Ann. Inst. Statist. Math.
Vol. 52, No. 4, 767–777 (2000)

NUMBERS OF SUCCESS-RUNS OF SPECIFIED LENGTH UNTIL CERTAIN STOPPING TIME RULES AND GENERALIZED BINOMIAL DISTRIBUTIONS OF ORDER k*

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(Received April 21, 1998; revised February 22, 1999)

Abstract. A new distribution called a generalized binomial distribution of order k is defined and some properties are investigated. A class of enumeration schemes for success-runs of a specified length including non-overlapping and overlapping enumeration schemes is rigorously studied. For each nonnegative integer μ less than the specified length of the runs, an enumeration scheme called μ-overlapping way of counting is defined. Let k and ℓ be positive integers satisfying $\ell < k$. Based on independent Bernoulli trials, it is shown that the number of $(\ell - 1)$-overlapping occurrences of success-run of length k until the n-th overlapping occurrence of success-run of length ℓ follows the generalized binomial distribution of order $(k - \ell)$. In particular, the number of non-overlapping occurrences of success-run of length k until the n-th success follows the generalized binomial distribution of order $(k - 1)$. The distribution remains unchanged essentially even if the underlying sequence is changed from the sequence of independent Bernoulli trials to a dependent sequence such as higher order Markov dependent trials. A practical example of the generalized binomial distribution of order k is also given.

Key words and phrases: Binomial distribution of order k, Markov chain, probability generating function, stopping time, success-run, waiting time.

1. Introduction

Let $X_1, X_2, \ldots$ be $\{0, 1\}$-valued i.i.d. random variables with $P(X_i = 1) = p$. We say success and failure for the outcomes "1" and "0", respectively. For a given positive integer k, let τ_k be the waiting time (number of trials) until the first occurrence of a success-run of length k in $X_1, X_2, \ldots$. The distribution of τ_k is called a geometric distribution of order k and denoted by $G_k(p)$. We also denote by $G_k(p, a)$ the shifted geometric distribution of order k so that its support begins with a. The distribution has a very long history (cf. Johnson *et al.* (1992)) and it has been investigated by many authors (Philippou *et al.* (1983), Aki and Hirano (1994, 1995)) especially since 1980's. It has also many applications such as start-up demonstration tests, the reliability of engineering consecutive systems (Hahn and Gage (1983), Chao *et al.* (1995), Hirano (1994), Viveros and Balakrishnan (1993) and Balakrishnan *et al.* (1997))

Among some properties of the geometric distribution of order k, the following property is very interesting: If ℓ is a positive integer less than k, the distribution of the

* This research was partially supported by the ISM Cooperative Research Program (97-ISM · CRP-A1).

number of overlapping success-runs of length ℓ until τ_k is also the shifted geometric distribution of order $(k-\ell)$, $G_{k-\ell}(p, k-\ell+1)$ (cf. Aki and Hirano (1994)). When we extend the study from based on an i.i.d. sequence to based on a dependent sequence, the corresponding waiting time distribution becomes complicated. However, surprisingly, the distribution of the number of overlapping success-runs of length ℓ until the first occurrence of a success-run of length k is essentially unchanged even if the underlying sequence $X_1, X_2, \dots$ is changed to a sequence of dependent trials such as Markov dependent or higher order Markov dependent trials (cf. Aki and Hirano (1994, 1995) and Hirano *et al.* (1997)). To be precise, the following statement holds. Let $X_{-m+1}, X_{-m+2}, \dots, X_0, X_1, X_2, \dots$ be $\{0,1\}$-valued m-th order Markov chain with

$$\pi_{x_1,\dots,x_m} = P(X_{-m+1} = x_1, X_{-m+2} = x_2, \dots, X_0 = x_m)$$
$$p_{x_1,\dots,x_m} = P(X_i = 1 \mid X_{i-m} = x_1, X_{i-m+1} = x_2, \dots, X_{i-1} = x_m)$$

for $x_1, \dots, x_m = 0, 1$ and $i = 1, 2, \dots$. Then, if $m \le \ell < k$, the distribution of the number of overlapping occurrences of success-runs of length ℓ until the first occurrence of the success-run of length k in $X_1, X_2, \dots$ is the shifted geometric distribution of order $(k-\ell)$, $G_{k-\ell}(p_{1,1,\dots,1}, k-\ell+1)$.

In the present paper, we shall study the distribution of the number of success-runs of length k until the n th overlapping occurrence of the success-run of length ℓ, while in the above case the distribution of the number of overlapping success-runs of length ℓ until the first occurrence of the success-run of length k is studied. In the problem the generalized binomial distribution of order k to be defined in Section 2 plays an important role like the geometric distribution of order k in the above case.

In order to obtain the corresponding distributional results, how to enumerate success-runs is also very important. We give here the definition of enumeration scheme for success-runs.

DEFINITION 1.1. Suppose we are given a $\{0,1\}$-valued sequence of finite length. Success-runs of length k are assumed to be enumerated along the sequence in order. For a nonnegative integer μ less than k, the μ-*overlapping* number of success-runs of length k is the number of success-runs each of which may have overlapping part of length at most μ with the previous success-run of length k that has been enumerated. Then 0-*overlapping* and $(k-1)$-*overlapping* numbers of success-runs of length k mean non-overlapping and overlapping numbers of success-runs of length k, respectively.

For example, let us enumerate the number of success-runs of length 3 in the sequence $SSSSFSSSSSSSF$. By the non-overlapping way of enumeration, the sequence $(SSS)SF(SSS)(SSS)SF$ contains 3 (0-overlapping) success-runs of length 3. By the overlapping way of enumeration, the sequence $(S[SS)S]F(S[S\{S)(S][S\}S)S]F$ contains 7 (2-overlapping) success-runs of length 3. The 1-overlapping number of success-runs of length 3 in the sequence $(SSS)SF(SS[S)S\{S]SS\}F$ is 4.

We shall show in Section 3 that the distribution of the number of $(\ell-1)$-overlapping occurrences of success-run of length k until the n-th overlapping occurrence of success-run of length ℓ is a generalized binomial distribution of order $(k-\ell)$. A practical example of the generalized binomial distribution of order k is given in Remark 3.2 of Section 3. Similarly as we stated above on the geometric distribution of order k, the distribution remains unchanged essentially even if we change the underlying sequence of $\{0,1\}$-valued i.i.d. random variables to a dependent sequence such as higher order Markov chain. This property will be rigorously stated and proved in Section 4.

The main tool for deriving the results in this paper is the so-called method of conditional probability generating functions. However, the method is used here by conditioning with some stopping time rules. A little advanced use like this is explained generally in the Appendix.

2. Definition of a new distribution and its p.g.f.

Let a, k, and n be arbitrarily fixed positive integers. Let $X_1, X_2, \dots$ be a sequence of $\{0,1\}$-valued i.i.d. random variables with $P(X_i = 1) = p = 1 - q$. We call X_i the i-th trial. At each trial, we define the score of the i-th trial by

$$s(i) = \begin{cases} a & \text{if a non-overlapping 1-run of length } k \text{ is observed at the } i\text{-th trial,} \\ 1 & \text{otherwise.} \end{cases}$$

For example, when $k = 3$, in the sequence 1111011111110, we have $s(1) = s(2) = 1$, $s(3) = a$, $s(4) = \cdots = s(7) = 1$, $s(8) = a$, $s(9) = s(10) = 1$, $s(11) = a$, $s(12) = s(13) = 1$.

DEFINITION 2.1. A distribution is called a generalized binomial distribution of order k, to be denoted by $B_k(n,p,a)$, if it is the distribution of the number of non-overlapping success-runs of length k while the sum of the scores is less than or equal to n.

Remark 2.1. When $a = 1$, the generalized binomial distribition of order k is the usual binomial distribution of order k with probability function

$$P(X = x) = \sum_{m=0}^{k-1} \sum_{x_1,\dots,x_k} \binom{x_1 + \cdots + x_k + x}{x_1, \dots, x_k, x} p^n \left(\frac{q}{p}\right)^{x_1+\cdots+x_k}$$

for $x = 0, 1, \dots, [n/k]$ where the inner summation is over $x_1, \dots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k = n - m - kx$ (cf. e.g. Hirano (1986) and Philippou and Makri (1986)) since the sum of the scores becomes the number of trials in this case. As we shall see later, $B_k(n,p,2)$ is very important in our problems. The binomial distribution of order k is generalized and there are alternative binomial distributions of order k (cf. e.g. Aki (1985), Ling (1988), Johnson *et al.* (1992) and Balasubramanian *et al.* (1995)). However, the generalized binomial distributions of order k with $a \neq 1$ in this section are new discrete distributions.

In the following propositions we derive the recurrence relations and the probability generating functions (p.g.f.'s) of $B_k(n,p,a)$.

PROPOSITION 2.1. *Let $\phi_n(t)$ be the p.g.f. of $B_k(n,p,a)$. Then,*

if $n < k + a - 1$, *then* $\phi_n(t) = 1$.
If $n = k + a - 1$, *then* $\phi_n(t) = p^k t + (1 - p^k)$.
If $n > k + a - 1$, *then*

$$\phi_n(t) = q\phi_{n-1}(t) + pq\phi_{n-2}(t) + \cdots + p^{k-1}q\phi_{n-k}(t) + p^k t\phi_{n-k-a+1}(t).$$

PROOF. If $n < k + a - 1$, then a success-run of length k can not occur. If $n = k + a - 1$, then only the case that $X_1 = X_2 = \cdots = X_k = 1$ is possible for occurrence of a success-run of length k, and hence $\phi_n(t) = p^k t + (1-p^k)$. When $n > k+a-1$, by considering the location of the first occurrence of "0", we obtain

$$\phi_n(t) = q\phi_{n-1}(t) + pq\phi_{n-2}(t) + \cdots + p^{k-1}q\phi_{n-k}(t) + p^k t\phi_{n-k-a+1}(t).$$

This completes the proof. □

PROPOSITION 2.2. *The p.g.f. of $B_k(n,p,a)$ is given by*

$$\begin{aligned}\phi_n(t) = &\sum_{m=0}^{k-1} \sum_{n_1+2n_2+\cdots+kn_k+(k+a-1)n_{k+1}=n-m} \binom{n_1+n_2+\cdots+n_{k+1}}{n_1,n_2,\dots,n_{k+1}} \\ &\times q^{n_1+\cdots+n_k} p^{n_2+2n_3+\cdots+(k-1)n_k+kn_{k+1}+m} t^{n_{k+1}} \\ &+ \sum_{m=k}^{k+a-2} \sum_{n_1+2n_2+\cdots+kn_k+(k+a-1)n_{k+1}=n-m} \binom{n_1+n_2+\cdots+n_{k+1}}{n_1,n_2,\cdots,n_{k+1}} \\ &\times q^{n_1+\cdots+n_k} p^{n_2+2n_3+\cdots+(k-1)n_k+kn_{k+1}+k} t^{n_{k+1}}.\end{aligned}$$

PROOF. We define $\Phi(z) = \sum_{n=0}^{\infty} \phi_n(t)z^n$ and $U(j) = \sum_{n=0}^{j} \phi_n(t)z^n$. Then, from Proposition 2.1 we obtain

$$\begin{aligned}\Phi(z) &= qz(\Phi(z) - U(k+a-2)) + pqz^2(\Phi(z) - U(k+a-3)) + \cdots \\ &\quad + p^{k-1}qz^k(\Phi(z) - U(a-1)) + p^k t z^{k+a-1}(\Phi(z) - U(0)) + U(k+a-1) \\ &= (qz + pqz^2 + \cdots + p^{k-1}qz^k + p^k t z^{k+a-1})\Phi(z) \\ &\quad + 1 + pz + p^2z^2 + \cdots + p^{k-1}z^{k-1} + f(a,z),\end{aligned}$$

where

$$f(a,z) = \begin{cases} p^k z^k + p^k z^{k+1} + \cdots + p^k z^{k+a-2} & (a \geq 2) \\ 0 & (a = 1). \end{cases}$$

Therefore, we have

$$\Phi(z) = \frac{1 + pz + p^2z^2 + \cdots + p^{k-1}z^{k-1} + f(a,z)}{1 - qz - pqz^2 - \cdots - p^{k-1}qz^k - p^k t z^{k+a-1}}.$$

If $|z| < 1$ and $|t| \leq 1$, then $|\sum_{m=1}^{k} p^{m-1}qz^m + p^k t z^{k+a-1}| < 1$. Then, by expanding the above equation we obtain the desired result. □

3. Numbers of success-runs until some stopping time rules

In this section, we consider the distributions of the numbers of non-overlapping occurrences of success-run of length k until some stopping time rules in a sequence of $\{0,1\}$-valued random variables.

For the moment, let $X_1, X_2, \ldots$ be i.i.d. random variables with $P(X_i = 1) = p = 1 - q = 1 - P(X_i = 0)$. Let n be a positive integer. Let τ be the waiting time of the n-th "1" in $X_1, X_2, \ldots$.

Then, we obtain the following proposition.

PROPOSITION 3.1. *The distribution of the number of non-overlapping occurrences of success-run of length k until τ is a generalized binomial distribution of order $(k-1)$, $B_{k-1}(n, p, 2)$.*

PROOF. We denote by $\phi_n(t)$ the p.g.f. of the distribution of the number of non-overlapping occurrences of success-run of length k until τ. Suppose we have currently success-run of length ℓ. Then, $\phi_m^{(\ell)}(t)$ denotes the p.g.f. of the conditional distribution of the number of non-overlapping occurrences of success-run of length k from this time until we observe the m-th "1" after this time.

If $n < k$ then $\phi_n(t) = 1$, since a success-run of length k can not be observed until τ.

If $n = k$ then $\phi_n(t) = p^{k-1}t + (1 - p^{k-1})$, since in this case a success-run of length k can be obseved only just after a failure-run of nonnegative length which begins from X_1.

Suppose that $n > k$. Then $\phi_n(t) = \phi_{n-1}^{(1)}(t)$ holds, since the first "1" necessarily occurs before the occurrence of the n-th "1".

Suppose we have currently success-run of length ℓ. If the next outcome is "1", then the current length of success-run becomes $(\ell + 1)$. If the next outcome is "0", then the current length of success-run becomes zero. However, the next "1" necessarily occurs if the observed number of "1"'s is less than n. Hence, we obtain

$$(3.1)\qquad \begin{cases} \phi_{n-1}^{(1)}(t) = p\phi_{n-2}^{(2)}(t) + q\phi_{n-2}^{(1)}(t) \\ \phi_{n-2}^{(2)}(t) = p\phi_{n-3}^{(3)}(t) + q\phi_{n-3}^{(1)}(t) \\ \qquad\vdots \\ \phi_{n-k+2}^{(k-2)}(t) = p\phi_{n-k+1}^{(k-1)}(t) + q\phi_{n-k+1}^{(1)}(t) \\ \phi_{n-k+1}^{(k-1)}(t) = pt\phi_{n-k}^{(0)}(t) + q\phi_{n-k}^{(1)}(t). \end{cases}$$

From (3.1), we have

$$\phi_{n-1}^{(1)}(t) = q\phi_{n-2}^{(1)}(t) + pq\phi_{n-3}^{(1)}(t) + \cdots + p^{k-2}q\phi_{n-k}^{(1)}(t) + p^{k-1}t\phi_{n-k-1}^{(1)}(t),$$

and hence we obtain

$$\phi_n(t) = q\phi_{n-1}(t) + pq\phi_{n-2}(t) + \cdots + p^{k-2}q\phi_{n-k+1}(t) + p^{k-1}t\phi_{n-k}(t).$$

Then, by using Proposition 2.1 we see that the distribution is the generalized binomial distribution of order $(k-1)$, $B_{k-1}(n, p, 2)$. □

Remark 3.1. Here, we explain a little advanced use of the so-called method of conditional p.g.f.'s. The method is a widely-used useful method for deriving complicated discrete distributions especially based on dependent trials and it is usually used by considering the condition of one-step ahead from every condition (cf. Ebneshahrashoob and Sobel (1990) and Aki *et al.* (1996)). We have, however, used the method by considering the condition not of one-step ahead but of random-step ahead for obtaining the formula (3.1). We refer the reader who is interested in the method of conditional p.g.f.'s in general situations to the Appendix.

Remark 3.2. A start-up demonstration test is a mechanism by which a vender demonstrates to the customer the reliability of a equipment with regard to its starting (cf. Hahn and Gage (1983), Viveros and Balakrishnan (1993) and Balakrishnan *et al.* (1997)). The vender repeats start-ups of the equipment until consecutive k successful start-ups are observed. Consider the following more practical start-up demonstration test. The equipment consumes the specified amount of fuel gas for one successful start-up and it does not consume fuel gas for unsuccessful start-ups. The equipment has in advance the amount of fuel gas necessary for n successful start-ups. Then, the probability that the number of success-runs of length k until the n-th success is zero means the probability that the start-up demonstration test does not successfully end. Further, if the vender repeats start-ups of the equipment until the fuel gas is completely consumed, then the number of non-overlapping runs of consecutive k successful start-ups follows the generalized binomial distribution of order $(k-1)$ with $a = 2$.

Next, we consider the distribution of number of occurrences of success-run of length k until the n-th overlapping occurrence of success-run of length ℓ, where k and ℓ are positive integers satisfying $\ell < k$ and $k > 2$.

Let τ_ℓ be the waiting time for the n-th overlapping occurrence of success-run of length ℓ in $X_1, X_2, \ldots$.

THEOREM 3.1. *Let $X_1, X_2, \ldots$ be a sequence of $\{0,1\}$-valued random variables with $P(X_i = 1) = p = 1 - q = 1 - P(X_i = 0)$. Then, the number of $(\ell-1)$-overlapping occurrences of success-run of length k until τ_ℓ follows the generalized binomial distribution of order $(k-\ell)$, $B_{k-\ell}(n,p,2)$.*

PROOF. Let $\phi_n(t)$ be the p.g.f. of the distribution of the number of $(\ell-1)$-*overlapping* occurrences of success-run of length k until τ_ℓ.

Suppose we are currently observing a success-run of length ν. Then, $\phi_j^{(\nu)}(t)$ denotes the p.g.f. of the conditional distribution of the number of $(\ell-1)$-*overlapping* occurrences of success-run of length k from this time until we observe the j-th overlapping occurrence of success-run of length ℓ after this time.

If $\ell + n - 1 < k$, then $\phi_n(t) = 1$, since a success-run of length k can not be observed until τ_ℓ.

We consider the case that $\ell + n - 1 = k$. In this case, a success-run of length k can be observed only when a success-run of length $(k-\ell)$ occurs just after the first occurrence of success-run of length ℓ. Since we continue observing until the n-th occurrence of success-run of length ℓ, we observe the first occurrence of success-run of length ℓ with probability one and hence the probability that a success-run of length k occurs in the case is $p^{k-\ell}$.

Suppose that $n > k - \ell + 1$. By considering the first occurrence of success-run of length ℓ, we have $\phi_n(t) = \phi_{n-1}^{(\ell)}(t)$. Suppose we have currently success-run of length ν $(\ell \le \nu < k-1)$. Then, if the next outcome is "1", the current length of success-run becomes $(\nu+1)$. If the next outcome is "0", then the current length of success-run becomes zero. However, the next success-run of length ℓ necessarily occurs if the observed number of overlapping occurrences of success-run of length ℓ is less than n.

Suppose we have currently success-run of length $(k-1)$. If the next outcome is "0", then the current length of success-run becomes zero and the next success-run of length ℓ necessarily occurs while the observed number of overlapping occurrences of success-run of length ℓ is less than n. If the next outcome is "1", then the current length of success-run becomes k and the success-run of length k is enumerated. In this case, if the next outcome is also "1", then the next overlapping success-run of length ℓ is observed here. Further, if success-run of length $(k-\ell)$ occurs just after the outcome, then we observe the next success-run of length k. Then, the success-runs of length k have the overlapping part of length $(\ell - 1)$.

Therefore, we obtain

$$
(3.2) \qquad \begin{cases} \phi_{n-1}^{(\ell)}(t) = p\phi_{n-2}^{(\ell+1)}(t) + q\phi_{n-2}^{(\ell)}(t) \\ \phi_{n-2}^{(\ell+1)}(t) = p\phi_{n-3}^{(\ell+2)}(t) + q\phi_{n-3}^{(\ell)}(t) \\ \vdots \\ \phi_{n-k+\ell+1}^{(k-2)}(t) = p\phi_{n-k+\ell}^{(k-1)}(t) + q\phi_{n-k+\ell}^{(\ell)}(t) \\ \phi_{n-k+\ell}^{(k-1)}(t) = pt\phi_{n-k+\ell-2}^{(\ell)}(t) + q\phi_{n-k+\ell-1}^{(\ell)}(t). \end{cases}
$$

From (3.2), we have

$$
\phi_{n-1}^{(\ell)}(t) = q\phi_{n-2}^{(\ell)}(t) + pq\phi_{n-3}^{(\ell)}(t) + \cdots + p^{k-\ell-1}q\phi_{n-k+\ell-1}^{(\ell)}(t) + p^{k-\ell}t\phi_{n-k+\ell-2}^{(\ell)}(t),
$$

and hence we obtain

$$
\phi_n(t) = q\phi_{n-1}(t) + pq\phi_{n-2}(t) + \cdots + p^{k-\ell-1}q\phi_{n-k+\ell}(t) + p^{k-\ell}t\phi_{n-k+\ell-1}(t).
$$

From Proposition 2.1, the distribution is the generalized binomial distribution of order $(k-\ell)$, $B_{k-\ell}(n, p, 2)$. □

4. Higher order Markov dependent trials

In this section, we study the corresponding problem based on higher order Markov dependent trials. Let m, ℓ and k be positive integers satisfying $m \le \ell < k$.

Let $X_{-m+1}, X_{-m+2}, \ldots, X_0, X_1, X_2, \ldots$ be $\{0,1\}$-valued m-th order Markov chain with

$$
\begin{aligned}
\pi_{x_1,\ldots,x_m} &= P(X_{-m+1} = x_1, X_{-m+2} = x_2, \ldots, X_0 = x_m) \\
p_{x_1,\ldots,x_m} &= P(X_i = 1 \mid X_{i-1} = x_m, X_{i-2} = x_{m-1} \ldots, X_{i-m} = x_1) \\
q_{x_1,\ldots,x_m} &= 1 - p_{x_1,\ldots,x_m},
\end{aligned}
$$

for $x_1, \ldots, x_m = 0, 1$ and $i = 1, 2, \ldots$. For $x_1, \ldots, x_m = 0, 1$, we assume that $0 < p_{x_1,\ldots,x_m}, q_{x_1,\ldots,x_m} < 1$.

Let τ_ℓ be the waiting time for the n-th overlapping occurrence of success-run of length ℓ in $X_1, X_2, \ldots$.

THEOREM 4.1. *Let $X_{-m+1}, X_{-m+2}, \ldots, X_0, X_1, X_2, \ldots$ be the above $\{0,1\}$-valued m-th order Markov dependent trials. Let m, ℓ and k be positive integers satisfying $m \le \ell < k$. Then, the number of $(\ell-1)$-overlapping occurrences of success-run of length k in $X_1, X_2, \ldots$ until τ_ℓ follows the generalized binomial distribution of order $(k-\ell)$, $B_{k-\ell}(n, p_{1,1,\ldots,1}, 2)$.*

PROOF. Let $\phi_n(t)$ be the p.g.f. of the distribution of the number of $(\ell-1)$-*overlapping* occurrences of success-run of length k in $X_1, X_2, \ldots, X_{\tau_\ell}$. We denote by $A_{x_1,\ldots,x_m}$ the event that $\{X_{-m+1} = x_1, X_{-m+2} = x_2, \ldots, X_0 = x_m\}$. We fix $A_{x_1,\ldots,x_m}$ and let $\phi_n^{A_{x_1,\ldots,x_m}}(t)$ be the conditional p.g.f. given $A_{x_1,\ldots,x_m}$. For $\nu = m, m+1, \ldots, k-1$, suppose we have currently success-run of length ν. Then, $\phi_j^{(\nu)}(t)$ denotes the p.g.f. of the conditional distribution of the number of $(\ell-1)$-*overlapping* occurrences of success-run of length k from this time until we observe the j-th overlapping occurrence of success-run of length ℓ after this time. Since $m \le \ell$, the conditional distribution does not depend on more than m-step past.

Starting from $A_{x_1,\dots,x_m}$, we observe the sequence until the n-th overlapping occurrence of success-run of length ℓ. Therefore, we observe the first occurrence of success-run of length ℓ with probability one, and hence we have

$$\phi_n^{A_{x_1,\dots,x_m}}(t) = \phi_{n-1}^{(\ell)}(t).$$

Note that the RHS of the equation does not depend on the initial condition $A_{x_1,\dots,x_m}$. Hence, we can write simply $\phi_n(t) = \phi_n^{A_{x_1,\dots,x_m}}(t)$. Similarly as the proof of Theorem 3.1, we obtain

$$(4.1)\qquad \begin{cases} \phi_{n-1}^{(\ell)}(t) = p_{1,1,\dots,1}\phi_{n-2}^{(\ell+1)}(t) + q_{1,1,\dots,1}\phi_{n-2}^{(\ell)}(t) \\ \phi_{n-2}^{(\ell+1)}(t) = p_{1,1,\dots,1}\phi_{n-3}^{(\ell+2)}(t) + q_{1,1,\dots,1}\phi_{n-3}^{(\ell)}(t) \\ \qquad\vdots \\ \phi_{n-k+\ell+1}^{(k-2)}(t) = p_{1,1,\dots,1}\phi_{n-k+\ell}^{(k-1)}(t) + q_{1,1,\dots,1}\phi_{n-k+\ell}^{(\ell)}(t) \\ \phi_{n-k+\ell}^{(k-1)}(t) = p_{1,1,\dots,1}t\phi_{n-k+\ell-2}^{(\ell)}(t) + q_{1,1,\dots,1}\phi_{n-k+\ell-1}^{(\ell)}(t). \end{cases}$$

From (4.1), we have

$$\begin{aligned} \phi_{n-1}^{(\ell)}(t) &= q_{1,1,\dots,1}\phi_{n-2}^{(\ell)}(t) + p_{1,1,\dots,1}q_{1,1,\dots,1}\phi_{n-3}^{(\ell)}(t) + \cdots \\ &\quad + p_{1,1,\dots,1}^{k-\ell-1}q_{1,1,\dots,1}\phi_{n-k+\ell-1}^{(\ell)}(t) + p_{1,1,\dots,1}^{k-\ell}t\phi_{n-k+\ell-2}^{(\ell)}(t), \end{aligned}$$

and hence we obtain

$$\begin{aligned} \phi_n(t) &= q_{1,1,\dots,1}\phi_{n-1}(t) + p_{1,1,\dots,1}q_{1,1,\dots,1}\phi_{n-2}(t) + \cdots \\ &\quad + p_{1,1,\dots,1}^{k-\ell-1}q_{1,1,\dots,1}\phi_{n-k+\ell}(t) + p_{1,1,\dots,1}^{k-\ell}t\phi_{n-k+\ell-1}(t). \end{aligned}$$

From Proposition 2.1, the distribution is the generalized binomial distribution of order $(k-\ell)$, $B_{k-\ell}(n, p_{1,1,\dots,1}, 2)$. □

Appendix

Let $\boldsymbol{N} = \{0,1,2,\dots\}$. Suppose $\{X_n\}_{n\in\boldsymbol{N}}$ is a sequence of $\{0,1\}$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_n = \sigma(\{X_0, X_1, \dots, X_n\})$ and let $\mathcal{F}_\infty = \vee_{n=0}^\infty \mathcal{F}_n$. Suppose we are given $(\mathcal{F}_n)$-stopping times τ_0, τ_1 and τ_2, i.e., they are measurable maps from Ω to $\bar{\boldsymbol{N}} = \{0,1,2,\dots,\infty\}$ such that $\{\tau_i \le n\} \in \mathcal{F}_n$ for all $n \in \boldsymbol{N}$ and $i = 0, 1$ and 2. For $i = 0, 1$ and 2, we set

$$\mathcal{F}_{\tau_i} = \{A \in \mathcal{F}_\infty \mid \forall n \in \boldsymbol{N}, A \cap \{\tau_i \le n\} \in \mathcal{F}_n\}.$$

Let $E_1, \dots, E_\ell$ be $\{0,1\}$-patterns of finite lengths. For integers a and b $(a < b)$, we denote by $\boldsymbol{N}(a,b)$ the vector of numbers of occurrences of E_1, $E_2, \dots$ and E_ℓ in X_{a+1},

$X_{a+2}, \ldots, X_b$. Under some assumptions, we now derive recurrence relations of conditional probability generating functions of the vector of numbers of occurrences of the pattrens until the stopping time τ_0.

ASSUMPTIONS

A1. $\tau_1 \le \tau_2 \le \tau_0$ hold almost surely.

A2. For $i = 1$ and 2, there exist $B_j^{(\tau_i)} \in \mathcal{F}_{\tau_i}$ $j = 1, \ldots, m$ such that $\Omega = \cup_{j=1}^m B_j^{(\tau_i)}$ (disjoint) and

$$\begin{aligned} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_i,\tau_0)} \mid \mathcal{F}_{\tau_i}] &= E[\boldsymbol{t}^{\boldsymbol{N}(\tau_i,\tau_0)} \mid \sigma(\{B_j^{(\tau_i)}\}_{j=1}^m)] \\ &= E[\boldsymbol{t}^{\boldsymbol{N}(\tau_i,\tau_0)} \mid B_j^{(\tau_i)}], \qquad \omega \in B_j^{(\tau_i)}, \quad \text{where} \end{aligned}$$

$E[\boldsymbol{t}^{\boldsymbol{N}(\tau_i,\tau_0)} \mid B_j^{(\tau_i)}] = \frac{1}{P(B_j^{(\tau_i)})} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_i,\tau_0)} 1_{B_j^{(\tau_i)}}]$ and $\boldsymbol{t}^{\boldsymbol{v}} = t_1^{v_1} t_2^{v_2} \cdots t_\ell^{v_\ell}$ for $\boldsymbol{t} = (t_1, \ldots, t_\ell)$ and $\boldsymbol{v} = (v_1, \ldots, v_\ell)$.

A3. The random variables $\boldsymbol{N}(0, \tau_1)$ and $\boldsymbol{N}(\tau_1, \tau_2)$ take only finite values $\{\boldsymbol{r}_{1j}\}_{j=1}^{k_1}$ and $\{\boldsymbol{r}_{2j}\}_{j=1}^{k_2}$, respectively. We define $R_i^{(1)} = \{\omega \mid \boldsymbol{N}(0, \tau_1) = \boldsymbol{r}_{1i}\}$, $i = 1, \ldots, k_1$ and $R_j^{(2)} = \{\omega \mid \boldsymbol{N}(\tau_1, \tau_2) = \boldsymbol{r}_{2j}\}$, $j = 1, \ldots, k_2$. Then, without loss of generality we assume $P(R_i^{(1)}) > 0$ and $P(R_j^{(2)}) > 0$.

THEOREM A.1. *Under the above assumptions, the following relations hold*

$$\text{(A.1)} \qquad E[\boldsymbol{t}^{\boldsymbol{N}(0,\tau_0)}] = \sum_{\ell=1}^{k_1} P(R_\ell^{(1)}) \boldsymbol{t}^{\boldsymbol{r}_{1\ell}} \sum_{j=1}^{m} P(B_j^{(\tau_1)} \mid R_\ell^{(1)}) E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid B_j^{(\tau_1)}]$$

and

$$\begin{aligned} \text{(A.2)} \qquad E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid B_{j_0}^{(\tau_1)}] &= \sum_{\ell=1}^{k_2} P(R_\ell^{(2)} \mid B_{j_0}^{(\tau_1)}) \boldsymbol{t}^{\boldsymbol{r}_{2\ell}} \\ &\quad \times \sum_{j=1}^{m} P(B_j^{(\tau_2)} \mid B_{j_0}^{(\tau_1)} \cap R_\ell^{(2)}) E[\boldsymbol{t}^{\boldsymbol{N}(\tau_2,\tau_0)} \mid B_j^{(\tau_2)}]. \end{aligned}$$

PROOF. Since $\tau_1 \le \tau_0$ holds almost surely, we can write $\boldsymbol{N}(0, \tau_0) = \boldsymbol{N}(0, \tau_1) + \boldsymbol{N}(\tau_1, \tau_0)$, and hence $\boldsymbol{t}^{\boldsymbol{N}(0,\tau_0)} = \boldsymbol{t}^{\boldsymbol{N}(0,\tau_1)} \cdot \boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)}$. Note that $\boldsymbol{t}^{\boldsymbol{N}(0,\tau_1)}$ is $\mathcal{F}_{\tau_1}$-measurable. This implies

$$E[\boldsymbol{t}^{\boldsymbol{N}(0,\tau_0)}] = E[\boldsymbol{t}^{\boldsymbol{N}(0,\tau_1)} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid \mathcal{F}_{\tau_1}]].$$

From the assumption A2, there exists an $\mathcal{F}_{\tau_1}$-measurable partition $\{B_j^{(\tau_1)}\}_{j=1}^m$ such that

$$E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid \mathcal{F}_{\tau_1}] = E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid \sigma(\{B_j^{(\tau_1)}\}_{j=1}^m)].$$

Further, from the assumption A3, $\boldsymbol{N}(0, \tau_1)$ takes only finite values $\{\boldsymbol{r}_{1i}\}_{i=1}^{k_1}$. Hence, for $\ell = 1, \ldots, k_1$ and $j = 1, \ldots, m$, $\boldsymbol{t}^{\boldsymbol{N}(0,\tau_1)} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid \mathcal{F}_{\tau_1}]$ has the constant value $\boldsymbol{t}^{\boldsymbol{r}_{1\ell}} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid B_j^{(\tau_1)}]$ over $R_\ell^{(1)} \cap B_j^{(\tau_1)}$. Therefore, we have

$$\begin{aligned} E[t^{\boldsymbol{N}(0,\tau_0)}] &= \sum_{\ell=1}^{k_1} \sum_{j=1}^{m} P(R_\ell^{(1)} \cap B_j^{(\tau_1)}) \boldsymbol{t}^{\boldsymbol{r}_{1\ell}} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid B_j^{(\tau_1)}] \\ &= \sum_{\ell=1}^{k_1} \boldsymbol{t}^{\boldsymbol{r}_{1\ell}} P(R_\ell^{(1)}) \sum_{j=1}^{m} P(B_j^{(\tau_1)} \mid R_\ell^{(1)}) E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid B_\ell^{(\tau_1)}]. \end{aligned}$$

Similarly, we can show (A.2). Since $\tau_1 \le \tau_2$ holds almost surely, $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$ holds. Then, we have

$$\begin{aligned} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid B_{j_0}^{(\tau_1)}] &= E[E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_0)} \mid \mathcal{F}_{\tau_2}] \mid B_{j_0}^{(\tau_1)}] \\ &= E[\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_2)} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_2,\tau_0)} \mid \mathcal{F}_{\tau_2}] \mid B_{j_0}^{(\tau_1)}]. \end{aligned}$$

From the assumptions A2 and A3, for $\ell = 1,\dots,k_2$ and $j = 1,\dots,m$, $\boldsymbol{t}^{\boldsymbol{N}(\tau_1,\tau_2)} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_2,\tau_0)} \mid \mathcal{F}_{\tau_2}]$ has the constant value $\boldsymbol{t}^{\boldsymbol{r}_{2\ell}} E[\boldsymbol{t}^{\boldsymbol{N}(\tau_2,\tau_0)} \mid B_j^{(\tau_2)}]$ over $R_\ell^{(2)} \cap B_j^{(\tau_2)}$. Consequently, we obtain (A.2), which completes the proof. □

Example A.1. Let $X_1, X_2, \dots$ be $\{0,1\}$-valued i.i.d. random variables with $P(X_i = 1) = p = 1 - q = 1 - P(X_i = 0)$. Let τ_1, τ_2 and τ_0 be the waiting times of the first, the second and the n-th occurrence of "1". E_1 denotes a success-run of length k $(n > k > 2)$. Let $N(a,b)$ be the number of non-overlapping occurrences of E_1 in $X_{a+1}, X_{a+2}, \dots, X_b$. Then, by definition $\tau_1 \le \tau_2 \le \tau_0$ hold. Since τ_1 comes before τ_0 almost surely, we have $E[t^{N(\tau_1,\tau_0)} \mid \mathcal{F}_{\tau_1}] = E[t^{N(\tau_1,\tau_0)}]$, i.e., $B_1^{(\tau_1)} = \Omega$. If we define $B_1^{(\tau_2)} = \{\omega \mid \tau_1(\omega) = \tau_2(\omega) - 1\}$ and $B_2^{(\tau_2)} = \{\omega \mid \tau_1(\omega) < \tau_2(\omega) - 1\}$, then we obtain

$$E[t^{N(\tau_2,\tau_0)} \mid \mathcal{F}_{\tau_2}] = \begin{cases} E[t^{N(\tau_2,\tau_0)} \mid B_1^{(\tau_2)}], & \omega \in B_1^{(\tau_2)} \\ E[t^{N(\tau_2,\tau_0)} \mid B_2^{(\tau_2)}], & \omega \in B_2^{(\tau_2)}. \end{cases}$$

Since $k > 2$, $N(0,\tau_1) = 0$ and $N(\tau_1,\tau_2) = 0$ hold almost surely. Therefore, we obtain from Theorem A.1,

$$E[t^{N(0,\tau_0)}] = E[t^{N(\tau_1,\tau_0)}]$$

and

$$\begin{aligned} E[t^{(N(\tau_1,\tau_0)}] &= P(B_1^{(\tau_2)}) E[t^{N(\tau_2,\tau_0)} \mid B_1^{(\tau_2)}] + P(B_2^{(\tau_2)}) E[t^{N(\tau_2,\tau_0)} \mid B_2^{(\tau_2)}] \\ &= pE[t^{N(\tau_2,\tau_0)} \mid B_1^{(\tau_2)}] + qE[t^{N(\tau_2,\tau_0)} \mid B_2^{(\tau_2)}]. \end{aligned}$$

The latter is the first equation in (3.1).

Example A.2. Let $X_{-m+1}, X_{-m+2}, \dots, X_0, X_1, X_2, \dots$ be $\{0,1\}$-valued m-th order Markov chain defined in Section 4. Let τ_1, τ_2 and τ_0 be the waiting times of the first, the second and the n-th overlappng occurrence of success-run of length ℓ. E_1 denotes a success-run of length k $(m \le \ell < k-1)$. Let $N(a,b)$ be the number of $(\ell-1)$-*overlapping* occurrences of E_1 in $X_{a+1}, X_{a+2}, \dots, X_b$. Then, by definition $\tau_1 \le \tau_2 \le \tau_0$ hold. Similarly as Example A.1, we have $E[t^{N(\tau_1,\tau_0)} \mid \mathcal{F}_{\tau_1}] = E[t^{N(\tau_1,\tau_0)}]$, i.e., $B_1^{(\tau_1)} = \Omega$. If we define $B_1^{(\tau_2)} = \{\omega \mid \tau_1(\omega) = \tau_2(\omega) - 1\}$ and $B_2^{(\tau_2)} = \{\omega \mid \tau_1(\omega) < \tau_2(\omega) - 1\}$, then we obtain

$$E[t^{N(\tau_2,\tau_0)} \mid \mathcal{F}_{\tau_2}] = \begin{cases} E[t^{N(\tau_2,\tau_0)} \mid B_1^{(\tau_2)}], & \omega \in B_1^{(\tau_2)} \\ E[t^{N(\tau_2,\tau_0)} \mid B_2^{(\tau_2)}], & \omega \in B_2^{(\tau_2)}. \end{cases}$$

Since $\ell < k-1$, $N(0,\tau_1) = 0$ and $N(\tau_1,\tau_2) = 0$ hold almost surely. Then, we have from Theorem A.1,

$$E[t^{N(0,\tau_0)}] = E[t^{N(\tau_1,\tau_0)}]$$

and

$$\begin{aligned}E[t^{(N(\tau_1,\tau_0)}] &= P(B_1^{(\tau_2)})E[t^{N(\tau_2,\tau_0)} \mid B_1^{(\tau_2)}] + P(B_2^{(\tau_2)})E[t^{N(\tau_2,\tau_0)} \mid B_2^{(\tau_2)}] \\ &= p_{1,1,\cdots,1}E[t^{N(\tau_2,\tau_0)} \mid B_1^{(\tau_2)}] + (1 - p_{1,1,\dots,1})E[t^{N(\tau_2,\tau_0)} \mid B_2^{(\tau_2)}].\end{aligned}$$

The latter is the first equation in (4.1).

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