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SOONER AND LATER WAITING TIME PROBLEMS BASED ON A DEPENDENT SEQUENCE

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Abstract. This paper introduces a new concept: a binary sequence of order (k, r), which is an extension of a binary sequence of order k and a Markov dependent sequence. The probability functions of the sooner and later waiting time random variables are derived in the binary sequence of order (k, r). The probability generating functions of the sooner and later waiting time distributions are also obtained. Extensions of these results to binary sequence of order (g, h) are also presented.

Key words and phrases: Binary sequence of order (k, r), binary sequence of order k, sooner and later waiting time problems, probability function, probability generating function.

1. Introduction

The origin of success run waiting time goes back as far as De Moivre's era (Johnson *et al.* (1992), pp. 426–432). Recently distribution theory on runs has been developed by many authors. Ebneshahrashoob and Sobel (1990) proposed a sooner and later problem for success and failure runs. They obtained the probability generating functions of the waiting time distributions for a run of "0" of length r or (and) a run of "1" of length k whichever comes sooner (later) in independent Bernoulli trials. Generalizations of this problem have been considered by Ling (1990, 1992), Balasubramanian *et al.* (1993), Ling and Low (1993), Aki and Hirano (1993), Uchida and Aki (1995), Antzoulakos and Philippou (1996), Aki *et al.* (1996), Fu (1996), Koutras and Papastavridis (1993) and Koutras and Alexandrou (1997).

Aki (1985) defined a concept of a binary sequence of order k which is an extension of independent trials with a constant success probability. Aki (1985) and Aki and Hirano (1988) obtained some exact probability functions and probability generating functions of discrete distributions of order k (such as the binomial, the geometric, the negative binomial distributions, etc.) in a binary sequence of order k. Antzoulakos and Philippou (1996) derived the probability functions and the probability generating functions of the sooner and later waiting time problems in a binary sequence of order k. Balakrishnan (1997) gave the probability generating functions of the joint distributions of numbers of success runs and failures until the first consecutive k successes in a binary sequence of order h.

Antzoulakos and Philippou (1996) discussed the sooner and later waiting time problems in a binary sequence of order k. We think that the binary sequence of order k is a stochastic model for statistical analyses on success runs of length k, in which we can take account of dependency caused by every success run of length less than k as follows: the

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success probability of the t-th trial depends on the number of consecutive successes just before the trial. If the number is 0, i.e. it is failure in the $(t-1)$-th trial, the success probability of the t-th trial is a constant. When we consider effectively the sooner and later waiting time problems between a success run of length k and a failure run of length r, we need to take consideration of dependency caused by every run whichever is a success or failure run, since we have to consider success and failure runs simultaneously. Naturally, the binary sequence of order k can be extended, and these results of Antzoulakos and Philippou (1996) can be improved. In this paper we consider the trials that the success probability of the t-th trial not only depends on the number of consecutive successes just before the trial, but also depends on the number of consecutive failures just before the trial, if the $(t-1)$-th trial is failure.

In this paper we define a new concept of dependent sequence: a binary sequence of order (k, r), which is an extension of a binary sequence of order k and a Markov chain dependent sequence. Then we derive exact formulas for the probability functions and the probability generating functions of the sooner and later waiting time problems in the case of a binary sequence of order (k, r). These results are extensions of the results of Antzoulakos and Philippou (1996). Finally, we discuss the sooner and later waiting time problems in a binary sequence of order (g, h).

2. A binary sequence of order (k, r)

In this section we define a binary sequence of order (k, r), which is an extension of a binary sequence of order k (Aki (1985)) and a Markov dependent sequence.

DEFINITION 2.1. An infinite sequence $\{X_n, n = 1, 2, \ldots\}$ of $\{0, 1\}$-valued random variables is said to be a binary sequence of order (k, r), if there exist positive integers k, r, real numbers $0 < p_1^{(1)}, \ldots, p_k^{(1)} < 1$, $0 < p_1^{(0)}, \ldots, p_r^{(0)} < 1$ and $0 < p_1 < 1$, such that

(1) $\Pr(X_1 = 1) = p_1$,

(2) $\Pr(X_n = 1 \mid X_{n-1} = 1, X_{n-2} = x_{n-2}, \ldots, X_1 = x_1) = p_j^{(1)}$, $n \geq 2$,
where $j = m - [\frac{m-1}{k}]k$, m is the smallest positive integer which satisfies $x_{n-m} = 0$, and if $x_{n-1} = x_{n-2} = \cdots = x_1 = 1$, let $m = n$, and

(3) $\Pr(X_n = 1 \mid X_{n-1} = 0, X_{n-2} = x_{n-2}, \ldots, X_1 = x_1) = p_i^{(0)}$, $n \geq 2$,
where $i = m - [\frac{m-1}{r}]r$, m is the smallest positive integer which satisfies $x_{n-m} = 1$, and if $x_{n-1} = \cdots = x_1 = 0$, let $m = n$, and $[x]$ denotes the greatest integer in x.

Remark 2.1.

(1) If $p_1 = p_1^{(1)}$, and $p_1^{(0)} = \cdots = p_r^{(0)} = p_1^{(1)}$, the binary sequence of order (k, r) is a binary sequence of order k with parameters $0 < p_1^{(1)}, p_2^{(1)}, \ldots, p_k^{(1)} < 1$.

(2) If $p_j^{(1)} = p^{(1)}$, $j = 1, \ldots, k$ and $p_i^{(0)} = p^{(0)}$, $i = 1, \ldots, r$, the binary sequence of order (k, r) is a Markov dependent sequence.

(3) A binary sequence of order $(1, 1)$ is a Markov dependent sequence.

(4) A binary sequence of order (k, r) must be a binary sequence of order (ik, jr), $i, j = 1, 2, 3, \ldots$.

Example 2.1. (Urn model) An urn contains a red and b white balls. Let k and r be two fixed positive integers such that $k \leq a$ and $r \leq b$. A ball is drawn at random and laid beside the urn. Suppose its colour is red (resp. white). A new random drawing is made from the urn. If it is the same colour (red (resp. white)), laid the red (white) ball beside the urn. But if it is different colour (white (resp. red)), laid the white (red) ball beside the urn and all red (white) balls outside the urn are replaced. This procedure is repeated, while the number of red (white) balls outside the urn is less than k (resp. r). If the number of red (white) balls outside the urn becomes $k(r)$, then all red (white) balls outside the urn are replaced and the above procedure is continued again. A binary sequence of order (k, r) is defined by recording 0 or 1 for each random drawing according to whether it is a white ball or a red ball. It is easy to see that $p_1 = \frac{a}{a+b}$, $p_{i+1}^{(1)} = \frac{a-i}{a+b-i}$, $(i = 1, \dots, k-1)$, $p_1^{(1)} = \frac{a}{a+b}$, and $p_{j+1}^{(0)} = \frac{a}{a+b-j}$, $(j = 1, \dots, r-1)$, $p_1^{(0)} = \frac{a}{a+b}$.

3. The sooner and later waiting time distributions

Let W_S and W_L be the waiting time random variables for a run of $k(\geq 2)$ successes or a run of $r(\geq 2)$ failures whichever comes sooner and later respectively, in a binary sequence of order (k, r). The outcomes "1" and "0" are usually called "success" and "failure", respectively.

We introduce some notations: $q_1 = 1 - p_1$, $q_i^{(0)} = 1 - p_i^{(0)}$, $(i = 1, 2, \dots, r)$, $q_i^{(1)} = 1 - p_i^{(1)}$, $(i = 1, 2, \dots, k)$, and

$$A(i) = \begin{cases} \dfrac{q_1}{p_1} & \text{if} \quad i = 0 \\ q_2^{(1)} & \text{if} \quad i = 1 \\ \left(\prod_{s=2}^{i} p_s^{(1)}\right) q_{i+1}^{(1)} & \text{if} \quad i = 2, 3, \dots, k-1 \\ \left(\prod_{s=2}^{k} p_s^{(1)}\right) q_1^{(1)} & \text{if} \quad i = k \end{cases}, \tag{3.1}$$

$$B(i) = \begin{cases} \dfrac{p_1}{q_1} & \text{if} \quad i = 0 \\ p_2^{(0)} & \text{if} \quad i = 1 \\ \left(\prod_{s=2}^{i} q_s^{(0)}\right) p_{i+1}^{(0)} & \text{if} \quad i = 2, 3, \dots, r-1 \\ \left(\prod_{s=2}^{r} q_s^{(0)}\right) p_1^{(0)} & \text{if} \quad i = r \end{cases}. \tag{3.2}$$

We can get the following theorems for the sooner waiting time W_S. Since the methods are similar to the methods of Antzoulakos and Philippou (1996), we omit the proofs in the paper. These proofs are given in Han and Aki (1997).

THEOREM 3.1. *Let W_S be a random variable denoting the waiting time until the first occurrence of a success run of length k or a failure run of length r in the binary sequence $\{X_n\}_{n=1}^{\infty}$ of order (k, r). For $w \geq \min\{k, r\}$, we have*

$$\begin{aligned} \Pr(W_S = w) = &\sum_{n=0}^{k-1} p_1 A(n) \sum_{1} \frac{(\sum_i \sum_j x_{ij})!}{\prod_i \prod_j x_{ij}!} \prod_{i=1}^{r-1} \prod_{j=1}^{k-1} (B(i)A(j))^{x_{ij}} \left(\frac{B(r)}{p_1^{(0)}}\right) \\ &+ \sum_{n=0}^{r-1} q_1 B(n) \sum_{2} \frac{(\sum_i \sum_j y_{ij})!}{\prod_i \prod_j y_{ij}!} \sum_{i=1}^{k-1} \sum_{j=1}^{r-1} (A(i)B(j))^{y_{ij}} \left(\frac{A(k)}{q_1^{(1)}}\right), \end{aligned}$$

where the inner summations $\sum_1$ *and* $\sum_2$ *are taken over all nonnegative integers* x_{ij} *and* y_{ij} *satisfying the conditions* $n+\sum_{i=1}^{r-1}\sum_{j=1}^{k-1}(i+j)x_{ij}+r=w$ *and* $n+\sum_{i=1}^{k-1}\sum_{j=1}^{r-1}(i+j)y_{ij}+k=w$, *respectively.*

THEOREM 3.2. *Let* W_S *be as in Theorem* 3.1 *and denote by* $G_S(t)$ *its probability generating function. We have*

$$G_S(t)=\left\{\left(\frac{B(r)t^r}{p_1^{(0)}t}\right)\sum_{n=0}^{k-1}(p_1tA(n)t^n)+\left(\frac{A(k)t^k}{q_1^{(1)}t}\right)\sum_{n=0}^{r-1}(q_1tB(n)t^n)\right\}$$
$$\times\left[1-\left(\sum_{i=1}^{r-1}B(i)t^i\right)\left(\sum_{j=1}^{k-1}A(j)t^j\right)\right]^{-1}.$$

Remark 3.1. By Remark 2.1, if $p_1^{(0)}=\cdots=p_r^{(0)}=p_1^{(1)}=p_1$ holds, Theorems 3.1 and 3.2 reduces to Theorems 2.1 and 2.2 of Antzoulakos and Philippou (1996). In a Markov dependent sequence, these results are similar to the results of Aki and Hirano (1993).

Similarly, the pobability function and the probability generating function of the later waiting time W_L may be easily established following the methodology employed by Antzoulakos and Philippou (1996) (see Han and Aki (1997)).

The above-mentioned results can be extended to a binary sequence of order (g,h). We consider the sooner and later waiting time problems for a success-runs of length k and a failure-runs of length r in a binary sequence of order (g,h).

We introduce

$$A^*(i)=\begin{cases}\dfrac{q_1}{p_1} & \text{i=0}\\[2ex] \dfrac{1}{p_1^{(1)}}(p_1^{(1)}\cdots p_g^{(1)})^{[i/g]}\left(\prod_{s=1}^{\{i/g\}g}p_s^{(1)}\right)q_{\{i/g\}g+1}^{(1)} & i\ge 1\end{cases}\tag{3.3}$$

and

$$B^*(i)=\begin{cases}\dfrac{p_1}{q_1} & \text{i=0}\\[2ex] \dfrac{1}{q_1^{(0)}}(q_1^{(0)}\cdots q_h^{(0)})^{[i/h]}\left(\prod_{s=1}^{\{i/h\}h}q_s^{(0)}\right)p_{\{i/h\}h+1}^{(0)} & i\ge 1\end{cases},\tag{3.4}$$

where $\{x\}=x-[x]$, and $\prod_{s=1}^{0}f(s)=1$.

For the sooner waiting time problem, because the contributions of $\overbrace{11\cdots1}^{n}$ $(0\le n\le k-1)$ and $\overbrace{00\cdots0}^{i}\overbrace{11\cdots1}^{j}$ $(1\le i\le r-1, 1\le j\le k-1)$ are $p_1A^*(n)$ and $B^*(i)A^*(j)$, respectively, we can obtain similar results of Theorems 3.1 and 3.2. It is only replacing $A(i)$, $B(i)$, $p_1^{(0)}$ and $q_1^{(1)}$ in Theorems 3.1 and 3.2, by $A^*(i)$, $B^*(i)$, $p_{\{r/h\}h+1}^{(0)}$ and $q_{\{k/g\}g+1}^{(1)}$, respectively.

For the later waiting time problem, we have

THEOREM 3.3. *Let W_L be a random variable denoting the waiting time until the occurrence of a success run of length k and a failure run of length r in the binary sequence $\{X_n\}_{n=1}^{\infty}$ of order (g,h). Then for $w \geq k+r$, we have*

$$\begin{aligned}\Pr(W_L = w) = &\sum_{n=0}^{\infty} p_1 A^\star(n) \sum_5 \frac{(\sum_i \sum_j x_{ij})!}{\prod_i \prod_j x_{ij}!} \prod_{i=1}^{r-1} \prod_{j=1}^{\infty} (B^\star(i)A^\star(j))^{x_{ij}} \\ &\times \left(\frac{B^\star(r)}{p^{(0)}_{\{r/h\}h+1}} \right) + \sum_{n=0}^{\infty} q_1 B^\star(n) \sum_6 \frac{(\sum_i \sum_j y_{ij})!}{\prod_i \prod_j y_{ij}!} \\ &\times \prod_{i=1}^{k-1} \prod_{j=1}^{\infty} (A^\star(i)B^\star(j))^{y_{ij}} \left(\frac{A^\star(k)}{q^{(1)}_{\{k/g\}g+1}} \right),\end{aligned}$$

where the inner summation $\sum_5$ is taken over all nonnegative integers x_{ij} satisfying the condition $n + \sum_{i=1}^{r-1} \sum_{j=1}^{\infty} (i+j)x_{ij} + r = w$, and for at least one (i,j), $j \geq k$ either $x_{ij} \geq 1$ or $n \geq k$, and the summation $\sum_6$ is taken over all nonnegative integers y_{ij} satisfying the condition $n + \sum_{i=1}^{k-1} \sum_{j=1}^{\infty} (i+j)y_{ij} + k = w$, and for at least one (i,j), $j \geq r$ either $y_{ij} \geq 1$ or $n \geq r$.

Since the proof of this Theorem is similar to Antzoulakos and Philippou (1996), we omit the proof here.

THEOREM 3.4. *Let W_L be as in Theorem 3.3 and denote by $G_L(t)$ its probability generating function. We have*

$$\begin{aligned}G_L(t) = &\left(\frac{B^\star(r)t^r}{p^{(0)}_{\{r/h\}h+1} t} \right) \sum_{n=0}^{\infty} (p_1 t A^\star(n) t^n) \left[1 - \left(\sum_{i=1}^{r-1} B^\star(i) t^i \right) \left(\sum_{j=1}^{\infty} A^\star(j) t^j \right) \right]^{-1} \\ &- \left(\frac{B^\star(r)t^r}{p^{(0)}_{\{r/h\}h+1} t} \right) \sum_{n=0}^{k-1} (p_1 t A^\star(n) t^n) \left[1 - \left(\sum_{i=1}^{r-1} B^\star(i) t^i \right) \left(\sum_{j=1}^{k-1} A^\star(j) t^j \right) \right]^{-1} \\ &+ \left(\frac{A^\star(k)t^k}{q^{(1)}_{\{k/g\}g+1} t} \right) \sum_{n=0}^{\infty} (q_1 t B^\star(n) t^n) \left[1 - \left(\sum_{i=1}^{\infty} B^\star(i) t^i \right) \left(\sum_{j=1}^{k-1} A^\star(j) t^j \right) \right]^{-1} \\ &- \left(\frac{A^\star(k)t^k}{q^{(1)}_{\{k/g\}g+1} t} \right) \sum_{n=0}^{r-1} (q_1 t B^\star(n) t^n) \left[1 - \left(\sum_{i=1}^{r-1} B^\star(i) t^i \right) \left(\sum_{j=1}^{k-1} A^\star(j) t^j \right) \right]^{-1}.\end{aligned}$$

PROOF. Let $W_L^{(0)}$ ($W_L^{(1)}$) be a random variable denoting the waiting time until a failure (success) run of length r (k) occurs later than a success (failure) run of length k (r) in the binary sequence of order (g,h), and let $G_L^{(0)}(t)$ and $G_L^{(1)}(t)$ be their probability generating functions, respectively. Then

$$G_L(t) = G_L^{(0)}(t) + G_L^{(1)}(t). \tag{3.5}$$

We have

$$
\begin{aligned}
G_L^{(0)}(t) &= \sum_{w=k+r}^{\infty} \Pr(W_L^{(0)} = w)t^w \\
&= \sum_{w=k+r}^{\infty} \left\{ \sum_{n=0}^{\infty} \left[(p_1 t A^\star(n) t^n) \sum_5 \frac{(\sum_i \sum_j x_{ij})!}{\prod_i \prod_j x_{ij}!} \right.\right. \\
&\qquad \left.\left. \times \prod_{i=1}^{r-1} \prod_{j=1}^{\infty} (B^\star(i) t^i A^\star(j) t^j)^{x_{ij}} \left(\frac{B^\star(r) t^r}{p_{\{r/h\}h+1}^{(0)} t} \right) \right] \right\} \\
&= \left(\frac{B^\star(r) t^r}{p_{\{r/h\}h+1}^{(0)} t} \right) \sum_{n=0}^{\infty} (p_1 t A^\star(n) t^n) \sum_{w=0}^{\infty} \sum_{*} \frac{(\sum_i \sum_j x_{ij})!}{\prod_i \prod_j x_{ij}!} \prod_{i=1}^{r-1} \prod_{j=1}^{\infty} (B^\star(i) t^i A^\star(j) t^j)^{x_{ij}},
\end{aligned}
$$

where the inner summation $\sum_*$ is taken over all nonnegative integers x_{ij} satisfying the condition $n + \sum_{i=1}^{r-1} \sum_{j=1}^{\infty} (i+j) x_{ij} = w + k$, and for at least one (i,j), $j \geq k$ either $x_{ij} \geq 1$ or $n \geq k$, $1 \leq i \leq r-1$ and $1 \leq j$.

Throughout the present proof, x_{ij}, $i = 1, \ldots, j = 1, \ldots$ denote nonnegative integers.

We denote

$$
D_n = \left\{ x_{ij} \,\middle|\, \begin{array}{l} n + \sum_{i=1}^{r-1} \sum_{j=1}^{\infty} (i+j) x_{ij} = w + k, \\ \text{and at least one}\,(i,j),\ j \geq k,\ x_{ij} \geq 1; \end{array} w \geq 1 \right\}, \tag{3.6}
$$

$$(n = 0, 1, \ldots, k-1),$$

$$
C_n = \left\{ x_{ij} \,\middle|\, n + \sum_{i=1}^{r-1} \sum_{j=1}^{\infty} (i+j) x_{ij} = w + k, w \geq 0 \right\}, \qquad (n = k, \ldots). \tag{3.7}
$$

We have

$$
\sum_{n=0}^{\infty} \sum_{w=0}^{\infty} \sum_{*} = \sum_{n=0}^{k-1} \sum_{D_n} + \sum_{n=k}^{\infty} \sum_{C_n}. \tag{3.8}
$$

Let

$$
D = \left\{ x_{ij} \,\middle|\, \begin{array}{l} \sum_{i=1}^{r-1} \sum_{j=1}^{\infty} x_{ij} = w, \\ \text{and at least one}\,(i,j),\ j \geq k,\ x_{ij} \geq 1; \end{array} w \geq 1 \right\}, \tag{3.9}
$$

$$
C = \left\{ x_{ij} \,\middle|\, \sum_{i=1}^{r-1} \sum_{j=1}^{\infty} x_{ij} = w, w \geq 0 \right\}. \tag{3.10}
$$

We have $D_n = D$, $(n = 0, 1, \ldots, k-1)$, and $C_n = C$, $(n = k, \ldots)$.

Hence, we have

$$
\sum_{n=0}^{\infty} \sum_{w=0}^{\infty} \sum_{*} = \sum_{n=0}^{k-1} \sum_{D} + \sum_{n=k}^{\infty} \sum_{C}. \tag{3.11}
$$

Noting

$$D = \left\{ x_{ij} \;\middle|\; \sum_{i=1}^{r-1}\sum_{j=1}^{\infty} x_{ij} = w, \text{and} \sum_{i=1}^{r-1}\sum_{j=1}^{k-1} x_{ij} < w, w \geq 1 \right\}, \tag{3.12}$$

we have

$$\begin{aligned}
&\sum_{D} \frac{(\sum_i \sum_j x_{ij})!}{\prod_i \prod_j x_{ij}!} \prod_{i=1}^{r-1}\prod_{j=1}^{\infty} (B^\star(i)t^i A^\star(j)t^j)^{x_{ij}} \\
&= \sum_{w=1}^{\infty} \left(\sum_{E_1} - \sum_{E_2} \right) \frac{(\sum_i \sum_j x_{ij})!}{\prod_i \prod_j x_{ij}!} \prod_{i=1}^{r-1}\prod_{j=1}^{\infty} (B^\star(i)t^i A^\star(j)t^j)^{x_{ij}} \\
&= \sum_{w=0}^{\infty} \left[\left(\sum_{i=1}^{r-1}\sum_{j=1}^{\infty} B^\star(i)t^i A^\star(j)t^j \right)^w - \left(\sum_{i=1}^{r-1}\sum_{j=1}^{k-1} B^\star(i)t^i A^\star(j)t^j \right)^w \right],
\end{aligned}$$

where $E_1 = \{x_{ij} \mid \sum_{i=1}^{r-1}\sum_{j=1}^{\infty} x_{ij} = w\}$ and $E_2 = \{x_{ij} \mid \sum_{i=1}^{r-1}\sum_{j=1}^{k-1} x_{ij} = w\}$, for $w \geq 0$, and

$$\sum_{C} \frac{(\sum_i \sum_j x_{ij})!}{\prod_i \prod_j x_{ij}!} \prod_{i=1}^{r-1}\prod_{j=1}^{\infty} (B^\star(i)t^i A^\star(j)t^j)^{x_{ij}} = \sum_{w=0}^{\infty} \left(\sum_{i=1}^{r-1}\sum_{j=1}^{\infty} B^\star(i)t^i A^\star(j)t^j \right)^w.$$

Hence, we obtain

$$\begin{aligned}
G_L^{(0)}(t) = &\left(\frac{B^\star(r)t^r}{p_{\{r/h\}h+1}t} \right) \sum_{n=0}^{\infty} (p_1 t A^\star(n)t^n) \left[1 - \left(\sum_{i=1}^{r-1} B^\star(i)t^i \right) \left(\sum_{j=1}^{\infty} A^\star(j)t^j \right) \right]^{-1} \\
&- \left(\frac{B^\star(r)t^r}{p_{\{r/h\}h+1}t} \right) \sum_{n=0}^{k-1} (p_1 t A^\star(n)t^n) \left[1 - \left(\sum_{i=1}^{r-1} B^\star(i)t^i \right) \left(\sum_{j=1}^{k-1} A^\star(j)t^j \right) \right]^{-1}.
\end{aligned}$$

A similar argument leads to $G_L^{(1)}(t)$, which completes the proof of the Theorem.

Further, we have

$$\sum_{j=1}^{\infty} A^\star(j)t^j = \frac{\sum_{j=1}^{g} A^\star(j)t^j}{1 - p_1^{(1)} \cdots p_g^{(1)} t^g}. \tag{3.13}$$

For the probability functions of the sooner and later waiting times, Theorems 3.1 and 3.3 may be rather complex. However, Theorems 3.2 and 3.4 are suitable for numerical and symbolic calculations by computer algebra systems.

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