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ON THE PROPORTIONAL ODDS MODEL IN SURVIVAL ANALYSIS

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Abstract. The proportional odds (PO) model with its property of convergent hazard functions is of considerable value in modeling survival data with non-proportional hazards. This paper explores the structure, implications, and properties of the PO model. Results proved include connections with geometric minima and maxima, ageing characteristics, and bounds on mean and variance of survival times.

Key words and phrases: Ageing characteristics, geometric distribution, random minima and maxima, residual life, stochastic orders, survival models.

1. Introduction

The analysis of survival data is mainly concerned with evaluating the risk or hazard of death at any time after an individual patients' entry into a clinical trial or other medical study. Typically, supplementary information; such as demographic, physiological, or lifestyle characteristics; is recorded for each individual. A key problem then is to explore how and to what extent the explanatory variables (observed in the form of supplementary information) affect the risk or hazard of death. Such knowledge is useful in devising treatment regimes and evaluating the prognosis for current or future patients with particular values of these explanatory variables. The most popular model for exploring the relationship between the survival experience of a patient and explanatory variables is Cox's proportional hazards (PH) model

$$\lambda_i(t) = \lambda_0(t) \exp\left(\sum_{j=1}^{p} \beta_j Z_{ji}\right)$$

where $\lambda_i(t)$ is the hazard function for the i-th individual, $\lambda_0(t)$ is the baseline hazard function, Z_{ji} is the value of the j-th explanatory variable on the i-th individual, and $\beta_1, \dots, \beta_p$ are the regression coefficients. As is well-known, the partial likelihood approach to estimating the β-coefficients does not require knowledge of the actual form of the baseline hazard function. Note, however, that in the PH model, $\lambda_i(t)/\lambda_0(t)$ is constant in t, thus ruling out situations in which the hazard ratio converges to 1 as $t \to \infty$. In practical applications, it is not uncommon for the hazard functions obtained for two or more groups to converge with time. The assumption of a constant hazard ratio is clearly unreasonable when initial effects, such as differences in the stage of disease or in treatment, disappear with time. Similarly, the demonstration of the effectiveness of a cure, when the mortality of the disease group approaches that of a disease-free control group, requires that the hazard ratio converge to 1.

One approach to allow for converging hazard functions is to include time-dependent explanatory variables in the Cox model. An alternative approach is to try the proportional odds (PO) model

$$\emptyset_i(t) = \emptyset_0(t) \exp\left(\sum_{j=1}^{p} \beta_j Z_{ji}\right)$$

where $\emptyset_i(t)$ are the odds of the i-th individual surviving beyond time t and $\emptyset_0(t)$ the corresponding baseline odds. The PO model, introduced by Bennett (1983), guarantees that the hazard ratio $\lambda_i(t)/\lambda_0(t)$ converges to 1. Bennett (1983) also describes how the PO model can be fitted using maximum likelihood estimation, and gives an example of its use in the analysis of a lung cancer trial. Collett (1994) has fitted the PO model to the data on the survival times of women with breast tumors that were negatively or positively stained. Dinse and Lagakos (1983) and Rossini and Tsiatis (1996) have used the PO model for analyzing interval censored data. The application of the PO model for the analysis of reliability data has been described by Crowder *et al.* (1991)

The objective of this paper is to explore the structure, implications and properties of the PO model. Section 2 gives a preliminary discussion of odds and proportional odds. The objective of Section 3 is two fold—firstly, it gives a key representation of the PO model in terms of geometric maxima and minima and, secondly, it establishes that in presence of proportional odds certain ageing characteristics of the control group are necessarily inherited by the treatment group. Finally, the results of Section 4 provide bounds on mean and variance of the survival time for treatment groups in terms of the corresponding quantities for the control group.

Throughout this paper, we need the well-known notions of increasing failure rate (IFR), decreasing failure rate (DFR), increasing failure rate average (IFRA),decreasing failure rate average (DFRA), new better than used (NBU), new worse than used (NWU), new better than used in expectation (NBUE) and new worse than used in expectation (NWUE) which are extensively used for modeling positive or negative ageing in reliability and survival analysis. See Barlow and Proschan (1975) for the motivations, definitions and extensive study of these notions.

2. The odds function and proportional odds

Throughout this paper, X will denote a survival time random variable with continuous distribution function $F(t) = P(X \leq t)$ and survivor function $\bar{F}(t) = P(X > t) = 1 - F(t)$. If, and when, needed, $f(t) = F'(t)$ will denote probability density function (pdf) of X and $\lambda(t) = f(t)/\bar{F}(t)$ the corresponding hazard function. The odds on surviving beyond time t are given by the "odds function"

$$\emptyset(t) = \frac{P(X > t)}{P(X \leq t)} = \frac{\bar{F}(t)}{F(t)}.$$

Clearly, $\emptyset(t)$ can be rewritten as $\tilde{\lambda}(t)/\lambda(t)$ where $\tilde{\lambda}(t) = \frac{d}{dt} \ln F(t)$ is the so called reversed hazard rate of X at the point t. (See Shaked and Shanthikumar (1994) for details about reversed hazard rates.) Thus, for infinitely small $\Delta t > 0$,

$$\emptyset(t) \cong \frac{P(t - \Delta t < X \leq t \mid X \leq t)}{P(t < X \leq t + \Delta t \mid X > t)}.$$

The odds function is a decreasing function of t. Two less obvious properties are given below.

PROPOSITION 1. (a) *If X is DFR (i.e. if $\lambda(t)$ is decreasing) then $\emptyset(t)$ is convex.*
(b) *If X is NBU (i.e. if $\bar{F}(t+s) \le \bar{F}(t)\bar{F}(s); t, s \ge 0$) then $\emptyset(t+s) \le \emptyset(t)\emptyset(s); t, s \ge 0$.*

PROOF. (a) Since $\emptyset(t) = \bar{F}(t)/F(t)$ is non-negative and decreasing, $\tilde{\lambda}(t) = \lambda(t)\emptyset(t)$ is decreasing whenever X is DFR. It follows that

$$\begin{aligned}\emptyset''(t) &= \emptyset(t)\{\lambda(t) + \tilde{\lambda}(t)\}^2 - \emptyset(t)\{\lambda'(t) + \tilde{\lambda}'(t)\} \\ &\ge 0.\end{aligned}$$

(b) We can write

$$\emptyset(t)\emptyset(s) = 1\Big/\left[\frac{1}{\bar{F}(t)\bar{F}(s)} - \left\{\frac{1}{\bar{F}(t)} + \frac{1}{\bar{F}(s)}\right\} + 1\right].$$

But,

$$\begin{aligned}&\frac{1}{\bar{F}(t)\bar{F}(s)} - \left\{\frac{1}{\bar{F}(t)} + \frac{1}{\bar{F}(s)}\right\} + 1 \\ &\quad \le \frac{1}{\bar{F}(t)\bar{F}(s)} - 1 \\ &\quad \le \frac{1}{\bar{F}(t+s)} - 1, \text{ when } X \text{ is NBU} \\ &\quad = \frac{1}{\emptyset(t+s)}\end{aligned}$$

which gives $\emptyset(t)\emptyset(s) \ge \emptyset(t+s)$. □

We now define

$$\begin{aligned}\emptyset(x \mid t) &= \frac{P(X > x+t \mid X > t)}{P(X \le x+t \mid X > t)} \\ &= \frac{\bar{F}_t(x)}{F_t(x)}\end{aligned}$$

as the conditional odds on survival to $x + t$ given survival to t. Here, $\bar{F}_t(x) = P(X - t > x \mid X > t) = \bar{F}(x+t)/\bar{F}(t)$ is the survivor function of the "residual life" and $F_t(x) = 1 - \bar{F}_t(x)$. Although this paper is not concerned with conditional odds, it is natural to compare conditional and unconditional odds. In this connection, we have the following proposition whose simple proof is omitted.

PROPOSITION 2. (a)

$$\frac{\emptyset(x \mid t) - \emptyset(x+t)}{\emptyset(x+t)} \ge \frac{1}{\emptyset(t)}.$$

(b) *If X is NBU(NWU) then $\emptyset(x \mid t) \le (\ge)\emptyset(x)$.*

To introduce the proportional odds (PO) model, let Y be another survival time random variable with continuous distribution function $G(t)$, survivor function $\bar{G}(t)$, and hazard function $\lambda_Y(t)$. Henceforth, we will denote the odds function $\emptyset(t) = \bar{F}(t)/F(t)$ by $\emptyset_X(t)$ and $\emptyset_Y(t) = \bar{G}(t)/G(t)$ will be the odds function for Y. We will say that the survival time random variables X and Y satisfy the proportional odds (PO) model with proportionality constant α if

$$\emptyset_Y(t) = \alpha\emptyset_X(t). \tag{2.1}$$

For interpretation purposes, one thinks of X as the survival time of a member of the control group and Y as that for a member of the treatment group. Following Bennett (1983), we take $\alpha = e^{\eta}$ where $\eta = \beta_1 Z_1 + \cdots + \beta_p Z_p$ and $Z_1, \ldots, Z_p$ are the values of the explanatory variables. This paper is not concerned with issues of statistical inference and, therefore, we will suppress this aspect of the PO model.

The odds function is a one-to-one increasing function of the survivor function. It follows from (2.1) that if $\alpha > (<)1$, then $\bar{G}(t) > (<)\bar{F}(t)$. The following proposition provides more information about the reduction in risk, excess risk, and the relative risk.

PROPOSITION 3. *Suppose* $\emptyset_Y(t) = \alpha\emptyset_X(t)$ *for all* $t > 0$.

(a) *If* $0 < \alpha \leq 1$ *then* $0 \leq G(t) - F(t) \leq (1-\alpha)/4\alpha$, *and* $G(t)/F(t)$ *is decreasing in* t.

(b) *If* $\alpha \geq 1$ *then* $0 \leq F(t) - G(t) \leq \frac{\alpha-1}{4}$, *and* $\bar{G}(t)/\bar{F}(t)$ *is increasing in* t.

PROOF. In terms of survival functions, the PO model is the statement

$$\bar{G}(t) = \frac{\alpha\bar{F}(t)}{1-\bar{\alpha}\bar{F}(t)},$$

where $\bar{\alpha} = 1 - \alpha$, so that

$$\bar{F}(t) - \bar{G}(t) = \frac{\bar{\alpha}F(t)\bar{F}(t)}{1-\bar{\alpha}\bar{F}(t)}.$$

(a) If $0 < \alpha \leq 1$ then $\bar{G}(t) < \bar{F}(t)$ and the excess in risk of dying by time t is

$$\begin{aligned} G(t) - F(t) &= \bar{F}(t) - \bar{G}(t) = (\bar{\alpha}/\alpha)\bar{G}(t)F(t) \\ &\leq (\bar{\alpha}/\alpha)\bar{F}(t)F(t) \\ &\leq \frac{\bar{\alpha}}{4\alpha}. \end{aligned}$$

Moreover,

$$\frac{G(t)}{F(t)} = \frac{\bar{F}(t) - \bar{G}(t)}{F(t)} + 1 = \frac{\bar{\alpha}\bar{F}(t)}{1-\bar{\alpha}\bar{F}(t)} + 1$$

is decreasing in t whenever $0 < \alpha \leq 1$.

(b) If $\alpha > 1$, the proof of the first part follows by writing $\emptyset_X(t) = \frac{1}{\alpha}\emptyset_Y(t)$.

Further,

$$\frac{\bar{G}(t)}{\bar{F}(t)} = \frac{(\alpha-1)F(t)}{1-\bar{\alpha}\bar{F}(t)} + 1$$

is increasing in t whenever $\alpha > 1$. □

To close this section, we note that (see Collett (1994)) if the PO model (2.1) holds then

$$\frac{\lambda_Y(t)}{\lambda_X(t)} = \frac{1}{1+(\alpha-1)\bar{F}(t)} = \frac{G(t)}{F(t)}.$$

Also,

$$\frac{\lambda_{\tilde{Y}}(t)}{\lambda_{\tilde{X}}(t)} = \frac{\bar{F}(t)}{\bar{G}(t)}.$$

Thus, if $\alpha > (<) 1$, the above hazard ratio function is an increasing (decreasing) function with value $1/\alpha$ at $t = 0$ and converging to 1 at $t = \infty$. It is this property of convergent hazard functions which makes the PO model particularly attractive. Furthermore, the PO model, when written in the form

(2.2) $$\bar{G}_\alpha(x) = \alpha\bar{F}(x)/(1-\bar{\alpha}\bar{F}(x)), \quad 0 < \alpha < \infty,$$

provides a method of introducing a new parameter α to expand any given family of distributions for added flexibility. This, in fact, is the stated motivation for Marshall and Olkin (1997) who discuss some aspects of the family (2.2) but without its PO implications.

3. Structure of the proportional odds model and hereditary ageing characteristics

Our first objective in this section is to show that the structure of the PO model is essentially that of a competing (complementary) risk with a geometrically distributed number of unknown competing (complementary) causes. The observed survival time when there are n competing (complementary) causes is of the type $X_{1:n} = \min\{X_1, \dots, X_n\}$ $(X_{n:n} = \max\{X_1, \dots, X_n\})$.

Throughout the rest of this paper, X_1, $X_2, \dots$ will be a sequence of independent and identically distributed (i.i.d.) random variables with common continuous survivor function $\bar{F}$ and $Y_1, Y_2, \dots$ will be an independent sequence of i.i.d. random variables having common continuous survivor function $\bar{G}$. Moreover, for any $\theta\epsilon(0,1)$, N_θ will denote a random variable, independent of the sequences $\{X_n\}$ and $\{Y_n\}$, such that $P(N_\theta = n) = (\bar{\theta})^{n-1}\theta, n = 1, 2, \dots$ where $\bar{\theta} = 1-\theta$. Then, N_θ has probability generating function (pgf)

$$\psi_\theta(u) = E\left(u^{N_\theta}\right) = \frac{\theta u}{1-\bar{\theta}u}, \quad 0 \le u \le 1.$$

For later use, we note also that $1 - \psi_\theta(1 - \psi_\theta(u)) = u$.

The following structural/representation theorem is useful in understanding the PO model as well as in deriving new results. The notation $=^d$ will mean equality in distribution.

THEOREM 4. *Suppose that* $\emptyset_Y(t) = \alpha\emptyset_X(t)$.
(a) *If* $0 < \alpha \le 1$ *then*

$$Y \stackrel{d}{=} X_{1:N_\alpha}, \quad X \stackrel{d}{=} Y_{N_\alpha:N_\alpha}.$$

(b) *If* $\alpha > 1$ *and* $\beta = 1/\alpha$ *then*

$$Y \stackrel{d}{=} X_{N_\beta:N_\beta}, \quad X \stackrel{d}{=} Y_{1:N_\beta}.$$

PROOF. (a) If $0 < \alpha < 1$, the PO assumption gives

$$\begin{aligned} P(Y > t) &= \bar{G}(t) = \frac{\alpha \bar{F}(t)}{1 - \bar{\alpha}\bar{F}(t)} = \psi_\alpha\left(\bar{F}(t)\right) \\ &= \sum_{n=1}^{\infty} \left\{\bar{F}(t)\right\}^n (\bar{\alpha})^{n-1} \alpha \\ &= \sum_{n=1}^{\infty} P\left(X_{1:N_\alpha} > t \mid N_\alpha = n\right) P\left(N_\alpha = n\right) \\ &= P\left(X_{1:N_\alpha} > t\right). \end{aligned}$$

Thus, $Y{=^d}X_{1:N_\alpha}$. Also,

$$\begin{aligned} P(X \le t) &= 1 - \bar{F}(t) = 1 - \left\{1 - \psi_\alpha\left(1 - \psi_\alpha\left(\bar{F}(t)\right)\right)\right\} \\ &= 1 - \left\{1 - \psi_\alpha\left(1 - \bar{G}(t)\right)\right\} \\ &= \psi_\alpha(G(t)) = P\left(Y_{N_\alpha:N_\alpha} \le t\right), \end{aligned}$$

so that $X{=^d}Y_{N_\alpha:N_\alpha}$.

(b) If $\alpha > 1$ then the proof follows from part (a) by writing $\emptyset_X(t) = \beta\emptyset_Y(t)$, where $\beta = 1/\alpha$. □

The above theorem is particularly satisfying because it establishes the connection between the PO model and geometric minima and maxima. The latter have received increasing attention in the literature. See Shaked and Wong (1997) and the references therein. It may also be of interest here to note that Marshall and Olkin (1997) showed that the expanded parametric family of the form (2.2) is geometric-extreme stable. We now show that certain stochastic orderings are preserved under proportional odds. In what follows, F^{-1} will mean the function

$$F^{-1}(u) = \sup\left\{x : F(x) \le u\right\}, \qquad 0 \le u \le 1.$$

THEOREM 5. *Suppose that $\emptyset_Y(t) = \alpha\emptyset_X(t)$ and $\emptyset_{Y^*}(t) = \alpha\emptyset_{X^*}(t)$ where the proportionality constant α is the same and X^*, Y^* have survival functions $\bar{F}^*, \bar{G}^*$, respectively.*

(a) *If $X \le_c X^*$ (i.e., if $F^{*-1}F(x)$ is convex in x on the support of F) then $Y \le_c Y^*$.*

(b) *If $X \le_* X^*$ (i.e., if $F^{*-1}F(x)/x$ is increasing in $x \ge 0$) then $Y \le_* Y^*$.*

(c) *If $X \le_{SU} X^*$ (i.e., if $F^{*-1}F$ has the superadditivity property $F^{*-1}F(x+y) \ge F^{*-1}F(x) + F^{*-1}F(y)$, $x, y \ge 0$) then $Y \le_{SU} Y^*$.*

(d) *If $X \le_{disp} X^*$ (i.e., if $F^{*-1}F(x) - x$ increases in x) then $Y \le_{disp} Y^*$.*

PROOF. It will be sufficient to show that $G^{*-1}G = F^{*-1}F$. If $0 < \alpha \le 1$ then the PO assumptions give

$$\bar{G}(t) = \psi_\alpha\left(\bar{F}(t)\right), \qquad \bar{G}^*(t) = \psi_\alpha\left(\bar{F}^*(t)\right)$$

so that

$$\begin{aligned} G^{*-1}G(t) &= G^{*-1}\left(1-\bar{G}(t)\right) \\ &= \left(\bar{G}^{*}\right)^{-1}\bar{G}(t) \\ &= \left(\bar{F}^{*}\right)^{-1}\psi_{\bar{\alpha}}^{-1}\psi_{\alpha}\bar{F}(t) \\ &= \left(\bar{F}^{*}\right)^{-1}\bar{F}(t) = F^{*-1}F(t). \end{aligned}$$

When $\alpha > 1$, the result follows from the relations

$$G(t) = \psi_{\beta}(F(t)), \qquad G^{*}(t) = \psi_{\beta}\left(F^{*}(t)\right)$$

where $\beta = 1/\alpha$. □

The following theorem shows that certain positive ageing characteristics of X must be inherited by Y if $\emptyset_Y(t) = \alpha\emptyset_X(t)$ with $\alpha > 1$. We first note that a nonnegative random variable X is IFR [IFRA,NBU] if, and only if, $X \leq_c [\leq_*, \leq_{SU}]\ X^*$, where X^* is any random variable having the exponential distribution $P(X^* > t) = \exp(-\mu t), t \geq 0$, for some $\mu > 0$.

THEOREM 6. *Suppose* $\emptyset_Y(t) = \alpha\emptyset_X(t)$ *where* $\alpha > 1$. *If* X *is IFR [IFRA,NBU] then* Y *is IFR [IFRA,NBU].*

PROOF. Let X^* and Y^* be random variables with respective survivor functions

$$\begin{aligned} \bar{F}^{*}(t) &= e^{-t}, \qquad t \geq 0, \\ \bar{G}^{*}(t) &= \frac{\alpha}{e^{t} - \bar{\alpha}}, \qquad t \geq 0. \end{aligned}$$

It can be easily seen that if $\alpha > 1$ then Y^* is IFR and hence IFRA and NBU. Furthermore, $\emptyset_{Y^*}(t) = \alpha\emptyset_{X^*}(t)$.

Suppose now that $\emptyset_Y(t) = \alpha\emptyset_X(t)$ where $\alpha > 1$ and X is IFR [IFRA,NBU]. Then,

$$\begin{aligned} X \text{ is } IFR[IFRA, NBU] &\leftrightarrow X \leq_c [\leq_*, \leq_{SU}]X^* \\ &\rightarrow Y \leq_c [\leq_*, \leq_{SU}]Y^*, \qquad \text{by Theorem 5.} \end{aligned}$$

But, Y^* being IFR [IFRA,NBU] implies that $Y^* \leq_c [\leq_*, \leq_{SU}]X^*$ which, when coupled with $Y \leq_c [\leq_*, \leq_{SU}]Y^*$, gives $Y \leq_c [\leq_*, \leq_{SU}]X^*$. That is, Y is IFR [IFRA,NBU]. □

The following theorem demonstrates that if $0 < \alpha < 1$, then certain negative ageing characteristics of X must be inherited by Y.

THEOREM 7. *Suppose* $\emptyset_Y(t) = \alpha\emptyset_X(t)$ *where* $0 < \alpha < 1$.

(a) *Let* $f(t)$, $g(t)$ *denote pdfs of* X *and* Y, *respectively. If* $\log f(t)$ *is convex then* $\log g(t)$ *is convex.*

(b) *If* X *is DFR[DFRA,NWU] then* Y *is DFR[DFRA,NWU].*

PROOF. (a) It follows from the PO assumption that

$$g(t) = \frac{\alpha f(t)}{\left\{1 - \bar{\alpha}\bar{F}(t)\right\}^{2}}$$

so that

$$\frac{d^2}{dt^2}\log g(t) = \frac{d^2}{dt^2}\log f(t) - 2\bar{\alpha}\frac{d}{dt}\left(\frac{f(t)}{1-\bar{\alpha}\bar{F}(t)}\right).$$

When $\log f(t)$ is convex, X is DFR and, hence $f(t) = \lambda(t)\bar{F}(t)$ is decreasing. Consequently, for $\bar{\alpha} = 1-\alpha > 0, f(t)/\left\{1-\bar{\alpha}\bar{F}(t)\right\}$ is decreasing and

$$\frac{d^2}{dt^2}\log g(t) \geq \frac{d^2}{dt^2}\log f(t) \geq 0.$$

That is, $\log g(t)$ is convex.

(b) (i) To see that if X is DFR then Y is DFR, it is sufficient to note that the PO assumption gives

$$\frac{\lambda_Y(t)}{\lambda_X(t)} = \frac{1}{1-\bar{\alpha}\bar{F}(t)},$$

which is decreasing when $\bar{\alpha} = 1-\alpha > 0$.

(ii) Suppose now that X is DFRA, i.e., $\bar{F}(ct) \leq \left\{\bar{F}(t)\right\}^c, 0 < c < 1$.

Now, as seen in Theorem 4, $\bar{G}(ct) = \psi_\alpha\left(\bar{F}(ct)\right)$. Since the pgf $\psi_\alpha(u)$ is increasing,

$$\begin{aligned}\bar{G}(ct) &\leq \psi_\alpha((\bar{F}(t))^c) \\ &= E[\{\bar{F}(t)\}^c]^{N_\alpha} \\ &= E(U^c)\end{aligned}$$

where $U = \{\bar{F}(t)\}^{N_\alpha}$. But, $\{E(U^c)\}^{1/c}$ is an increasing function of $c > 0$ because the random variable U is non-negative. Hence, $E(U^c) \leq \{E(U)\}^c$ whenever $0 < c < 1$. Consequently, for $0 < c < 1$,

$$\begin{aligned}\bar{G}(ct) &\leq \{E(U)\}^c \leq \{E((\bar{F}(t))^{N_\alpha})\}^c \\ &= \{\bar{G}(t)\}^c.\end{aligned}$$

Thus, Y is DFRA.

(iii) Suppose now that X is NWU, i.e., $\bar{F}(s+t) \geq \bar{F}(s)\bar{F}(t), s, t \geq 0$ Let us write

$$h_x(N_\alpha) = \left\{\bar{F}(x)\right\}^{N_\alpha}.$$

Then,

$$\begin{aligned}\bar{G}(s+t) &= \psi_\alpha(\bar{F}(s+t)) \\ &\geq \psi_\alpha(\bar{F}(s)\bar{F}(t)) \\ &= E[\{\bar{F}(s)\bar{F}(t)\}^{N_\alpha}] \\ &= E[h_s(N_\alpha)h_t(N_\alpha)] \\ &\geq E[h_s(N_\alpha)]Et[h_t(N_\alpha)t] \\ &= E[\{\bar{F}(s)\}^{N_\alpha}]E[\{\bar{F}(t)\}^{N_\alpha}] \\ &= \bar{G}(s)\bar{G}(t)\end{aligned}$$

where the last inequality holds because the random variables $h_s(N_\alpha)$ and $h_t(N_\alpha)$ are monotone in the same direction. This proves that Y is also NWU. □

4. Bounds on mean and variance in the proportional odds model

The objective of this section is to obtain bounds on $E(Y)$ and $\text{Var}(Y)$ in terms of $E(X)$ and $\text{Var}(X)$, respectively, when X and Y have proportional odds. We shall continue to use the notations of the previous sections. Our bounds on $E(Y)$ will be in the setting of harmonic new better (worst) than used in expectation (HNBUE,HNWUE) distribution for X. We say that a survival random variable X with finite mean $\mu = E(X) = \int_0^\infty \bar{F}(x)dx$ is a HNBUE(HNWUE) random variable if

$$\int_t^\infty \bar{F}(x)dx \leq (\geq) \mu \exp(-t/\mu), \qquad t \geq 0.$$

For more information about HNBUE(HNWUE) random variables, see Basu and Kirmani (1986) and the references therein.

THEOREM 8. *Suppose $\emptyset_Y(t) = \alpha \emptyset_X(t)$ where $E(X) = \mu$.*
(a) *If $0 < \alpha < 1$ and X is HNBUE(HNWUE) then*

$$E(Y) \geq (\leq) \left(-\frac{\alpha \ln \alpha}{\bar{\alpha}}\right) \mu.$$

(b) *If $\alpha > 1$ and X is HNBUE(HNWUE) then*

$$E(Y) \leq (\geq) \left(-\frac{\alpha \ln \alpha}{\bar{\alpha}}\right) \mu.$$

PROOF. (a) Under the PO model with $0 < \alpha < 1, Y =^d X_{1:N_\alpha}$ so that $E(Y) = E(X_{1:N_\alpha})$. Now, let $W_1, W_2, \ldots$ be i.i.d. random variables with common exponential distribution of mean μ (the same mean as that of X). When X is HNBUE(HNWUE)

$$\int_0^t \bar{F}(x)dx \geq (\leq) \int_0^t e^{-x/\mu} dx$$

so that, by Theorem 7.3 of Barlow and Proschan (1975),

$$\int_0^t \{\bar{F}(x)\}^n dx \geq (\leq) \int_0^t e^{-nx/\mu} dx$$

and hence, for all $n = 1, 2, \ldots$,

$$E(X_{1:n}) = \int_0^\infty \{\bar{F}(x)\}^n dx \geq (\leq) \frac{\mu}{n}.$$

Consequently,

$$\begin{aligned} E(Y) &= E(X_{1:N_\alpha}) \geq (\leq) \mu E\left(\frac{1}{N_\alpha}\right) \\ &= \mu \left(-\frac{\alpha \ln \alpha}{\bar{\alpha}}\right) \end{aligned}$$

where we have used the easily verified fact that, for a geometric random variable N_α, $E\,(1/N_\alpha) = -\frac{\alpha \ln \alpha}{\bar{\alpha}}$.

(b) When $\alpha > 1, Y =^d X_{N_\beta : N_\beta}$ where $\beta = 1/\alpha$. Further, when X is HNBUE(HNWUE)

$$\int_t^\infty \bar{F}(x)dx \leq (\geq) \int_t^\infty e^{-x/\mu} dx,$$

so that, by Theorem 7.4 of Barlow and Proschan (1975),

$$\int_t^\infty \{1 - F^n(x)\}dx \leq (\geq) \int_t^\infty \{1 - (1 - e^{-x/\mu})^n\}dx.$$

Hence, for $n = 1, 2, \ldots$

$$\begin{aligned} E\,(X_{n:n}) &= \int_0^\infty \{1 - F^n(x)\}dx \\ &\leq (\geq) \int_0^\infty \{1 - (1 - e^{-x/\mu})^n\}dx \\ &= \mu \sum_{i=1}^n \frac{1}{i}. \end{aligned}$$

It follows that

$$\begin{aligned} E(Y) &= E\left(X_{N_\beta : N_\beta}\right) \\ &\leq (\geq) \mu E \left(\sum_{i=1}^{N_\beta} \frac{1}{i} \right) \\ &= \mu \sum_{n=1}^\infty \left(\sum_{i=1}^n \frac{1}{i} \right) (\bar{\beta})^{n-1} \beta, \qquad \bar{\beta} = 1 - \beta = -\bar{\alpha}/\alpha \\ &= \frac{\beta\mu}{\bar{\beta}} \sum_{i=1}^\infty \frac{1}{i} \left(\frac{(\bar{\beta})^i}{1 - \bar{\beta}} \right) \\ &= \frac{\mu}{\bar{\beta}} \{-\ln(1 - \bar{\beta})\} \\ &= \left(-\frac{\alpha\mu}{\bar{\alpha}}\right) \ln \alpha. \end{aligned}$$

□

It may be noted here that, for any $\alpha > 0$,

$$\min\{\alpha, 1\} < -\frac{\alpha \ln \alpha}{\bar{\alpha}} < \max\{\alpha, 1\}.$$

Further, Theorem 8 is of wide applicability because the HNBUE(HNWUE) class contains all IFR(DFR), IFRA(DFRA), NBU(NWU), and NBUE(NWUE) distributions. For example, let X and Y have survival functions

$$\bar{F}(t) = \exp\{-(\lambda t)^\beta\}, \quad t \geq 0, \quad \beta > 0$$

and

$$\bar{G}(t) = \frac{\alpha \exp\{-(\lambda t)^{\beta}\}}{1 - \bar{\alpha} \exp\{-(\lambda t)^{\beta}\}}, \quad t \geq 0, \quad \beta > 0, \ \alpha > 0$$

respectively, Then Theorem 8 applies because $\emptyset_Y(t) = \alpha \emptyset_X(t)$ and X is HNBUE (HNWUE) if $\beta \geq (\leq) 1$.

Our next result provides bounds on $P(Y - t > x \mid Y > t)$ in terms of $P(X - t > x \mid X > t)$ when X and Y have proportional odds.

PROPOSITION 9. *Let* $\bar{F}_t(x) = P(X - t > x \mid X > t) = \frac{\bar{F}(x+t)}{\bar{F}(t)}$, $x > 0$ *and* $\bar{G}_t(x) = P(Y - t > x \mid Y > t) = \frac{\bar{G}(x+t)}{\bar{G}(t)}$, $x > 0$. *If* $\emptyset_Y(t) = \alpha \emptyset_X(t)$ *then*

$$\min\{\alpha, 1\} \bar{F}_t(x) \leq \bar{G}_t(x) \leq \max\{\alpha, 1\} \bar{F}_t(x).$$

PROOF. It is easy to verify that

$$\min\{\alpha, 1\} \leq \frac{1 - \bar{\alpha}\bar{F}(t)}{1 - \bar{\alpha}\bar{F}(x+t)} \leq \max\{\alpha, 1\}.$$

The result then follows on noting that, under the PO assumption,

$$\bar{G}_t(x) = \left\{ \frac{1 - \bar{\alpha}\bar{F}(t)}{1 - \bar{\alpha}\bar{F}(x+t)} \right\} \bar{F}_t(x). \qquad \square$$

The above result immediately provides bounds on the mean residual life (mrl) function of Y in terms of the mrl function of X. Writing

$$\mu_X(t) = E(X - t \mid X > t) = \int_0^{\infty} \bar{F}_t(x) dx,$$

and

$$\mu_Y(t) = E(Y - t \mid Y > t) = \int_0^{\infty} \bar{G}_t(x) dx,$$

we have the following result.

COROLLARY 10. *If* $\emptyset_Y(t) = \alpha \emptyset_X(t)$ *and* $E(X) < \infty$ *then*

$$\min\{\alpha, 1\} \mu_X(t) \leq \mu_Y(t) \leq \max\{\alpha, 1\} \mu_X(t).$$

Finally, we obtain bounds on variance of Y in terms of that of X when the odds are proportional and X is IMRL (that is, the mrl function $\mu_X(t)$ is increasing). However, we

Table 1. $E(\frac{1}{N_\theta^2})$ for selected values of θ.

θ	$E(\frac{1}{N_\theta^2})$
0.05	0.0758228
0.10	0.144413
0.20	0.268699
0.50	0.582241
0.80	0.844015
0.90	0.92356
0.95	0.962147
1	$\pi^2/6 = 1.64493$

first note that if N_θ has the geometric distribution $P(N_\theta = n) = (\bar{\theta})^{n-1}\theta$, $n = 1, 2, \ldots$ then

$$E\left(\frac{1}{N_\theta^2}\right) = \frac{\theta}{\bar{\theta}}\sum_{n=1}^{\infty}\frac{(\bar{\theta})^n}{n^2} = \frac{\theta}{\bar{\theta}}Li_2(\bar{\theta})$$

where

$$Li_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2} = \int_z^0 \frac{\ln(1-t)}{t}dt$$

is the so called Dilogarithm function (see Wolfram (1996)). $Li_2(1-Z)$ is sometimes also known as Spence's integral. The Dilogarithm function can be easily plotted and tabled with the help of **Mathematica**, see Wolfram (1996). For selected values of θ, Table 1 gives values of $E(\frac{1}{N_\theta^2})$ obtained with the help of **Mathematica**.

We now prove

THEOREM 11. *Suppose $\emptyset_Y(t) = \alpha\emptyset_X(t)$ and X is IMRL with $E(X^2) < \infty$.*
(a) *If $0 < \alpha < 1$ then*

$$\alpha^3 \operatorname{Var}(X) \le \operatorname{Var}(Y) \le \operatorname{Var}(X)E\left(\frac{1}{N_\alpha^2}\right) \le \left(-\frac{\alpha\ln\alpha}{\bar{\alpha}}\right)\operatorname{Var}(X).$$

(b) *If $\alpha > 1$ then*

$$\operatorname{Var}(X) \le \operatorname{Var}(Y) \le \alpha^3 \operatorname{Var}(X).$$

PROOF. The key formula in our proof is a result of Pyke (1965) according to which

$$\operatorname{Var}(X) = E\left[\mu_X^2(X)\right].$$

(a) When $0 < \alpha < 1, Y =^d X_{1:N_\alpha}$ and, by Corollary 10,

$$\alpha\mu_X(t) \le \mu_Y(t) \le \mu_X(t).$$

Further, as shown in part (b) of Lemma 1 of Kirmani (1996), if X is IMRL then

$$\mu_{X_{1:n}}(t) \leq (1/n)\mu_X(t), \qquad n = 1, 2, 3, \ldots.$$

Here, $X_{1:n} = \min\{X_1, \ldots, X_n\}$ and $X_1, \ldots, X_n$ are i.i.d. as X. We also note that $X_{1:n}$ is stochastically smaller than X and hence, when X is IMRL

$$E\left(\mu_X^2\left(X_{1:n}\right)\right) \leq E\left(\mu_X^2(X)\right).$$

Putting together the various facts given above, we get

$$\begin{aligned}
\operatorname{Var}(Y) &= E(\mu_Y^2(Y)) \\
&= E(\mu^2_{X_{1:N_\alpha}}(X_{1:N_\alpha})) \\
&\leq E\left(\frac{1}{N_\alpha^2}\mu_X^2\left(X_{1:N_\alpha}\right)\right) \\
&\leq E\left(\frac{1}{N_\alpha^2}\mu_X^2(X)\right) \\
&= E\left(\mu_X^2(X)\right) E\left(\frac{1}{N_\alpha^2}\right) \\
&\leq \operatorname{Var}(X) E\left(\frac{1}{N_\alpha}\right) \\
&= \left(-\frac{\alpha \ln \alpha}{\bar{\alpha}}\right) \operatorname{Var}(X).
\end{aligned}$$

Also, for $0 < \alpha < 1$ and increasing $\mu_X(t)$:

$$\begin{aligned}
\operatorname{Var}(Y) &= E\left(\mu_Y^2(Y)\right) \\
&\geq \alpha^2 E\left(\mu_X^2(Y)\right), \text{by Corollary 10} \\
&= \alpha^2\left[\mu_X^2(0) + \int_0^\infty 2\mu_X(t)\mu_X'(t)P(Y > t)dt\right] \\
&\geq \alpha^2\left[\mu_X^2(0) + \int_0^\infty 2\mu_X(t)\mu_X'(t)\alpha P(X > t)dt\right], \text{by Proposition 9} \\
&\geq \alpha^3\left[\mu_X^2(0) + \int_0^\infty 2\mu_X(t)\mu_X'(t)P(X > t)dt\right] \\
&= \alpha^3 E\left[\mu_X^2(X)\right] \\
&= \alpha^3 \operatorname{Var}(X).
\end{aligned}$$

(b) The proof of $\operatorname{Var}(Y) \leq \alpha^3 \operatorname{Var}(X)$ is similar to the one given above. To prove that $\operatorname{Var}(Y) \leq \operatorname{Var}(X)$ for $\alpha > 1$, note that

$$\begin{aligned}
\operatorname{Var}(Y) &= E\left(\mu_Y^2(Y)\right) \\
&\geq E\left(\mu_X^2(Y)\right), \qquad \text{because } \mu_Y(t) \geq \mu_X(t) \text{ by Corollary 10} \\
&\geq E\left(\mu_X^2(X)\right), \qquad \text{because } \mu_X(t) \text{ is increasing and } Y \geq_{st} X \\
&= \operatorname{Var}(X),
\end{aligned}$$

where $Y \geq_{st} X$ means that $P(Y \geq t) \geq P(X \geq t)$ for all t, see Shaked and Shanthikumar (1994). □

To see the usefulness of the above theorem, let X and Y be as in the paragraph following the proof of Theorem 8. For $0 \leq \beta \leq 1$, X is IMRL. The variance of Y must be calculated numerically and there is no table available in the literature for obtaining its values for any choices of $0 \leq \beta \leq 1$ and $\alpha > 0$.

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