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LÉVY-DRIVEN CARMA PROCESSES *

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Abstract. Properties and examples of continuous-time ARMA (CARMA) processes driven by Lévy processes are examined. By allowing Lévy processes to replace Brownian motion in the definition of a Gaussian CARMA process, we obtain a much richer class of possibly heavy-tailed continuous-time stationary processes with many potential applications in finance, where such heavy tails are frequently observed in practice. If the Lévy process has finite second moments, the correlation structure of the CARMA process is the same as that of a corresponding Gaussian CARMA process. In this paper we make use of the properties of general Lévy processes to investigate CARMA processes driven by Lévy processes $\{W(t)\}$ without the restriction to finite second moments. We assume only that $W(1)$ has finite r-th absolute moment for some strictly positive r. The processes so obtained include CARMA processes with marginal symmetric stable distributions.

Key words and phrases: Lévy process, CARMA process, stochastic differential equation, stable process.

1. Introduction

A zero-mean Gaussian CARMA(p,q) process $\{Y(t)\}$ with $0 \le q < p$ and coefficients $a_1, \dots, a_p$, $b_0, \dots, b_q$, is defined (see e.g. Brockwell and Davis (1996)) to be a stationary solution of the (suitably interpreted) p-th order linear differential equation,

$$a(D)Y(t) = b(D)DW(t), \qquad t \ge 0, \tag{1.1}$$

where D denotes differentiation with respect to t, $\{W(t)\}$ is standard Brownian motion,

$$a(z) := z^p + a_1 z^{p-1} + \cdots + a_p,$$
$$b(z) := b_0 + b_1 z + \cdots + b_p z^p,$$

and the coefficients b_j satisfy $b_q \ne 0$ and $b_j = 0$ for $q < j \le p$. Since the derivatives $D^j W(t)$ do not exist in the usual sense, we interpret (1.1) as being equivalent to the *observation* and *state* equations,

$$Y(t) = \boldsymbol{b}'\boldsymbol{X}(t), \tag{1.2}$$

and

$$d\boldsymbol{X}(t) - A\boldsymbol{X}(t)dt = \boldsymbol{e}dW(t), \tag{1.3}$$

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where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}$$

and we assume that $\boldsymbol{X}(0)$ is a Gaussian random vector such that

$$\boldsymbol{X}(0) \text{ is independent of } \{W(t), t \geq 0\}. \tag{1.4}$$

The state equation (1.3) is an Ito differential equation for $\boldsymbol{X}(t)$. If $p = 1$, A is defined to be $-a_1$. Because of the linearity of (1.3), its solution has the simple form,

$$\boldsymbol{X}(t) = e^{At}\boldsymbol{X}(0) + \int_0^t e^{A(t-u)}\boldsymbol{e}\,dW(u), \tag{1.5}$$

where the integral is defined as the L^2 limit of approximating Riemann-Stieltjes sums. The process $\{\boldsymbol{X}(u), u \geq 0\}$ also satisfies the relations,

$$\boldsymbol{X}(t) = e^{A(t-s)}\boldsymbol{X}(s) + \int_s^t e^{A(t-u)}\boldsymbol{e}\,dW(u), \qquad \text{for all} \quad t > s \geq 0, \tag{1.6}$$

which clearly show (by the independence of increments of $\{W(t)\}$) that $\{\boldsymbol{X}(u)\}$ is Markov.

It is well-known (see e.g. Brockwell (2000*a*)) that the equations (1.4) and (1.6) have a weakly stationary solution if and only if the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A (which are the same as the zeroes of the autoregressive polynomial $z^p + a_1 z^{p-1} + \cdots + a_p$) all have negative real parts, i.e. if and only if

$$\Re(\lambda_i) < 0, \qquad i = 1, \ldots, p. \tag{1.7}$$

If $\{\boldsymbol{X}(t)\}$ is such a solution then it is easy to show that

$$E(\boldsymbol{X}(0)) = \boldsymbol{0} \tag{1.8}$$

and

$$E(\boldsymbol{X}(0)\boldsymbol{X}'(0)) = \Sigma := \int_0^\infty e^{Ay}\boldsymbol{e}\boldsymbol{e}'e^{A'y}\,dy. \tag{1.9}$$

Conversely if (1.4), (1.7), (1.8) and (1.9) are satisfied, then the process $\{\boldsymbol{X}(t)\}$ defined by (1.5) is weakly stationary and satisfies the relations,

$$E[\boldsymbol{X}(t)] = \boldsymbol{0}, \qquad t \geq 0,$$

and

$$E[\boldsymbol{X}(t+h)\boldsymbol{X}(t)'] = e^{Ah}\Sigma, \qquad h \geq 0.$$

From (1.2) the mean and autocovariance function of the CARMA(p, q) process $\{Y(t)\}$ are then given by

$$E[Y(t)] = 0, \qquad t \geq 0$$

and

$$\gamma_Y(h) = E[Y(t+h)Y(t)] = \boldsymbol{b}'e^{A|h|}\Sigma\boldsymbol{b}. \tag{1.10}$$

If in addition the zeroes of the autoregressive polynomial are all distinct then the autocovariance function of $\{Y(t)\}$ has the simple form (see Brockwell (2000a)),

$$\gamma_Y(h) = \sum_{\lambda:a(\lambda)=0} \frac{e^{\lambda|h|}b(\lambda)b(-\lambda)}{a'(\lambda)a(-\lambda)}.$$

If $\boldsymbol{X}(0)$ satisfies (1.4), (1.8) and (1.9) and is also Gaussian, then $\{\boldsymbol{X}(t)\}$ and $\{Y(t)\}$ are strictly stationary and Gaussian.

The aim of this paper is to extend these results to CARMA processes driven by general Lévy processes instead of Brownian motion. CARMA processes driven by second-order Lévy processes have been treated by Brockwell (2000b). They are defined as follows.

DEFINITION 1.1. If $\{W(t)\}$ is a second order Lévy process, and p and q are integers such that $0 \le q < p$, then $\{Y(t), t \ge 0\}$ is a **second-order Lévy-driven CARMA**(p,q) **process** with parameters $a_1, \dots, a_p$, $b_0, \dots, b_q$, if and only if $\{Y(t)\}$ satisfies (1.2) with $\{\boldsymbol{X}(t)\}$ a strictly stationary second-order solution of the equations (1.4) and (1.6).

It was shown by Brockwell (2000b) that conditions (1.7) are necessary and sufficient for the existence of such a process. Under these conditions the finite dimensional joint characteristic functions of the process were determined. If $\{W(t)\}$ is scaled so that $EW(t) = ct$ and $E[(W(t) - W(s))^2] = t - s$ for $t \ge s \ge 0$, then $EY(t) = b_0c/a_p$ and $\gamma_Y(h)$ is given by (1.10). Notice that with this scaling, γ_Y depends only on the coefficients $a_1, \dots, a_p$ and $b_0, \dots, b_q$.

In this paper we drop the second-order assumption on $\{W(t)\}$, and show that the conditions (1.7) and $E|W(1)|^r < \infty$ for some $r > 0$ are sufficient for the existence of a general (possibly infinite variance) Lévy-driven CARMA process defined as follows.

DEFINITION 1.2. If $\{W(t)\}$ is a Lévy process and p and q are integers such that $0 \le q < p$, then $\{Y(t), t \ge 0\}$ is a **Lévy-driven CARMA**(p,q) **process** with parameters $a_1, \dots, a_p$, $b_0, \dots, b_q$, if and only if $\{Y(t)\}$ satisfies (1.2) with $\{\boldsymbol{X}(t)\}$ a strictly stationary solution of the equations (1.4) and (1.6).

Under the same conditions we also derive the joint characteristic functions of such processes, which include CARMA processes with symmetric stable marginal distributions.

The results rely on properties of stochastic integrals with respect to Lévy processes. An excellent account of these processes emphasizing the properties most relevant to stochastic integration is contained in the book of Protter (1991). Some of these properties are briefly outlined in the following section. Further useful references on the properties of Lévy processes are the lecture notes of Ito (1969) and the books of Bertoin (1996) and Küchler and Sørensen (1997).

First order stochastic differential equations with non-negative Lévy input process have been widely used in storage theory (Cinlar and Pinsky (1972), Harrison and Resnick (1976), Brockwell *et al.* (1982)) and more recently as a basis for non-Gaussian stochastic volatility models by Barndorff-Nielsen and Shephard (1999), who consider a wide variety of such models and their financial applications. Second-order Lévy-driven CARMA processes are of particular interest because they have the same autocovariance functions as corresponding Gaussian processes but exhibit a wide range of non-Gaussian marginal distributions such as the more heavy tailed distributions frequently encountered in financial data.

2. Lévy-driven CARMA processes

2.1 *Lévy Processes*

For a detailed account of the pertinent properties of Lévy processes see Protter (1991). We give here an account of the bare essentials needed for our results. Suppose we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le\infty}, P)$, where $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and $(\mathcal{F}_t)$ is right-continuous.

DEFINITION 2.1. An adapted process $\{W(t)\}_{0\le t\le\infty}$ with $W(0) = 0$ a.s. is said to be a **Lévy process** if

(i) $W(t) - W(s)$ is independent of $\mathcal{F}_s$, $0 \le s < t < \infty$,
(ii) $W(t) - W(s)$ has the same distribution as W_{t-s} and
(iii) $W(t)$ is continuous in probability.

Every Lévy process has a unique modification which is càdlàg (right continuous with left limits) and which is also a Lévy process. We shall therefore assume that our Lévy process has these properties.

The characteristic function of $W(t)$, $\phi_t(\theta) := E[\exp(i\theta W(t))]$, has the form

$$\phi_t(\theta) = \exp(t\xi(\theta)), \qquad \theta \in \mathbb{R}, \tag{2.1}$$

where

$$\xi(\theta) = i\theta m - \frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}_0}\left(e^{i\theta x} - 1 - \frac{ix\theta}{1+x^2}\right)\nu(dx), \tag{2.2}$$

for some $m \in \mathbb{R}$, $\sigma \ge 0$, and measure ν on the Borel subsets of $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$. The measure ν is called the Lévy measure of the process W and has the property,

$$\int_{\mathbb{R}_0} \frac{u^2}{1+u^2}\nu(du) < \infty.$$

If ν is the zero measure then $\{W(t)\}$ is Brownian motion with $E(W(t)) = mt$ and $\mathrm{Var}(W(t)) = \sigma^2 t$. If $m = \sigma^2 = 0$ and $\nu(\mathbb{R}_0) < \infty$, then $W(t) = at + P(t)$, where $\{P(t)\}$ is a compound Poisson process with jump-rate $\nu(\mathbb{R}_0)$, jump-size distribution $\nu/\nu(\mathbb{R}_0)$, and $a = -\int_{\mathbb{R}_0}\frac{u}{1+u^2}\nu(du)$. Another important example is the gamma process $\{W(t)\}$, for which

$$\xi(\theta) = \int_{\mathbb{R}_0}(e^{i\theta x} - 1)\nu(dx), \tag{2.3}$$

$\nu(du) = \alpha u^{-1}e^{-\beta u}du$, $u > 0$, and $W(t)$ has probability density function given by $\beta^{\alpha t}x^{\alpha t-1}e^{-\beta x}/\Gamma(\alpha t)$, $x > 0$. This is an example of a Lévy process whose sample-paths have (a.s.) infinitely many jumps in every interval of positive length. If $\{W_1(t)\}$ and $\{W_2(t)\}$ are two independent and identically distributed gamma processes then $W_1 - W_2$

is a symmetrized gamma process with Lévy measure, $\nu(du) = \frac{1}{2}\alpha|u|^{-1}e^{-\beta|u|}du$. For the non-decreasing stable process $X(t)$ with

$$E[\exp(i\theta X(t))] = \exp[-t\beta(-i\theta)^{\alpha}/\Gamma(1-\alpha)], \qquad \beta > 0, \quad 0 < \alpha < 1,$$

ξ also has the form (2.3), but with

$$\nu(du) = \alpha\beta u^{-1-\alpha}du, \qquad u > 0.$$

This is another example of a Lévy process which in each finite interval has infinitely many jumps with probability 1. Moreover it has infinite moments of all orders greater than or equal to α.

The most crucial result from the point of view of stochastic integration is Theorem 40 of Protter (1991), namely,

THEOREM 2.1. *Let W be a Lévy process. Then $W(t) = Y(t) + Z(t)$ where Y and Z are Lévy processes, Y is a martingale with bounded jumps, $Y \in L^p$ for all $p \geq 1$ and Z has paths of bounded variation on compacts.*

PROOF. See Protter (1991), p. 31.

2.2 *Existence of the Lévy driven CARMA process*

THEOREM 2.2. *If $\{W(t)\}$ is a Lévy process with characteristic function (2.1) and $E|W(1)|^r < \infty$ for some $r > 0$, then the Lévy-driven CARMA process specified by Definition 1.2 exists if condition (1.7) is satisfied, in which case the cumulant generating function of $Y(t_1), Y(t_2), \ldots, Y(t_n)$, $(0 \leq t_1 < t_2 < \cdots < t_n)$ is*

$$\text{(2.4)} \quad \ln E[\exp(i\theta_1 Y(t_1) + \cdots + i\theta_n Y(t_n))]$$
$$= \int_0^{\infty} \xi\left(\sum_{i=1}^{n} \theta_i \boldsymbol{b}' e^{A(t_i+u)}\right) \boldsymbol{e} du + \int_0^{t_1} \xi\left(\sum_{i=1}^{n} \theta_i \boldsymbol{b}' e^{A(t_i-u)}\right) \boldsymbol{e} du$$
$$+ \int_{t_1}^{t_2} \xi\left(\sum_{i=2}^{n} \theta_i \boldsymbol{b}' e^{A(t_i-u)}\right) \boldsymbol{e} du + \cdots + \int_{t_{n-1}}^{t_n} \xi\left(\sum_{i=2}^{n} \theta_i \boldsymbol{b}' e^{A(t_i-u)}\right) \boldsymbol{e} du.$$

In particular, the marginal distribution of $Y(t)$ has cgf,

$$\text{(2.5)} \qquad \ln E[\exp(i\theta Y(t))] = \int_0^{\infty} \xi(\theta \boldsymbol{b}' e^{Au}\boldsymbol{e})du.$$

PROOF. If (1.7) is satisfied and $\boldsymbol{X}(0)$ is any $p \times 1$ random vector, then the first term on the right of (1.5) converges in probability to zero as $t \to \infty$ and the integral term, by Theorem 2.1, can be expressed as

$$\boldsymbol{V}(t) = \int_0^t e^{A(t-u)}\boldsymbol{e}dW(u) = \int_0^t e^{A(t-u)}\boldsymbol{e}dY(u) + \int_0^t e^{A(t-u)}\boldsymbol{e}dZ(u),$$

where the first integral is defined as the L^2 limit of approximating Riemann-Stieltjes sums and the second is defined pathwise. The time-homogeneity of the Lévy process implies that $\boldsymbol{V}(t)$ has the same distribution as

$$\text{(2.6)} \qquad \boldsymbol{U}(t) = \int_0^t e^{Au}\boldsymbol{e}dW(u) = \int_0^t e^{Au}\boldsymbol{e}dY(u) + \int_0^t e^{Au}\boldsymbol{e}dZ(u).$$

The first integral, $\boldsymbol{U}_1(t)$ in (2.6), converges in L^2 as $t \to \infty$ to a random vector $\boldsymbol{U}_1$ (since for all $t > s$, $E\|\boldsymbol{U}_1(t) - \boldsymbol{U}_1(s)\|^2 \le c\int_s^\infty \|e^{Au}\boldsymbol{e}\|^2 du$ where c is independent of s, and the last expression converges to zero as $s \to \infty$ by (1.7)). We show next that the second integral, $\boldsymbol{U}_2(t)$ in (2.6), converges in probability as $t \to \infty$ to a random vector $\boldsymbol{U}_2$. To do this we write

$$(2.7) \qquad \boldsymbol{U}_2(t) = \sum_{j=1}^{[t]} e^{jA}\boldsymbol{\xi}_j + \int_{[t]}^t e^{Au}\boldsymbol{e}dZ(u),$$

where $[t]$ denotes the integer part of t and $\boldsymbol{\xi}_j$, $j = 1, 2, \ldots$, are the i.i.d. random vectors,

$$\boldsymbol{\xi}_j = \int_{j-1}^j e^{A(u-j)}\boldsymbol{e}dZ(u).$$

By (1.7), the integral on the right of (2.7) converges to zero in probability as $t \to \infty$. The sum on the right of (2.7) converges almost surely as $t \to \infty$ since the i-th component, $S_{ij}(t)$, of the j-th summand satisfies

$$|S_{ij}(t)| \le Ke^{j\lambda}|\eta_j|,$$

where K is a positive constant, $\lambda < 0$, and $\{\eta_j\}$ is an i.i.d. sequence with $E|\eta_j|^m = M < \infty$ for some $m \in (0,1)$. Hence

$$E\left(\sum_{j=1}^\infty |S_i(t)|\right)^m \le K^m E\left(\sum_{j=1}^\infty e^{j\lambda}|\eta_j|\right)^m \le MK^m \sum_{j=1}^\infty e^{j\lambda m} < \infty.$$

Thus each component of the sum in (2.7) converges absolutely with probability 1 and so $\boldsymbol{U}_2(t)$ converges in probability as $t \to \infty$.

The arguments of the preceding paragraph show that $\boldsymbol{U}(t)$ and hence $\boldsymbol{X}(t)$ converge in distribution as $t \to \infty$ to $\boldsymbol{U}_1 + \boldsymbol{U}_2$. The cgf of $\boldsymbol{U}(t)$ is easily calculated since

$$\boldsymbol{U}(t) = p \lim_{\Delta \to 0} \sum_{i=1}^n \exp(Au_i)\boldsymbol{e}(W(u_i) - W(u_{i-1})),$$

where $0 = u_0 < u_1 < \cdots < u_n = t$ and $\Delta = \max(u_i - u_{i-1})$. Hence, for all $\boldsymbol{\theta} \in \mathbb{R}^p$,

$$\begin{aligned}\ln E[\exp(i\boldsymbol{\theta}'\boldsymbol{U}(t))] &= \lim_{\Delta \to 0} \sum_{i=1}^n \xi(\boldsymbol{\theta}'e^{Au}\boldsymbol{e})(u_i - u_{i-1}) \\ &= \int_0^t \xi(\boldsymbol{\theta}'e^{Au}\boldsymbol{e})du,\end{aligned}$$

and so $\boldsymbol{X}(t)$ converges in distribution as $t \to \infty$ to a random vector with cgf,

$$(2.8) \qquad \kappa(\boldsymbol{\theta}) = \int_0^\infty \xi(\boldsymbol{\theta}'e^{Au}\boldsymbol{e})du.$$

This implies that the distribution defined by (2.8) is the unique stationary distribution for the Markov process $\{\boldsymbol{X}(t)\}$ defined by (1.4) and (1.6). Hence if $\boldsymbol{X}(0)$ has cgf (2.8), the process $\{\boldsymbol{X}(t)\}$ is strictly stationary. Since $Y(t) = \boldsymbol{b}'\boldsymbol{X}(t)$, the process $\{Y(t)\}$ is then strictly stationary with cgf (2.5).

To determine the joint stationary distribution of $\boldsymbol{X}(t_1), \boldsymbol{X}(t_2), \ldots, \boldsymbol{X}(t_n)$, $0 \le t_1 < t_2 < \cdots < t_n$, we write

$$(2.9)\quad \sum_{k=1}^{n} i\boldsymbol{\theta}_k'\boldsymbol{X}(t_k) = i\sum_{k=1}^{n}\boldsymbol{\theta}_k'\boldsymbol{X}(0) + \int_0^{t_1}\sum_{k=1}^{n}\boldsymbol{\theta}_k' e^{A(t_k-u)}\boldsymbol{e}dW(u) + \int_{t_1}^{t_2}\sum_{k=2}^{n}\boldsymbol{\theta}_k' e^{A(t_k-u)}\boldsymbol{e}dW(u) + \cdots + \int_{t_{n-1}}^{t_n}\boldsymbol{\theta}_n' e^{A(t_n-u)}\boldsymbol{e}dW(u).$$

Since the terms on the right are independent, we find, by the same calculations which led to (2.8), that the joint cgf of $\boldsymbol{X}(t_1), \ldots, \boldsymbol{X}(t_n)$ is

$$(2.10)\quad \ln E[\exp(i\boldsymbol{\theta}_1'\boldsymbol{X}(t_1) + \cdots + i\boldsymbol{\theta}_n'\boldsymbol{X}(t_n))]$$
$$= \int_0^\infty \xi\left(\sum_{i=1}^{n}\boldsymbol{\theta}_i' e^{A(t_i+u)}\right)\boldsymbol{e}du + \int_0^{t_1}\xi\left(\sum_{i=1}^{n}\boldsymbol{\theta}_i' e^{A(t_i-u)}\right)\boldsymbol{e}du + \int_{t_1}^{t_2}\xi\left(\sum_{i=2}^{n}\boldsymbol{\theta}_i' e^{A(t_i-u)}\right)\boldsymbol{e}du + \cdots + \int_{t_{n-1}}^{t_n}\xi\left(\sum_{i=2}^{n}\boldsymbol{\theta}_i' e^{A(t_i-u)}\right)\boldsymbol{e}du.$$

Since $Y(t_i) = \boldsymbol{b}'X(t_i)$, $i = 1, 2, \ldots, n$, we find at once from (2.10) that the joint cgf of $Y(t_1), \ldots, Y(t_n)$ is as specified in (2.4). □

3. Examples

Example 1. In the special case when ν is the zero measure, $\{\sigma^{-1}[Y(t) - b_0 m/a_p]\}$ is the Gaussian CARMA(p, q) process defined in Section 1.

Example 2. If $\{W(t)\}$ is a compound Poisson process with finite jump-rate λ and bilateral exponential jump-size distribution with probability density, $f(x) = \frac{1}{2}\beta e^{-\beta|x|}$, then by Theorem 2.1, the corresponding CAR(1) process has marginal cgf,

$$\kappa(\theta) = \int_0^\infty \xi(\theta e^{-a_1 u})du,$$

where $\xi(\theta) = \lambda\theta^2/(\beta^2 + \theta^2)$. Straightforward evaluation of the integral gives

$$\kappa(\theta) = -\frac{\lambda}{2a_1}\ln\left(1 + \frac{\theta^2}{\beta^2}\right),$$

showing that $Y(t)$ is distributed as the difference between two independent gamma distributed random variables with exponent $\lambda/(2a_1)$ and scale parameter β. In particular, if $\lambda = 2a_1$, the marginal distribution is bilateral exponential.

Example 3. If $\{W(t)\}$ is a symmetric stable process, then

$$\ln Ee^{i\theta W(t)} = -ct|\theta|^\alpha, \qquad c > 0, \quad 0 < \alpha \le 2.$$

Since $E|W(1)|^r < \infty$ for $r < \alpha$, we can apply Theorem 2.1 to deduce that if (1.7) holds then the corresponding Lévy-driven CARMA process $Y(t)$ exists and has the symmetric stable marginal distribution determined by

$$\ln Ee^{i\theta Y(t)} = -ct|\theta|^\alpha \int_0^\infty |\boldsymbol{b}' e^{Au}\boldsymbol{e}|^\alpha du.$$

Example 4. Numerous examples of marginal distributions for CAR(1) processes driven by non-negative Lévy processes can be found in the paper of Barndorff-Nielsen and Shephard (1999) who use them in conjunction with stochastic volatility models. Some other examples with applications in storage theory can be found in the papers of Cinlar and Pinsky (1972), Harrison and Resnick (1976) and Brockwell *et al.* (1982). By modelling non-negative processes $\{Y(t)\}$ as CARMA (rather than CAR(1)) processes driven by second-order non-negative Lévy processes, we can enlarge the class of potential autocorrelation functions, but at the same time the CARMA parameters are constrained by the non-negativity of $\{Y(t)\}$ to satisfy the condition,

$$\boldsymbol{b}' e^{At}\boldsymbol{e} \geq 0 \quad \text{for all} \quad t \geq 0. \tag{3.1}$$

An interesting problem in this connection (raised by Neil Shephard) is to characterize the class of possible CARMA correlation functions,

$$\rho_Y(h) = \boldsymbol{b}' \ e^{A|h|}\Sigma \ \boldsymbol{b}/(\boldsymbol{b}'\Sigma \ \boldsymbol{b}), \tag{3.2}$$

when the constraint (3.2) is imposed.

Example 5. On the left side of Fig. 1 are the histogram and sample autocorrelation function of the absolute daily returns $(100\ln(P(t)/P(t-1)))$ on the Hang Seng Index for the period July 1st, 1997-April 9th, 1999. It has been observed by Granger *et al.* (1999) that, as in this example, such absolute daily returns frequently follow an approximately exponential distribution with a slowly decaying positive autocorrelation function. The sample autocorrelation function can be well approximated by that of a CARMA(2,1) model with coefficients $a_1 = 2.66$, $a_2 = .30$, $b_0 = 1.0$ and $b_1 = 2.80$, estimated by maximization of the Gaussian likelihood. In an attempt to approximate the empirical marginal distribution, the two parameters of a gamma process $\{W(t)\}$ were adjusted so that the simulated marginal distribution of the corresponding gamma-driven CARMA(2,1) process had approximately the appropriate shape. A good match (shown on the right side of Fig. 1) was obtained by choosing the distribution of $W(1)$ to have exponent .060 and scale parameter 10. A more systematic approach to maximum likelihood estimation for such models is described in Section 4. The lack of evidence for long memory in the sample autocorrelation function of this data is very likely due to its relatively short length and is consistent with the suggestion of Granger *et al.* (1999) that the long memory observed in longer realizations of these series may be due to shifting levels.

Although the ad hoc procedure used in Example 5 gives a good match between the model and empirical marginal distributions and autocorrelation functions, this does not necessarily mean that the model gives a good representation of the dynamics of the process. A more systematic approach to the fitting of such models is needed. In the

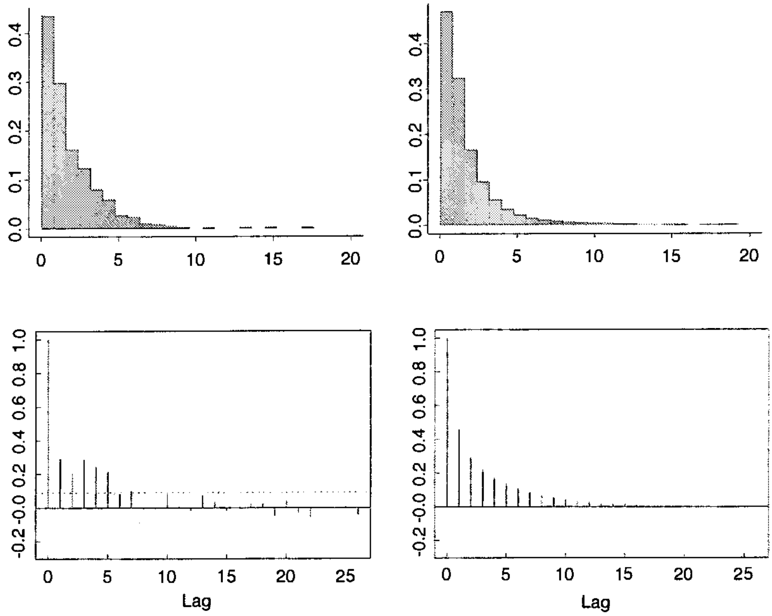

Fig. 1. The figures on the left show the histogram (top) and sample autocorrelation function of the absolute daily returns on the Hang Seng Index, July 1, 1997–April 9, 1999. The figures on the right are the corresponding graphs for the model defined in Example 5. The top right graph is based on 10,000 simulated values generated by the model.

following section we describe a simulation-based method for computing the likelihood under a Lévy-driven CARMA model for cases where the state vector has absolutely continuous transition function.

4. Inference for Lévy-driven CARMA processes

For linear CARMA processes, maximization of the Gaussian likelihood based on observations $Y(t_1), \ldots, Y(t_N)$ can be carried out very conveniently using the Kalman recursions and the innovations form of the likelihood (see Jones (1981, 1985)). However in order to distinguish between CARMA processes driven by different Lévy processes, it is important to develop estimation methods based on the *exact* as opposed to Gaussian likelihood.

A simulation-based method is described below. It is closely analogous to the method used by Brockwell and Williams (1997) to model daily returns on the Australian All-ordinaries Index using a threshold CAR(2) process. (A comparison in terms of AIC of the performance of such non-linear models with ARCH and GARCH models is given by Brockwell (2000*b*).)

To compute the likelihood for a linear Lévy-driven CARMA process we first observe that the state-space representation of the process can be reexpressed as

$$Y(t) = [\,1\ 0\ \cdots\ 0\,]\boldsymbol{Y}(t), \qquad t \geq 0, \tag{4.1}$$

where $\boldsymbol{Y}(t) = B\boldsymbol{X}(t)$ is a stationary solution of the vector AR(1) equation,

$$d\boldsymbol{Y}(t) = BAB^{-1}\boldsymbol{Y}(t)dt + Be\,dW(t), \qquad t \geq 0 \tag{4.2}$$

and

$$B = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{p-1} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{if} \quad b_0 \neq 0.$$

(If $b_0 = 0$ and i is the smallest integer such that $b_i \neq 0$, then we replace the first component of the $(i+1)$-th row in the definition of B by 1.)

The state process in the representation (4.1) and (4.2),

$$\boldsymbol{Y}(t) = \begin{bmatrix} Y(t) \\ \boldsymbol{V}(t) \end{bmatrix},$$

is Markov. If $\{\boldsymbol{Y}(t)\}$ has a transition density with respect to Lebesgue measure and if we denote by $p(y_{r+1}, \boldsymbol{v}_{r+1}, t_{r+1} - t_r \mid y_r, \boldsymbol{v}_r)$, the density of $(Y(t_{r+1}), \boldsymbol{V}(t_{r+1})')'$, given that $\boldsymbol{Y}(t_r) = (y_r, \boldsymbol{v}_r')'$, then the joint probability density f_r of the random variables, $Y(t_r), \boldsymbol{V}(t_r), Y(t_{r-1}), Y(t_{r-2}), \ldots, Y(t_1)$ satisfies the recursions,

$$\begin{aligned} f_{r+1}&(y_{r+1}, \boldsymbol{v}_{r+1}, y_r, y_{r-1}, \ldots, y_1) \\ &= \int p(y_{r+1}, \boldsymbol{v}_{r+1}, t_{r+1} - t_r \mid y_r, \boldsymbol{v}_r) f_r(y_r, \boldsymbol{v}_r, y_{r-1}, \ldots, y_1) d\boldsymbol{v}_r. \end{aligned} \tag{4.3}$$

For a given set of observed values $y_1, \ldots, y_N$, at times $t_1, \ldots, t_N$, the functions $f_2, \ldots, f_N$ are functions of $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_N$ respectively. These functions can easily be computed recursively from (4.3) in terms of f_1 and the functions $p(y_{r+1}, \cdot, t_{r+1} - t_r \mid y_r, \cdot)$. The likelihood of the observations $y_1, \ldots, y_N$, is then clearly

$$L(\theta; y_1, \ldots, y_N) = \int_{\boldsymbol{v}_N} f_N(\boldsymbol{v}_n) d\boldsymbol{v}_n. \tag{4.4}$$

The filtered value of the unobserved vector $\boldsymbol{V}(t_r)$, $r = 1, \ldots, N$, (i.e. the conditional expectation of $\boldsymbol{V}(t_r)$ given $Y(t_i) = y_i$, $i = 1, \ldots, r$) is readily obtained from the function f_r as

$$\tilde{\boldsymbol{v}}_r = \frac{\int \boldsymbol{v} f_r(\boldsymbol{v}) d\boldsymbol{v}}{\int f_r(\boldsymbol{v}) d\boldsymbol{v}}. \tag{4.5}$$

The Markov property of $\{\boldsymbol{Y}(t)\}$ then gives the predictor of $Y(t_{r+1})$,

$$\tilde{y}_{r+1} = m((y_r, \tilde{\boldsymbol{v}}_r')', t_{r+1} - t_r). \tag{4.6}$$

The preceding calculations are all dependent on the determination of the transition densities, $p(y_{r+1}, \boldsymbol{v}_{r+1}, t_{r+1} - t_r \mid y_r, \boldsymbol{v}_r)$ with y_{r+1} and y_r equal to the observations at times t_{r+1} and t_r, $r = 1, \ldots, N-1$ respectively, and the evaluation of the successive

$(p-1)$-dimensional integrals appearing in the recursions (4.3). For $p = 2$, these are one-dimensional integrals and, by discretizing v, can be evaluated as a sequence of matrix multiplications. Since $\boldsymbol{v}_1$ is not observed, we take $f_1(y, \boldsymbol{v})$ to be the Dirac delta function assigning mass one to $(y_1, \mathbf{0}')'$. The likelihood in (4.4) is then the density of $Y(t_2), \ldots, Y(t_n)$, *conditional* on $Y(t_1) = y_1$ and $\boldsymbol{V}(t_1) = \mathbf{0}$.

The method used by Brockwell and Williams (1997) for fitting Gaussian non-linear CAR(2) models was to replace $p(y_{r+1}, \boldsymbol{v}_{r+1}, t_{r+1} - t_r \mid y_r, \boldsymbol{v}_r)$ by a Gaussian transition density with moments calculated by Euler approximation.

However the transition density itself (provided it exists) can be replaced by a simulation-based kernel estimate and the likelihood as computed from (4.4) maximized numerically with respect to the unknown parameters. Work in this direction is currently in progress.

5. Concluding remarks

This work was motivated by the widely recognized need in financial modelling for the use of heavy-tailed models. Such heavy tails can also be generated by non-linear models driven by Gaussian noise. In fact it was shown by Ozaki (1985), Section 3.2, that for any generalized Pearson density W (i.e. satisfying $W'(x) = c(x)W(x)/d(x)$ with c and d analytic), there is a corresponding non-linear diffusion model with W as its limiting density. This provides an alternative method for generating continuous-time stationary time series models with prescribed marginal density. The linear structure of Lévy-driven CARMA models and (in the second-order case) the simple characterization of their second-order properties does however give them some advantages in the modelling of empirical data. A natural further extension is to allow non-linearity (e.g. of threshold type) in the definition of the Lévy driven CARMA processes. This raises a host of further questions which will be considered at a later date.

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