ($s = 1$) are given in Ahsanullah ((1995), p. 45) (see also Arnold *et al.* (1998), p. 127). It turns out that for $s \geq 2$ all estimators are different whereas in the case $s = 1$ the BLUEs and the UMVUEs coincide. The mean squared errors of the estimators in the record model are subsumed in Table 2.

The preceding results can be utilized to construct statistical tests an confidence bounds w.r.t. the parameters μ and ϑ. For instance, we consider the location parameter μ. We want to decide whether the location parameter is given by $\mu_0 = 0$. The corresponding decision problem reads

$$H_0 : \mu = 0 \leftrightarrow A : \mu \neq 0.$$

Table 2. Mean squared errors in the record model divided by ϑ^2.

MSE/ ϑ^2	μ	ϑ
MLE	$\frac{2}{s^2}$	$\frac{1}{R}$
UMVUE	$\frac{R}{(R-1)s^2}$	$\frac{1}{R-1}$
BLUE	$\frac{R}{s(R-s)}$	$\frac{1}{R-s}$

Considering the MLEs of μ and ϑ, we make use of the ratio of the MLEs, i.e., $T = \tilde{\mu}/\tilde{\vartheta}$. By Theorem 3.5, $\tilde{\mu}$ and $\tilde{\vartheta}$ are independent and the distribution of T does not depend on the parameter ϑ (cf. Remark 3.3). Hence, the ratio $(R/[(R-1)\sum_{i=1}^{s} n_i\alpha_{i1}])T$ follows an F-distribution with 2 and $2(R-1)$ degrees of freedom. For a similar result in the uncensored case we refer to David ((1981), p. 153).

The present result leads to classical tests. A similar observation holds for the test procedures proposed in Cramer and Kamps (1996), which are applied to decide whether the observed $(n-r+1)$-out-of-n system is an ordinary one or a sequential one. It turns out that the considered short-cut tests are well known in the context of testing homogeneity of variances from normally distributed populations. In particular, Test A is known as Hartley's test and the likelihood ratio test coincides with Bartlett's test, respectively.

4. Comparing two sequential k-out-of-n systems

The estimation of $P(X < Y)$, where X and Y are independent random variables is considered, e.g., in stress-strength models and in the comparison of two treatments. If X and Y are stochastically independent and exponentially distributed with the same location parameter μ, i.e., $X \sim \text{Exp}(\mu, \vartheta_1)$ and $Y \sim \text{Exp}(\mu, \vartheta_2)$, then $P(X < Y)$ is given by the ratio

$$P = P(X < Y) = \frac{\vartheta_2}{\vartheta_1 + \vartheta_2}.$$

In a previous paper (see Cramer and Kamps (1997*b*)), the authors derived the UMVUE of P when sampling is from two Weinman multivariate exponential distributions $\text{WME}_{n_1}(\mu, \vartheta_1, \tilde{\alpha})$, $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{n_1})$, and $\text{WME}_{n_2}(\mu, \vartheta_2, \tilde{\beta})$, $\tilde{\beta} = (\beta_1, \ldots, \beta_{n_2})$ (see (2.3)), respectively, with a common location parameter μ. Subsequently, consider two independent samples of type II censored sequential order statistics $X_*^{(1)}, \ldots, X_*^{(r_1)}$ and $Y_*^{(1)}, \ldots, Y_*^{(r_2)}$ based on exponential distributions $\text{Exp}(\mu, \vartheta_1/\alpha_i)$, $\alpha_i > 0$, $1 \le i \le n_1$, and $\text{Exp}(\mu, \vartheta_2/\beta_j)$, $\beta_j > 0$, $1 \le j \le n_2$, respectively.

In case of a known location parameter the UMVUE of P is given by the piecewise defined estimator

$$P^* = \begin{cases} F\left(1 - r_2, 1; r_1; \dfrac{n_1\alpha_1(W_1 - \mu)}{n_2\beta_1(W_2 - \mu)}\right), & \text{if } n_1\alpha_1(W_1 - \mu) \le n_2\beta_1(W_2 - \mu) \\ 1 - F\left(1 - r_1, 1; r_2; \dfrac{n_2\beta_1(W_2 - \mu)}{n_1\alpha_1(W_1 - \mu)}\right), & \text{if } n_1\alpha_1(W_1 - \mu) > n_2\beta_1(W_2 - \mu) \end{cases},$$

where $W_1 = \frac{1}{n_1\alpha_1}\sum_{i=1}^{r_1}(n_1 - i + 1)\alpha_i(X_*^{(i)} - X_*^{(i-1)})$, $W_2 = \frac{1}{n_2\beta_1}\sum_{j=1}^{r_2}(n_2 - j + 1)\beta_j(Y_*^{(j)} - Y_*^{(j-1)})$, $X_*^{(0)} = Y_*^{(0)} = 0$ and $F(a, b; c; x)$ denotes the hypergeometric function.

A similar, but in general more complicated representation holds in the case of an unknown location parameter (cf. Cramer and Kamps (1997*b*)). In this situation, we restrict ourselves to a special case, i.e., $n_1\alpha_1 = n_2\beta_1$, where the UMVUE $\tilde{P}$ of $P(X < Y)$ simplifies considerably. Denoting by $Z = \min\{X_*^{(1)}, Y_*^{(1)}\}$ the minimum of the combined sample, we obtain the representation

$$\tilde{P} = \frac{(r_1-1)(W_2-Z)}{(r_1-1)(W_2-Z)+(r_2-1)(W_1-Z)}. \tag{4.1}$$

Although this assumption seems to be very restrictive, it includes some important particular cases. First of all, if the estimation of P is based on the observation of record values of the underlying distributions, we have $\alpha_1 = 1/n_1$ and $\beta_1 = 1/n_2$ (see Table 1). Hence, the preceding assumption is fulfilled and the simplified estimator given in (4.1) can be used. Moreover, if all observed values are upper record values, the estimators W_1 and W_2 are simple, too. We find $W_1 = X_*^{(r_1)}$ and $W_2 = Y_*^{(r_2)}$ (see Table 1), such that only the largest and the smallest observed record values in each sample are necessary to estimate $P(X < Y)$.

Another interesting model leading to the ratio (4.1) is given by two progressive type II censored samples with the same sample size N. From Table 1 we conclude that $\alpha_1 = N/n_1$ and $\beta_1 = N/n_2$. Thus, for minimum variance unbiased estimation of $P(X < Y)$ by means of data from two arbitrary censoring schemes $(R_1, \ldots, R_{n_1})$ and $(S_1, \ldots, S_{n_2})$ with $N = \sum_{i=1}^{n_1} R_i + n_1 = \sum_{i=1}^{n_2} S_i + n_2$, representation (4.1) can be utilized.

An important feature of the estimator $\tilde{P}$ is given by the fact that it is possible to calculate its density function φ, i.e.,

$$\varphi(t) = c \cdot \frac{(1-t)^{r_1-2} t^{r_2-2}}{\left(1 + \left(\frac{(r_2-1)\vartheta_1}{(r_1-1)\vartheta_2} - 1\right) t\right)^{r_1+r_2-1}}, \qquad t \in (0,1),$$

where c is a normalizing constant. This is a particular Gauss hypergeometric distribution (cf. Armero and Bayarri (1994), Johnson *et al.* (1995), p. 253). Some properties, such as moments of arbitrary order, the mode and some symmetry properties are derived in Cramer and Kamps (1997*b*).

Remark 4.1. (i) The set-up leading to the estimator $\tilde{P}$ in the case of an unknown location parameter extends an approach of Bai and Hong (1992), who consider the case of non-censored data from independent exponential distributions ($\alpha_1 = \cdots = \alpha_{n_1} = \beta_1 = \cdots = \beta_{n_2} = 1$). The derivation of the estimator presented in Bai and Hong (1992) contains an error, and therefore the resulting estimator is not the UMVUE of $P(X < Y)$. In this situation, a correct derivation of the UMVUE of $P(X < Y)$ and the estimator itself are shown in Cramer and Kamps (1997*a*).

(ii) For special choices of the parameters α_i, β_j ($\alpha_1 = \cdots = \alpha_{n_1} = \beta_1 = \cdots = \beta_{n_2} = 1$) and r_1, r_2 the representation of the UMVUE P^* leads to the results of Tong (1974) for the non-censored case and to those of Bartoszewicz (1977) for the type II censored case.

The results can be applied in comparing two independent sequential k-out-of-n systems in step r based on $\text{Exp}(\mu, \vartheta_1/\alpha_i)$ and $\text{Exp}(\mu, \vartheta_2/\alpha_i)$, $1 \le i \le n$, respectively, with known parameters $\alpha_1, \dots, \alpha_n$. After the $(r-1)$-st failure in both systems, the lifetime distributions of the remaining components are given by $(T_1 \sim)$ $\mathrm{Exp}(\mu, \vartheta_1/\alpha_r)$ and $(T_2 \sim)$ $\mathrm{Exp}(\mu, \vartheta_2/\alpha_r)$, respectively, which lead to the quantity of interest $P(T_1 < T_2) = P$.

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