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SIMULTANEOUS ESTIMATION OF SEVERAL STRATUM MEANS UNDER ERROR-IN-VARIABLES SUPERPOPULATION MODELS

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Abstract. This paper considers simultaneous estimation of means from several strata under error-in-variables superpopulation models. Necessary and sufficient conditions for an estimator to be admissible in the class of linear estimators are obtained.

Key words and phrases: Error-in-variables superpopulation models, mean vector of strata, admissibility, linear estimator.

1. Introduction

In large scale survey sampling, it is often necessary to consider the estimation of means from several strata (cf., Ghosh and Lahiri (1987)). A common method of estimating a vector of stratum means is to consider a two step procedure whereby the investigator estimates the mean of every stratum *separately* in the first stage, and combines the estimates in the second stage to form the vector of estimates of stratum means. Of more practical interest and relevance is the *simultaneous* estimation of the means from several strata. This latter problem was considered by Ghosh and Lahiri (1987, 1988) and Chattopadhyay and Datta (1994). These authors considered empirical and hierarchical Bayes estimators of mean vectors of strata. Chattopadhyay and Datta (1994) also extended the analysis to error-in-variables superpopulation models. They assumed, however, that the reliability ratio is known in their analysis. In this paper, we relax this latter assumption, and assume instead, that the reliability ratio can be either known or unknown. For each case, we discuss the admissibility of the estimators for estimating simultaneously a vector of stratum means under error-in-variables models. Further, we obtain admissible estimators of the means in the class of linear estimators.

2. Model framework and main results

Suppose that a finite population U consists of L strata, and the size of i-th stratum is $N_i, i = 1, \ldots, L$. Let Y_{ij} be the characteristic value of the j-th unit in the i-th stratum, $\bar{Y}_i$ be the population mean of the i-th stratum and $\bar{Y} = (\bar{Y}_1, \ldots, \bar{Y}_L)'$. Let s_i be a sample of size $n_i (< N_i)$ from the i-th stratum and denote $f_i = n_i/N_i$, $i = 1, \ldots, L$. Following Chattopadhyay and Datta (1994), we assume that Y_{ij} is observed with errors as y_{ij}, and

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the error-in-variables superpopulation models are written as,

$$\begin{cases} \theta_i = \mu + a_i, & E(a_i) = 0, \quad E(a_i^2) = \dfrac{1}{r\lambda}, \\ Y_{ij} = \theta_i + e_{ij}, & E(e_{ij}) = 0, \quad E(e_{ij}^2) = \dfrac{1}{r}, \\ y_{ij} = Y_{ij} + v_{ij}, & E(v_{ij}) = 0, \quad E(v_{ij}^2) = \dfrac{1}{r\delta}, \end{cases} \tag{2.1}$$

where $i = 1, \ldots, L$, $j = 1, \ldots, N_i$, θ_i is the superpopulation mean of the i-th stratum, μ is the overall population mean, μ, r and λ are unknown parameters and δ is the reliability ratio. Furthermore, $\{a_i\}$, $\{e_{ij}\}$ and $\{v_{ij}\}$ are mutually independent. Values of δ for characteristics such as age, unemployment and per capita income are given in Fuller (1987). For the case of $L = 1$, the above model has been considered by Bolfarine (1991), Mukhopadhyay (1994), Zou and Liang (1997), among others.

Chattopadhyay and Datta (1994) gave hierarchical and empirical Bayes estimators of a vector of stratum means, $\bar{Y}$, by assuming that δ is known and $a_i \sim N(0, 1/(r\lambda))$, $e_{ij} \sim N(0, 1/r)$ and $v_{ij} \sim N(0, 1/(r\delta))$. In the subsequent discussion, we assume, instead, that δ can be either known or unknown. On the basis of the squared error loss function $\|T - \bar{Y}\|^2$, where T is any estimator of $\bar{Y}$, we give all admissible linear estimators of $\bar{Y}$ in the class of linear estimators $\mathcal{L}$, where

$$\mathcal{L} \triangleq \left\{ T = (T_1, \ldots, T_L)' : T_i = \sum_{j \in s_i} \omega_{js_i} y_{ij} + \omega_{0s_i}, \omega_{js_i}, \omega_{0s_i} \in R^1 \right\}.$$

THEOREM 1. *Assuming that δ is known, a necessary and sufficient condition for the estimator $T = (\sum_{j\in s_1} \omega_{js_1} y_{1j} + \omega_{0s_1}, \ldots, \sum_{j \in s_L} \omega_{js_L} y_{Lj} + \omega_{0s_L})'$ to be admissible in the class of linear estimators $\mathcal{L}$ is that there exist α and a_i satisfying* (a) $\frac{f_i}{1+\delta^{-1}} \le a_i \le 1$ *and* (b) $\frac{1}{n_i}(a_i - \frac{f_i}{1+\delta^{-1}})(a_j - 1) = \frac{1}{n_j}(a_j - \frac{f_j}{1+\delta^{-1}})(a_i - 1)$ *for $1 \le i \ne j \le L$, such that*
(i) $\omega_{js_i} = a_i/n_i$ $(\forall j \in s_i)$, *and* (ii) $\omega_{0s_i} = (a_i - 1)\alpha$,
where $i = 1, \ldots, L$.

Remark 1. It is readily seen from Theorem 1 that an admissible linear estimator in the class $\mathcal{L}$ must belong to one of the following three types:

(i) $T_0 = (\frac{f_1}{1+\delta^{-1}}\bar{y}_1 + \frac{f_1-(1+\delta^{-1})}{1+\delta^{-1}}\alpha, \ldots, \frac{f_L}{1+\delta^{-1}}\bar{y}_L + \frac{f_L-(1+\delta^{-1})}{1+\delta^{-1}}\alpha)'$, $\alpha \in R^1$;
(ii) $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_L)'$; and
(iii) $\hat{T} = (\frac{tn_1+f_1}{tn_1+(1+\delta^{-1})}\bar{y}_1 + \frac{f_1-(1+\delta^{-1})}{tn_1+(1+\delta^{-1})}\alpha, \ldots, \frac{tn_L+f_L}{tn_L+(1+\delta^{-1})}\bar{y}_L + \frac{f_L-(1+\delta^{-1})}{tn_L+(1+\delta^{-1})}\alpha)'$, $t > 0$.

Remark 2. Interestingly, the estimator $\hat{T}$ given in Remark 1 (iii) includes the estimator (3.12) of Chattopadhyay and Datta (1994). In fact, if $\alpha = -\mu$ and $t = 1/\lambda_0$, then the i-th component of the estimator $\hat{T}$ is exactly the same as the estimator (3.12) of Chattopadhyay and Datta (1994).

Remark 3. Remark 1 states that $\bar{y}$ is an admissible estimator in the class $\mathcal{L}$. It can be proved that $\bar{y}$ is a unique optimal estimator in the class of model unbiased linear estimators. Now, a necessary and sufficient condition for the estimator $T = (\sum_{j\in s_1} \omega_{js_1} y_{1j} + \omega_{0s_1}, \ldots, \sum_{j\in s_L} \omega_{js_L} y_{Lj} + \omega_{0s_L})'$ to be model unbiased is that for $i = 1, \ldots, L$,

$$\sum_{j \in s_i} \omega_{js_i} = 1 \quad \text{and} \quad \omega_{0s_i} = 0. \tag{2.2}$$

Making use of (2.2), it is obvious that $\bar{y}$ is model unbiased. Hence,

$$(2.3)\quad E\|T-\bar{Y}\|^2 = \sum_{i=1}^{L}\left\{\left[\sum_{j\in s_i}\left(\omega_{js_i}-\frac{1}{N_i}\right)^2+\frac{N_i-n_i}{N_i^2}\right]\frac{1}{r}+\sum_{j\in s_i}\omega_{js_i}^2\cdot\frac{1}{r\delta}\right\}$$

$$=\sum_{i=1}^{L}\left\{\left[\sum_{j\in s_i}\left(\left(\omega_{js_i}-\frac{1}{n_i}\right)^2+\left(\frac{1}{n_i}-\frac{1}{N_i}\right)^2\right)+\frac{N_i-n_i}{N_i^2}\right]\frac{1}{r}\right.$$

$$\left.+\sum_{j\in s_i}\left(\left(\omega_{js_i}-\frac{1}{n_i}\right)^2+\left(\frac{1}{n_i}\right)^2\right)\frac{1}{r\delta}\right\}$$

$$\geq\sum_{i=1}^{L}\left\{\left[\sum_{j\in s_i}\left(\frac{1}{n_i}-\frac{1}{N_i}\right)^2+\frac{N_i-n_i}{N_i^2}\right]\frac{1}{r}+\sum_{j\in s_i}\left(\frac{1}{n_i}\right)^2\frac{1}{r\delta}\right\}$$

$$=E\|\bar{y}-\bar{Y}\|^2,$$

and the equality of (2.3) holds if and only if $\omega_{js_i}=1/n_i$ ($j\in s_i$, $i=1,\ldots,L$). This shows that $\bar{y}$ is unique optimal in the class of model unbiased linear estimators. Obviously, $\bar{y}$ cannot be optimal in the class of general linear estimators $\mathcal{L}$.

Remark 4. If we let $A=\operatorname{diag}(a_1,\ldots,a_L)$, $d=(d_1,\ldots,d_L)'$, where a_i ($i=1,\ldots,L$) satisfies conditions (a) and (b) of Theorem 1, and $d_i=(a_i-1)\alpha(\alpha\in R^1)$, then the estimator $A\bar{y}+d$ is admissible in the class of the estimators of the form $A^*\bar{y}+d^*$, where A^* is an arbitrary $L\times L$ matrix and d^* is a $L\times 1$ vector (proof available upon request from the authors). Essentially, this result implies that if the estimator T is admissible in the class $\mathcal{L}$, then it is also admissible when the information of other strata is "borrowed" for estimating the mean of a particular stratum.

THEOREM 2. *If δ is unknown, then a necessary and sufficient condition for the estimator $T=(\sum_{j\in s_1}\omega_{js_1}y_{1j}+\omega_{0s_1},\ldots,\sum_{j\in s_L}\omega_{js_L}y_{Lj}+\omega_{0s_L})'$ to be admissible in the class of linear estimators $\mathcal{L}$ is that there exist α and a_i satisfying* (a) $\eta f_i\leq a_i\leq 1$ *and* (b) $\frac{1}{n_i}(a_i-\eta f_i)(a_j-1)=\frac{1}{n_j}(a_j-\eta f_j)(a_i-1)$ $(1\leq i\neq j\leq L)$ *for some $\eta\in[0,1]$, such that*

(i) $\omega_{js_i}=a_i/n_i$ $(\forall j\in s_i)$, *and* (ii) $\omega_{0s_i}=(a_i-1)\alpha$,

where $i=1,\ldots,L$.

Remark 5. It can be seen from Theorem 2 that for the case of an unknown δ, an admissible estimator in the class $\mathcal{L}$ must belong to one of the following three types:

(i) $T_0=(\eta f_1\bar{y}_1+(\eta f_1-1)\alpha,\ldots,\eta f_L\bar{y}_L+(\eta f_L-1)\alpha)'$, $\alpha\in R^1$;

(ii) $\bar{y}=(\bar{y}_1,\ldots,\bar{y}_L)'$;

(iii) $\hat{T}=(\frac{tn_1+\eta f_1}{tn_1+1}\bar{y}_1+\frac{\eta f_1-1}{tn_1+1}\alpha,\ldots,\frac{tn_L+\eta f_L}{tn_L+1}\bar{y}_L+\frac{\eta f_L-1}{tn_L+1}\alpha)'$, $t>0$, where $\eta\in[0,1]$.

Remark 6. Since the proof given in Remark 3 does not require the assumption that δ is known, it follows immediately that when δ is unknown, the estimator $\bar{y}$ is still unique optimal in the class of model unbiased linear estimators.

Remark 7. The statement of Remark 4 applies also to the case where δ is unknown. In other words, if an estimator satisfies Theorem 2, then it is also admissible if the information of other strata is "borrowed" for estimating the mean of a particular stratum (proof available upon request from the authors).

3. Proof of Theorem 1

In order to prove Theorems 1 and 2, we require the following supporting lemma, the proof of which is straightforward and is omitted.

LEMMA 1. *The risk of the estimator* $T = (\sum_{j\in s_1}\omega_{js_1}y_{1j} + \omega_{0s_1},\ldots,\sum_{j\in s_L}\omega_{js_L}y_{Lj}+\omega_{0s_L})'$ *is*

$$E\|T-\bar{Y}\|^2=\sum_{i=1}^{L}\left\{\left[\sum_{j\in s_i}\left(\omega_{js_i}-\frac{1}{N_i}\right)^2+\frac{N_i-n_i}{N_i^2}\right]\frac{1}{r}+\sum_{j\in s_i}\omega_{js_i}^2\cdot\frac{1}{r\delta}+\left(\sum_{j\in s_i}\omega_{js_i}-1\right)^2\frac{1}{r\lambda}+\left[\left(\sum_{j\in s_i}\omega_{js_i}-1\right)\mu+\omega_{0s_i}\right]^2\right\}. \tag{3.1}$$

The following gives the steps for proving the necessity part of Theorem 1:

(1^0) *If T is admissible, then there exists a_i $(i=1,\ldots,L)$ such that $\omega_{js_i}=a_i/n_i$* $(\forall j\in s_i,\ i=1,\ldots,L)$. Suppose, instead, that the latter condition is not satisfied, then there exists i_0 such that $\omega_{js_{i_0}}$ $(j\in s_{i_0})$ is not a constant. Define

$$\omega'_{js_{i_0}}=\begin{cases}\dfrac{1}{n_{i_0}}\displaystyle\sum_{j\in s_{i_0}}\omega_{js_{i_0}}, & j\in s_{i_0},\\ \omega_{0s_{i_0}}, & j=0,\end{cases} \tag{3.2}$$

and

$$\omega'_{js_i}=\omega_{js_i},\qquad j\in s_i\quad\text{or}\quad j=0,\ i\neq i_0. \tag{3.3}$$

Using Lemma 1, it is easily shown that the corresponding estimator $T_1 = (\sum_{j\in s_1}\omega'_{js_1}y_{1j}+\omega'_{0s_1},\ldots,\sum_{j\in s_L}\omega'_{js_L}y_{Lj}+\omega'_{0s_L})'$ is superior to T, which contradicts the admissibility of T.

Thus, using Lemma 1, the risk of the estimator T can be expressed as

$$E\|T-\bar{Y}\|^2=\sum_{i=1}^{L}\left\{\left[n_i\left(\frac{a_i}{n_i}-\frac{1}{N_i}\right)^2+\frac{N_i-n_i}{N_i^2}\right]\frac{1}{r}+\frac{a_i^2}{n_i}\cdot\frac{1}{r\delta}+(a_i-1)^2\frac{1}{r\lambda}+[(a_i-1)\mu+\omega_{0s_i}]^2\right\}. \tag{3.4}$$

(2^0) To show the necessity of the condition $\frac{f_i}{1+\delta^{-1}}\le a_i\le 1$, $i=1,\ldots,L$, suppose there exists a_1 such that $a_1<\frac{f_1}{1+\delta^{-1}}$. Further, we define

$$a_i''=\begin{cases}\dfrac{f_1}{1+\delta^{-1}}, & i=1,\\ a_i, & i\ge 2,\end{cases} \tag{3.5}$$

and

$$\omega''_{0s_i}=\begin{cases}\dfrac{a_1''-1}{a_1-1}\omega_{0s_1}, & i=1,\\ \omega_{0s_i}, & i\ge 2.\end{cases} \tag{3.6}$$

It follows from the definition of a_1'' that $(a_1''-1)^2<(a_1-1)^2$, and

$$n_1\left(\frac{a_1''}{n_1}-\frac{1}{N_1}\right)^2+\frac{a_1''^2}{n_1}\cdot\frac{1}{\delta}<n_1\left(\frac{a_1}{n_1}-\frac{1}{N_1}\right)^2+\frac{a_1^2}{n_1}\cdot\frac{1}{\delta}.$$

Using (3.4), we can see that the corresponding estimator $T_2=(a_1''\bar{y}_1+\omega_{0s_1}'',\ldots,a_L''\bar{y}_L+\omega_{0s_L}'')'$ is superior to T, which contradicts the admissibility of T.

Further, if there exists $a_1>1$, then by using (3.4), it is readily seen that the estimator $T_3=(\bar{y}_1,a_2\bar{y}_2+\omega_{0s_2},\ldots,a_L\bar{y}_L+\omega_{0s_L})'$ has smaller risk than T, which contradicts the admissibility of T.

(3^0) Next, we show that the condition that there exists α such that $\omega_{0s_i}=(a_i-1)\alpha$ for $i=1,\ldots,L$ is necessary for the admissibility of T. First note that if $a_i=1$, then $\omega_{0s_i}=0$. We assume, without loss of generality, that $a_1=1$. If $\omega_{0s_1}\neq 0$, then it is readily seen from (3.4) that the estimator $T_4=(\bar{y}_1,a_2\bar{y}_2+\omega_{0s_2},\ldots,a_L\bar{y}_L+\omega_{0s_L})'$ is superior to T, which contradicts the admissibility of T. This indicates that if q, the number of a_i $(i=1,\ldots,L)$ such that $a_i\neq 1$, is less than or equal to 1, then there must exist α such that $\omega_{0s_i}=(a_i-1)\alpha$ for $i=1,\ldots,L$. Suppose $q\geq 2$, and without loss of generality, we assume that $a_i\neq 1$, $i=1,\ldots,q$, and $a_{q+1}=\cdots=a_L=1$ $(q\leq L)$. It can be shown that there must exist α such that $\omega_{0s_i}=(a_i-1)\alpha$ for $i=1,\ldots,q$. Otherwise, there exist i_1 and i_2 $(1\leq i_1\neq i_2\leq q)$ such that $\omega_{0s_{i_1}}/(a_{i_1}-1)\neq\omega_{0s_{i_2}}/(a_{i_2}-1)$. Without loss of generality, we assume $\omega_{0s_1}/(a_1-1)\neq\omega_{0s_2}/(a_2-1)$. Denote

$$\omega_{0s_i}'''=\begin{cases}\dfrac{(a_1-1)^2\omega_{0s_1}+(a_1-1)(a_2-1)\omega_{0s_2}}{(a_1-1)^2+(a_2-1)^2}, & i=1;\\[2ex] \dfrac{(a_2-1)(a_1-1)\omega_{0s_1}+(a_2-1)^2\omega_{0s_2}}{(a_1-1)^2+(a_2-1)^2}, & i=2;\\[2ex] \omega_{0s_i}, & i\geq 3.\end{cases}\tag{3.7}$$

Now, the estimator $T_5=(a_1\bar{y}_1+\omega_{0s_1}''',\ldots,a_L\bar{y}_L+\omega_{0s_L}''')'$ is superior to T, since from (3.4) and Jensen's inequality, we have

$$\begin{aligned}\Delta &\triangleq E\|T_5-\bar{Y}\|^2-E\|T-\bar{Y}\|^2\\ &=\{[(a_1-1)\mu+\omega_{0s_1}''']^2+[(a_2-1)\mu+\omega_{0s_2}''']^2\}\\ &\quad-\{[(a_1-1)\mu+\omega_{0s_1}]^2+[(a_2-1)\mu+\omega_{0s_2}]^2\}\\ &=[(a_1-1)^2+(a_2-1)^2]\Bigg[\frac{(a_1-1)^2}{(a_1-1)^2+(a_2-1)^2}\left(\mu+\frac{\omega_{0s_1}}{a_1-1}\right)\\ &\qquad+\frac{(a_2-1)^2}{(a_1-1)^2+(a_2-1)^2}\left(\mu+\frac{\omega_{0s_2}}{a_2-1}\right)\Bigg]^2\\ &\quad-\{[(a_1-1)\mu+\omega_{0s_1}]^2+[(a_2-1)\mu+\omega_{0s_2}]^2\}\\ &<[(a_1-1)^2+(a_2-1)^2]\Bigg[\frac{(a_1-1)^2}{(a_1-1)^2+(a_2-1)^2}\left(\mu+\frac{\omega_{0s_1}}{a_1-1}\right)^2\\ &\qquad+\frac{(a_2-1)^2}{(a_1-1)^2+(a_2-1)^2}\left(\mu+\frac{\omega_{0s_2}}{a_2-1}\right)^2\Bigg]\\ &\quad-\{[(a_1-1)\mu+\omega_{0s_1}]^2+[(a_2-1)\mu+\omega_{0s_2}]^2\}\\ &=0,\end{aligned}\tag{3.8}$$

which contradicts the admissibility of T.

(4^0) Finally, we prove that a_i $(i = 1, \ldots, L)$ must satisfy condition (b). Otherwise, there exist i_0 and j_0 where $i_0 \neq j_0$, say $i_0 = 1$ and $j_0 = 2$, such that

$$\frac{1}{n_1}\left(a_1 - \frac{f_1}{1+\delta^{-1}}\right)(a_2 - 1) \neq \frac{1}{n_2}\left(a_2 - \frac{f_2}{1+\delta^{-1}}\right)(a_1 - 1). \tag{3.9}$$

Without loss of generality, we assume

$$\frac{1}{n_1}\left(a_1 - \frac{f_1}{1+\delta^{-1}}\right)(a_2 - 1) > \frac{1}{n_2}\left(a_2 - \frac{f_2}{1+\delta^{-1}}\right)(a_1 - 1), \tag{3.10}$$

which implies $a_1 < 1$.

Let k and ε be constants such that

$$\frac{a_2 - 1}{a_1 - 1} < k < \frac{a_2 - f_2/(1+\delta^{-1})}{n_2} \Big/ \frac{a_1 - f_1/(1+\delta^{-1})}{n_1}, \qquad \left(\frac{c}{0} \hat{=} +\infty, c > 0\right),$$

and

$$0 < \varepsilon < 2 \cdot \min\left\{ \frac{\dfrac{a_2 - f_2/(1+\delta^{-1})}{n_2} - \dfrac{k[a_1 - f_1/(1+\delta^{-1})]}{n_1}}{\dfrac{k^2}{n_1} + \dfrac{1}{n_2}}, \frac{(a_2 - 1) - k(a_1 - 1)}{k^2 + 1} \right\}.$$

Denote

$$\begin{cases} a_1^* = a_1 + k\varepsilon, \\ a_2^* = a_2 - \varepsilon. \end{cases} \tag{3.11}$$

Then

$$\begin{cases} \left[n_1\left(\dfrac{a_1^*}{n_1} - \dfrac{1}{N_1}\right)^2 + \dfrac{a_1^{*2}}{n_1} \cdot \dfrac{1}{\delta}\right] + \left[n_2\left(\dfrac{a_2^*}{n_2} - \dfrac{1}{N_2}\right)^2 + \dfrac{a_2^{*2}}{n_2} \cdot \dfrac{1}{\delta}\right] \\ \qquad < \left[n_1\left(\dfrac{a_1}{n_1} - \dfrac{1}{N_1}\right)^2 + \dfrac{a_1^2}{n_1} \cdot \dfrac{1}{\delta}\right] + \left[n_2\left(\dfrac{a_2}{n_2} - \dfrac{1}{N_2}\right)^2 + \dfrac{a_2^2}{n_2} \cdot \dfrac{1}{\delta}\right]; \\ (a_1^* - 1)^2 + (a_2^* - 1)^2 < (a_1 - 1)^2 + (a_2 - 1)^2. \end{cases} \tag{3.12}$$

Using (3.4) and (3.12), we can see that the estimator $T^* = (a_1^*\bar{y}_1 + (a_1^* - 1)\alpha, a_2^*\bar{y}_2 + (a_2^* - 1)\alpha, a_3\bar{y}_3 + (a_3 - 1)\alpha, \ldots, a_L\bar{y}_L + (a_L - 1)\alpha)'$ is superior to T, which contradicts the admissibility of T.

The sufficiency part of Theorem 1 can be established in a manner similar to the proof of sufficiency of Theorem 2 and is omitted.

4. Proof of Theorem 2

It follows directly from the proof of Theorem 1 that there exist constants α and a_i $(i = 1, \ldots, L)$ such that conditions (i) and (ii) of Theorem 2 hold. Now we prove that it is necessary that a_i satisfies conditions (a) and (b) for T to be admissible in the class of linear estimators $\mathcal{L}$.

(1^0) First note that for all $i = 1, \ldots, L$, $0 \le a_i \le 1$. In fact, it is easily seen from the proof of Theorem 1 that $a_i \le 1$. If there exists $a_1 < 0$, then by writing

$$a_i' = \begin{cases} 0, & i = 1, \\ a_i, & i \ge 2, \end{cases} \tag{4.1}$$

and

$$\omega_{0s_i}' = \begin{cases} \dfrac{\omega_{0s_1}}{1 - a_1}, & i = 1, \\ \omega_{0s_i}, & i \ge 2, \end{cases} \tag{4.2}$$

and using Lemma 1 or (3.4), we can see that the corresponding estimator $T' = (a_1'\bar{y}_1 + \omega_{0s_1}', \ldots, a_L'\bar{y}_L + \omega_{0s_L}')'$ is superior to the estimator T, which contradicts the admissibility of T.

(2^0) Next, it can be shown that for all $i \ne j$,

$$\left[\frac{a_i}{n_i}(a_j - 1) - \frac{a_j}{n_j}(a_i - 1)\right]\left[\frac{a_i - f_i}{n_i}(a_j - 1) - \frac{a_j - f_j}{n_j}(a_i - 1)\right] \le 0. \tag{4.3}$$

Otherwise, there must exist $i_1 \ne j_1$, say $i_1 = 1$, $j_1 = 2$, such that

$$\left[\frac{a_1}{n_1}(a_2 - 1) - \frac{a_2}{n_2}(a_1 - 1)\right]\left[\frac{a_1 - f_1}{n_1}(a_2 - 1) - \frac{a_2 - f_2}{n_2}(a_1 - 1)\right] > 0. \tag{4.4}$$

Without loss of generality, we assume $\frac{a_1}{n_1}(a_2 - 1) > \frac{a_2}{n_2}(a_1 - 1)$, so that $a_1 < 1$, and

$$\frac{a_1 - f_1}{n_1}(a_2 - 1) > \frac{a_2 - f_2}{n_2}(a_1 - 1). \tag{4.5}$$

Let k be a constant such that

$$\frac{a_2 - 1}{a_1 - 1} < k < \min\left\{\frac{a_2/n_2}{a_1/n_1}, \frac{(a_2 - f_2)/n_2}{(a_1 - f_1)/n_1}\right\},$$

when $a_1 > f_1$; and

$$\frac{a_2 - 1}{a_1 - 1} < k < \frac{a_2/n_2}{a_1/n_1},$$

when $a_1 \le f_1$.

Note that when $a_1 < f_1$, we have

$$\frac{(a_2 - f_2)/n_2}{(a_1 - f_1)/n_1} < \frac{a_2 - 1}{a_1 - 1}.$$

From (4.5), we have $a_2 > f_2$ when $a_1 = f_1$, so there exists ε such that

$$0 < \varepsilon < 2 \cdot \min\left\{\frac{\dfrac{a_2}{n_2} - \dfrac{ka_1}{n_1}}{\dfrac{k^2}{n_1} + \dfrac{1}{n_2}}, \frac{\dfrac{a_2 - f_2}{n_2} - \dfrac{k(a_1 - f_1)}{n_1}}{\dfrac{k^2}{n_1} + \dfrac{1}{n_2}}, \frac{(a_2 - 1) - k(a_1 - 1)}{k^2 + 1}\right\}.$$

Denote

$$\begin{cases} a_1^* = a_1 + k\varepsilon, \\ a_2^* = a_2 - \varepsilon. \end{cases}$$

Then

$$
(4.6)\quad \begin{cases} n_1\left(\frac{a_1^*}{n_1}-\frac{1}{N_1}\right)^2+n_2\left(\frac{a_2^*}{n_2}-\frac{1}{N_2}\right)^2 < n_1\left(\frac{a_1}{n_1}-\frac{1}{N_1}\right)^2+n_2\left(\frac{a_2}{n_2}-\frac{1}{N_2}\right)^2, \\ \frac{a_1^{*2}}{n_1}+\frac{a_2^{*2}}{n_2} < \frac{a_1^2}{n_1}+\frac{a_2^2}{n_2}, \\ (a_1^*-1)^2+(a_2^*-1)^2 < (a_1-1)^2+(a_2-1)^2. \end{cases}
$$

Using (3.4) and (4.6), it is obvious that the estimator $T^* = (a_1^*\bar{y}_1 + (a_1^* - 1)\alpha, a_2^*\bar{y}_2 + (a_2^* - 1)\alpha, a_3\bar{y}_3 + (a_3 - 1)\alpha, \ldots, a_L\bar{y}_L + (a_L - 1)\alpha)'$ is better than T, which contradicts the admissibility of T.

(3^0) Now we prove that a_i must satisfy both conditions (a) and (b). Otherwise, for each $\eta \in [0,1]$, either (a) or (b) is not satisfied. In particular, for $\eta = 0$, conditions (a) and (b) cannot hold simultaneously. Since $0 \le a_i \le 1$, so for $\eta = 0$, condition (a) is satisfied, which means that condition (b) cannot hold. That is, there exist i_0 and j_0 where $i_0 \neq j_0$, such that

$$
(4.7)\qquad \frac{a_{i_0}}{n_{i_0}}(a_{j_0}-1) \neq \frac{a_{j_0}}{n_{j_0}}(a_{i_0}-1).
$$

It can be seen from (4.3) that $a_i < 1$ for all $i = 1, \ldots, L$. In fact, if there exists i_1 such that $a_{i_1} = 1$, then it is readily seen from (4.3) that $a_i = 1$ for all $i = 1, \ldots, L$. In this case, both conditions (a) and (b) hold for all $\eta \in [0,1]$, which contradicts the above assumption. Making use of this result and (4.7), it is obvious that $\frac{a_i}{n_i(a_i-1)}$ $(i = 1, \ldots, L)$ are not all equal. So, without loss of generality, we can assume

$$
(4.8)\qquad \frac{a_1}{n_1(a_1-1)} = \cdots = \frac{a_{i-1}}{n_{i-1}(a_{i-1}-1)} > \frac{a_i}{n_i(a_i-1)} \ge \frac{a_{i+1}}{n_{i+1}(a_{i+1}-1)} \ge \cdots \ge \frac{a_L}{n_L(a_L-1)},
$$

and

$$
(4.9)\qquad \frac{a_1-f_1}{n_1(a_1-1)} = \min_{1\le j\le i-1}\left\{\frac{a_j-f_j}{n_j(a_j-1)}\right\}.
$$

Then from (4.3), we have

$$
(4.10)\qquad \frac{a_1-f_1}{n_1(a_1-1)} \le \frac{a_i-f_i}{n_i(a_i-1)}.
$$

In the subsequent analysis, we assume $L \ge 3$. The case of $L = 2$ is relatively simple and is omitted.

It can be shown that there must exist $l \neq (1, i)$ such that

$$
(4.11)\qquad \left[\frac{a_1}{n_1(a_1-1)} - \frac{a_i}{n_i(a_i-1)}\right]\left[\frac{a_i-f_i}{n_i(a_i-1)} - \frac{a_l-f_l}{n_l(a_l-1)}\right] \neq \left[\frac{a_i}{n_i(a_i-1)} - \frac{a_l}{n_l(a_l-1)}\right]\left[\frac{a_1-f_1}{n_1(a_1-1)} - \frac{a_i-f_i}{n_i(a_i-1)}\right].
$$

If (4.11) is not satisfied, then it is easy to see that for all $l \neq (1, i)$ and $\eta \in [0,1]$,

$$
(4.12)\qquad \left[\frac{a_1}{n_1(a_1-1)} - \frac{a_i}{n_i(a_i-1)}\right]\left[\frac{a_i-\eta f_i}{n_i(a_i-1)} - \frac{a_l-\eta f_l}{n_l(a_l-1)}\right] = \left[\frac{a_i}{n_i(a_i-1)} - \frac{a_l}{n_l(a_l-1)}\right]\left[\frac{a_1-\eta f_1}{n_1(a_1-1)} - \frac{a_i-\eta f_i}{n_i(a_i-1)}\right].
$$

Let us denote

$$g(\eta) \hat{=} \frac{a_1 - \eta f_1}{n_1(a_1 - 1)} - \frac{a_i - \eta f_i}{n_i(a_i - 1)}.$$

From (4.8) and (4.10), we have $g(0) > 0$ and $g(1) \leq 0$. If $g(1) < 0$, then there exists $\eta_0 \in (0,1)$, such that $g(\eta_0) = 0$. If $g(1) = 0$ and we let $\eta_0 = 1$, then we have $g(\eta_0) = 0$. In other words, there must exist $\eta_0 \in (0,1]$, such that $g(\eta_0) = 0$. Making use of this result and (4.12), we know that for all $l \neq (1,i)$,

$$\frac{a_i - \eta_0 f_i}{n_i(a_i - 1)} = \frac{a_l - \eta_0 f_l}{n_l(a_l - 1)}. \tag{4.13}$$

Note that $g(\eta_0) = 0$, hence for all $i \neq j$,

$$\frac{a_i - \eta_0 f_i}{n_i(a_i - 1)} = \frac{a_j - \eta_0 f_j}{n_j(a_j - 1)}. \tag{4.14}$$

This shows that for $\eta = \eta_0$, condition (b) is satisfied. Therefore, condition (a) cannot hold for $\eta = \eta_0$. That is, there exists i_0 such that $a_{i_0} < \eta_0 f_{i_0}$, which, when used in conjunction with (4.14), leads to $a_j < \eta_0 f_j$ for all $j = 1, \ldots, L$, which implies the inadmissibility of the estimator T. In fact, from (4.8), we see that $a_i \neq 0$ and

$$\frac{a_1/n_1}{a_i/n_i} < \frac{a_1 - 1}{a_i - 1}.$$

Making use of this result and (4.14), it can be seen that

$$\frac{a_1/n_1}{a_i/n_i} < \frac{(a_1 - \eta_0 f_1)/n_1}{(a_i - \eta_0 f_i)/n_i}. \tag{4.15}$$

Therefore,

$$\frac{a_1/n_1}{a_i/n_i} < \frac{(a_1 - f_1)/n_1}{(a_i - f_i)/n_i}. \tag{4.16}$$

Hence, there exists k such that

$$\frac{a_1/n_1}{a_i/n_i} < k < \min\left\{\frac{a_1 - 1}{a_i - 1}, \frac{(a_1 - f_1)/n_1}{(a_i - f_i)/n_i}\right\}. \tag{4.17}$$

Now let ε be a constant which satisfies

$$0 < \varepsilon < 2 \cdot \min\left\{\frac{\frac{ka_i}{n_i} - \frac{a_1}{n_1}}{\frac{1}{n_1} + \frac{k^2}{n_i}}, \frac{\frac{k(a_i - f_i)}{n_i} - \frac{a_1 - f_1}{n_1}}{\frac{1}{n_1} + \frac{k^2}{n_i}}, \frac{k(a_i - 1) - (a_1 - 1)}{1 + k^2}\right\}, \tag{4.18}$$

and define

$$\begin{cases} a_1^* = a_1 + \varepsilon, \\ a_i^* = a_i - k\varepsilon. \end{cases}$$

Then equation (4.6) still holds if the subscript 2 is replaced by i everywhere in the equation. This shows that the estimator T is inadmissible if (4.11) is not satisfied.

In what follows we use (4.11) to prove that T is not an admissible estimator, which contradicts the admissibility of T. We consider the following three cases.

Case 1. Assume that

$$\left[\frac{a_i}{n_i(a_i-1)}-\frac{a_l}{n_l(a_l-1)}\right]\left[\frac{a_1-f_1}{n_1(a_1-1)}-\frac{a_i-f_i}{n_i(a_i-1)}\right]<0.$$

It is obvious that

$$\frac{a_1-f_1}{n_1(a_1-1)}<\frac{a_i-f_i}{n_i(a_i-1)}. \tag{4.19}$$

So, we have

$$\frac{a_i}{n_i(a_i-1)}>\frac{a_l}{n_l(a_l-1)}. \tag{4.20}$$

Hence, $a_l \neq 0$ and from (4.3),

$$\frac{a_i-f_i}{n_i(a_i-1)}\leq\frac{a_l-f_l}{n_l(a_l-1)}. \tag{4.21}$$

From (4.21) and (4.19), we see that $a_1=f_1$ and $a_l=f_l$ cannot hold simultaneously.

Denote

$$A \triangleq \frac{\frac{a_i}{n_i}(a_l-1)-\frac{a_l}{n_l}(a_i-1)}{\frac{a_1}{n_1}(a_i-1)-\frac{a_i}{n_i}(a_1-1)}>0,$$

and

$$B \triangleq \frac{\frac{a_i-f_i}{n_i}(a_l-1)-\frac{a_l-f_l}{n_l}(a_i-1)}{\frac{a_1-f_1}{n_1}(a_i-1)-\frac{a_i-f_i}{n_i}(a_1-1)}\geq 0.$$

1) If

$$\begin{aligned}&\left[\frac{a_1}{n_1(a_1-1)}-\frac{a_i}{n_i(a_i-1)}\right]\left[\frac{a_i-f_i}{n_i(a_i-1)}-\frac{a_l-f_l}{n_l(a_l-1)}\right]\\&\quad<\left[\frac{a_i}{n_i(a_i-1)}-\frac{a_l}{n_l(a_l-1)}\right]\left[\frac{a_1-f_1}{n_1(a_1-1)}-\frac{a_i-f_i}{n_i(a_i-1)}\right],\end{aligned} \tag{4.22}$$

then $A<B$. Let $\delta>0$, such that $A<\delta<B$, and $\delta\cdot\frac{a_1-f_1}{n_1}+\frac{a_l-f_l}{n_l}\neq 0$. Further, let k be a constant such that

$$\frac{a_i-1}{\delta(a_1-1)+(a_l-1)}<k<\min\left\{\frac{\frac{a_i}{n_i}}{\frac{\delta a_1}{n_1}+\frac{a_l}{n_l}},\frac{\frac{a_i-f_i}{n_i}}{\frac{\delta(a_1-f_1)}{n_1}+\frac{a_l-f_l}{n_l}}\right\}, \tag{4.23}$$

when $\delta\cdot\frac{a_1-f_1}{n_1}+\frac{a_l-f_l}{n_l}>0$; or

$$\frac{a_i-1}{\delta(a_1-1)+(a_l-1)}<k<\frac{a_i/n_i}{\delta a_1/n_1+a_l/n_l}, \tag{4.24}$$

when $\delta \cdot \frac{a_1-f_1}{n_1} + \frac{a_l-f_l}{n_l} < 0$.

Note that when $\delta \cdot \frac{a_1-f_1}{n_1} + \frac{a_l-f_l}{n_l} < 0$,

$$\frac{\frac{a_i-f_i}{n_i}}{\frac{\delta(a_1-f_1)}{n_1}+\frac{a_l-f_l}{n_l}} < \frac{a_i-1}{\delta(a_1-1)+(a_l-1)}. \tag{4.25}$$

Thus, there exists ε such that

$$0<\varepsilon<2\cdot\min\left\{\frac{-\frac{\delta k(a_1-f_1)}{n_1}+\frac{a_i-f_i}{n_i}-\frac{k(a_l-f_l)}{n_l}}{\frac{\delta^2k^2}{n_1}+\frac{1}{n_i}+\frac{k^2}{n_l}}, \frac{-\frac{\delta k a_1}{n_1}+\frac{a_i}{n_i}-\frac{k a_l}{n_l}}{\frac{\delta^2k^2}{n_1}+\frac{1}{n_i}+\frac{k^2}{n_l}}, \frac{-\delta k(a_1-1)+(a_i-1)-k(a_l-1)}{\delta^2k^2+1+k^2}\right\}. \tag{4.26}$$

Let's define

$$\begin{cases} a_1^* = a_1+\delta k\varepsilon, \\ a_i^* = a_i-\varepsilon, \\ a_l^* = a_l+k\varepsilon. \end{cases} \tag{4.27}$$

Then we have

$$\begin{cases} n_1\left(\frac{a_1^*}{n_1}-\frac{1}{N_1}\right)^2+n_i\left(\frac{a_i^*}{n_i}-\frac{1}{N_i}\right)^2+n_l\left(\frac{a_l^*}{n_l}-\frac{1}{N_l}\right)^2 \\ \quad < n_1\left(\frac{a_1}{n_1}-\frac{1}{N_1}\right)^2+n_i\left(\frac{a_i}{n_i}-\frac{1}{N_i}\right)^2+n_l\left(\frac{a_l}{n_l}-\frac{1}{N_l}\right)^2, \\ \frac{a_1^{*2}}{n_1}+\frac{a_i^{*2}}{n_i}+\frac{a_l^{*2}}{n_l} < \frac{a_1^2}{n_1}+\frac{a_i^2}{n_i}+\frac{a_l^2}{n_l}, \\ (a_1^*-1)^2+(a_i^*-1)^2+(a_l^*-1)^2 < (a_1-1)^2+(a_i-1)^2+(a_l-1)^2. \end{cases} \tag{4.28}$$

Further, define

$$a_j^{**} = \begin{cases} a_1^*, & j=1, \\ a_i^*, & j=i, \\ a_l^*, & j=l, \\ a_j, & j\neq 1,i,l. \end{cases} \tag{4.29}$$

Then the estimator $T^{**} = (a_1^{**}\bar{y}_1+(a_1^{**}-1)\alpha, \ldots, a_L^{**}\bar{y}_L+(a_L^{**}-1)\alpha)'$ is superior to the estimator T.

2) When

$$\left[\frac{a_1}{n_1(a_1-1)}-\frac{a_i}{n_i(a_i-1)}\right]\left[\frac{a_i-f_i}{n_i(a_i-1)}-\frac{a_l-f_l}{n_l(a_l-1)}\right] > \left[\frac{a_i}{n_i(a_i-1)}-\frac{a_l}{n_l(a_l-1)}\right]\left[\frac{a_1-f_1}{n_1(a_1-1)}-\frac{a_i-f_i}{n_i(a_i-1)}\right], \tag{4.30}$$

we have $B < A$. Let $\delta > 0$, such that $B < \delta < A$, and $\delta \cdot \frac{a_1 - f_1}{n_1} + \frac{a_l - f_l}{n_l} \neq 0$. Further, let k be a constant such that

$$\max\left\{\frac{\frac{a_i}{n_i}}{\frac{\delta a_1}{n_1} + \frac{a_l}{n_l}}, \frac{\frac{a_i - f_i}{n_i}}{\frac{\delta(a_1 - f_1)}{n_1} + \frac{a_l - f_l}{n_l}}\right\} < k < \frac{a_i - 1}{\delta(a_1 - 1) + (a_l - 1)}, \tag{4.31}$$

when $\delta \cdot \frac{a_1 - f_1}{n_1} + \frac{a_l - f_l}{n_l} > 0$; or

$$\frac{a_i/n_i}{\delta a_1/n_1 + a_l/n_l} < k < \frac{a_i - 1}{\delta(a_1 - 1) + (a_l - 1)}, \tag{4.32}$$

when $\delta \cdot \frac{a_1 - f_1}{n_1} + \frac{a_l - f_l}{n_l} < 0$.

Similar to 1), there exists ε such that

$$0 < \varepsilon < 2 \cdot \min\left\{\frac{\frac{\delta k(a_1 - f_1)}{n_1} - \frac{a_i - f_i}{n_i} + \frac{k(a_l - f_l)}{n_l}}{\frac{\delta^2 k^2}{n_1} + \frac{1}{n_i} + \frac{k^2}{n_l}}, \frac{\frac{\delta k a_1}{n_1} - \frac{a_i}{n_i} + \frac{k a_l}{n_l}}{\frac{\delta^2 k^2}{n_1} + \frac{1}{n_i} + \frac{k^2}{n_l}}, \frac{\delta k(a_1 - 1) - (a_i - 1) + k(a_l - 1)}{\delta^2 k^2 + 1 + k^2}\right\}. \tag{4.33}$$

Define

$$\begin{cases} a_1^* = a_1 - \delta k \varepsilon, \\ a_i^* = a_i + \varepsilon, \\ a_l^* = a_l - k\varepsilon. \end{cases} \tag{4.34}$$

As in 1), we can see that the corresponding estimator T^{**} is superior to the estimator T.

Case 2. Assume that

$$\left[\frac{a_i}{n_i(a_i - 1)} - \frac{a_l}{n_l(a_l - 1)}\right]\left[\frac{a_1 - f_1}{n_1(a_1 - 1)} - \frac{a_i - f_i}{n_i(a_i - 1)}\right] = 0.$$

1) If $\frac{a_1 - f_1}{n_1(a_1 - 1)} = \frac{a_i - f_i}{n_i(a_i - 1)}$, then we must have $l \geq i + 1$ in (4.11). Otherwise, $2 \leq l \leq i - 1$. It is easy to see from (4.8) that

$$\frac{a_l}{n_l(a_l - 1)} > \frac{a_i}{n_i(a_i - 1)}. \tag{4.35}$$

From (4.3), we have

$$\frac{a_l - f_l}{n_l(a_l - 1)} \leq \frac{a_i - f_i}{n_i(a_i - 1)}. \tag{4.36}$$

Thus, from (4.9), we get

$$\frac{a_l - f_l}{n_l(a_l-1)} = \frac{a_i - f_i}{n_i(a_i-1)}, \tag{4.37}$$

which shows that both sides of (4.11) are equal to zero. This is, of course, impossible. Thus, given that $l \geq i+1$, it is easily seen that a_l is not equal to zero.

If

$$\left[\frac{a_1}{n_1(a_1-1)} - \frac{a_i}{n_i(a_i-1)}\right]\left[\frac{a_i - f_i}{n_i(a_i-1)} - \frac{a_l - f_l}{n_l(a_l-1)}\right] > 0, \tag{4.38}$$

then we have

$$\frac{a_i - f_i}{n_i(a_i-1)} > \frac{a_l - f_l}{n_l(a_l-1)}, \tag{4.39}$$

which implies

$$\frac{a_i}{n_i(a_i-1)} \leq \frac{a_l}{n_l(a_l-1)}. \tag{4.40}$$

So from (4.8), we obtain

$$\frac{a_i}{n_i(a_i-1)} = \frac{a_l}{n_l(a_l-1)}. \tag{4.41}$$

Thus, from (4.39) and the above assumption, we get

$$\left[\frac{a_1 - f_1}{n_1(a_1-1)} - \frac{a_l - f_l}{n_l(a_l-1)}\right]\left[\frac{a_1}{n_1(a_1-1)} - \frac{a_l}{n_l(a_l-1)}\right] > 0, \tag{4.42}$$

which contradicts (4.3). Therefore, we must have

$$\left[\frac{a_1}{n_1(a_1-1)} - \frac{a_i}{n_i(a_i-1)}\right]\left[\frac{a_i - f_i}{n_i(a_i-1)} - \frac{a_l - f_l}{n_l(a_l-1)}\right] < 0. \tag{4.43}$$

Hence,

$$\frac{a_i - f_i}{n_i(a_i-1)} < \frac{a_l - f_l}{n_l(a_l-1)}, \tag{4.44}$$

which implies that $a_1 = f_1$ and $a_l = f_l$ cannot hold simultaneously.

Note that $l \geq i+1$. So, $A \geq 0$. Let $\delta > A$ such that $\delta \cdot \frac{a_1 - f_1}{n_1} + \frac{a_l - f_l}{n_l} \neq 0$. Further, let k and ε be constants and define a_1^*, a_i^*, a_l^* and T^{**} as in part 1) of Case 1. It then be seen that the corresponding estimator T^{**} has smaller risk than T.

2) If $\frac{a_1 - f_1}{n_1(a_1-1)} < \frac{a_i - f_i}{n_i(a_i-1)}$, then we have $\frac{a_i}{n_i(a_i-1)} = \frac{a_l}{n_l(a_l-1)}$.

When

$$\left[\frac{a_1}{n_1(a_1-1)} - \frac{a_i}{n_i(a_i-1)}\right]\left[\frac{a_i - f_i}{n_i(a_i-1)} - \frac{a_l - f_l}{n_l(a_l-1)}\right] < 0, \tag{4.45}$$

we have

$$\frac{a_i - f_i}{n_i(a_i-1)} < \frac{a_l - f_l}{n_l(a_l-1)}, \tag{4.46}$$

which implies that $a_1 = f_1$ and $a_l = f_l$ cannot hold simultaneously.

Let δ be a constant such that $0 < \delta < B$, and $\delta \cdot \frac{a_1 - f_1}{n_1} + \frac{a_l - f_l}{n_l} \neq 0$. As in part 1) of Case 1, we can see that the estimator T is inadmissible.

When

$$\left[\frac{a_1}{n_1(a_1-1)} - \frac{a_i}{n_i(a_i-1)}\right]\left[\frac{a_i-f_i}{n_i(a_i-1)} - \frac{a_l-f_l}{n_l(a_l-1)}\right] > 0, \tag{4.47}$$

we have

$$\frac{a_i - f_i}{n_i(a_i-1)} > \frac{a_l - f_l}{n_l(a_l-1)}. \tag{4.48}$$

Clearly, $a_1 = f_1$ and $a_i = f_i$ cannot hold simultaneously. In addition, it is obvious that $a_i \neq 0$.

Now let δ be a constant such that ($\frac{c}{0} \hat{=} +\infty$, where $c > 0$)

$$0 < \delta < \frac{\frac{a_i - f_i}{n_i}(a_l - 1) - \frac{a_l - f_l}{n_l}(a_i - 1)}{\frac{a_l - f_l}{n_l}(a_1 - 1) - \frac{a_1 - f_1}{n_1}(a_l - 1)}, \tag{4.49}$$

and $\delta \cdot \frac{a_1 - f_1}{n_1} + \frac{a_i - f_i}{n_i} \neq 0$.

Let k and ε be constants and define a_1^*, a_i^*, a_l^* and T^{**} as in part 1) of Case 1 except that i and l are replaced by l and i respectively. It then follows that the estimator T^{**} is superior to the estimator T.

Case 3. Assume that

$$\left[\frac{a_i}{n_i(a_i-1)} - \frac{a_l}{n_l(a_l-1)}\right]\left[\frac{a_1-f_1}{n_1(a_1-1)} - \frac{a_i-f_i}{n_i(a_i-1)}\right] > 0.$$

It follows from (4.3) that

$$\frac{a_1 - f_1}{n_1(a_1-1)} < \frac{a_i - f_i}{n_i(a_i-1)}. \tag{4.50}$$

So we have

$$\frac{a_i}{n_i(a_i-1)} < \frac{a_l}{n_l(a_l-1)}, \tag{4.51}$$

and $a_1 = f_1$ and $a_i = f_i$ cannot hold simultaneously.

From (4.3) and (4.8), we get

$$\frac{a_i - f_i}{n_i(a_i-1)} \geq \frac{a_l - f_l}{n_l(a_l-1)}, \tag{4.52}$$

$$\frac{a_1}{n_1(a_1-1)} = \frac{a_l}{n_l(a_l-1)}. \tag{4.53}$$

It follows from (4.11), (4.53) and (4.9) that

$$\frac{a_1 - f_1}{n_1(a_1 - 1)} < \frac{a_l - f_l}{n_l(a_l - 1)}. \tag{4.54}$$

Let δ be a constant such that

$$\delta > \frac{\frac{a_i - f_i}{n_i}(a_l - 1) - \frac{a_l - f_l}{n_l}(a_i - 1)}{\frac{a_l - f_l}{n_l}(a_1 - 1) - \frac{a_1 - f_1}{n_1}(a_l - 1)} (\geq 0), \tag{4.55}$$

and $\delta \cdot \frac{a_1 - f_1}{n_1} + \frac{a_i - f_i}{n_i} \neq 0$.

Now let k and ε be constants and define a_1^*, a_i^*, a_l^* and T^{**} as in part 2) of Case 1 except that i and l are replaced by l and i respectively. It can be shown then that T is not an admissible estimator. This completes the proof for the necessity part of Theorem 2.

The proof given above shows that if a linear estimator, say T^{***}, cannot be written in the form $T^* = (a_1^*\bar{y}_1 + \omega_{0s_1}^*, \ldots, a_L^*\bar{y}_L + \omega_{0s_L}^*)'$, then there must exist an estimator of the form T^*, which is superior to T^{***}. So in order to show the admissibility of the estimator T, it suffices to prove that there exists no estimator of the form T^* which is superior to T. We consider the following three cases.

Case 1. Assume that $0 < \eta < 1$.

(1^0) If there exists i_0 such that $a_{i_0} = \eta f_{i_0}$, then for all $i = 1, \ldots, L$, $a_i = \eta f_i$. Suppose that

$$E\|T^* - \bar{Y}\|^2 \leq E\|T - \bar{Y}\|^2,$$

which is equivalent to

$$\begin{aligned} \sum_{i=1}^{L} \left\{ n_i \left(\frac{a_i^*}{n_i} - \frac{1}{N_i} \right)^2 \cdot \frac{1}{r} + \frac{a_i^{*2}}{n_i} \cdot \frac{1}{r\delta} + (a_i^* - 1)^2 \cdot \frac{1}{r\lambda} + [(a_i^* - 1)\mu + \omega_{0s_i}^*]^2 \right\} \\ \leq \sum_{i=1}^{L} \left\{ n_i \left(\frac{a_i}{n_i} - \frac{1}{N_i} \right)^2 \cdot \frac{1}{r} + \frac{a_i^2}{n_i} \cdot \frac{1}{r\delta} \right. \\ \left. + (a_i - 1)^2 \cdot \frac{1}{r\lambda} + [(a_i - 1)\mu + \omega_{0s_i}]^2 \right\}. \end{aligned} \tag{4.56}$$

Multiplying r on both sides of (4.56), and letting $r \to 0^+$ and $\lambda \to +\infty$, we get

$$\sum_{i=1}^{L} \left[n_i \left(\frac{a_i^*}{n_i} - \frac{1}{N_i} \right)^2 + \frac{a_i^{*2}}{n_i} \cdot \frac{1}{\delta} \right] \leq \sum_{i=1}^{L} \left[n_i \left(\frac{a_i}{n_i} - \frac{1}{N_i} \right)^2 + \frac{a_i^2}{n_i} \cdot \frac{1}{\delta} \right]. \tag{4.57}$$

Letting $\delta^{-1} = (1 - \eta)/\eta$ in (4.57), we obtain

$$\sum_{i=1}^{L} \left[n_i \left(\frac{a_i^*}{n_i} - \frac{1}{N_i} \right)^2 + \frac{a_i^{*2}}{n_i} \cdot \frac{1 - \eta}{\eta} \right] \leq \sum_{i=1}^{L} \frac{n_i}{N_i^2}(1 - \eta). \tag{4.58}$$

That is,

$$\sum_{i=1}^{L} \frac{1}{n_i \eta}(a_i^* - \eta f_i)^2 \le 0. \tag{4.59}$$

Hence, we have $a_i^* = \eta f_i = a_i$, $i = 1, \ldots, L$, and (4.56) can be written as,

$$\sum_{i=1}^{L} [(a_i - 1)\mu + \omega_{0s_i}^*]^2 \le \sum_{i=1}^{L} [(a_i - 1)\mu + \omega_{0s_i}]^2. \tag{4.60}$$

If $\mu = -\alpha$, then from (4.60), we obtain $\omega_{0s_i}^* = (a_i - 1)\alpha = \omega_{0s_i}$, $i = 1, \ldots, L$. This shows that T is admissible.

(2^0) For the case where there exists i_1 such that $a_{i_1} = 1$, the proof can be obtained along the same lines as in part (1^0) and is omitted.

(3^0) Now assume that $\eta f_i < a_i < 1$ for all $i = 1, \ldots, L$. Suppose that $E\|T^* - \bar{Y}\|^2 \le E\|T - \bar{Y}\|^2$, that is, equation (4.56) holds. Multiplying r on both sides of (4.56), and letting $r \to 0^+$, we have

$$\sum_{i=1}^{L} \left[n_i \left(\frac{a_i^*}{n_i} - \frac{1}{N_i} \right)^2 + \frac{a_i^{*2}}{n_i} \cdot \frac{1}{\delta} + (a_i^* - 1)^2 \cdot \frac{1}{\lambda} \right] \le \sum_{i=1}^{L} \left[n_i \left(\frac{a_i}{n_i} - \frac{1}{N_i} \right)^2 + \frac{a_i^2}{n_i} \cdot \frac{1}{\delta} + (a_i - 1)^2 \cdot \frac{1}{\lambda} \right], \tag{4.61}$$

which is equivalent to

$$\sum_{i=1}^{L} (a_i^* - a_i) \left[\left(\frac{a_i^* + a_i}{n_i} - \frac{2}{N_i} \right) + \frac{a_i^* + a_i}{n_i} \cdot \frac{1}{\delta} + (a_i^* + a_i - 2) \cdot \frac{1}{\lambda} \right] \le 0. \tag{4.62}$$

If we write

$$\frac{1}{\delta} = \frac{1 - \eta}{\eta}, \qquad \frac{1}{\lambda} = \frac{a_i - \eta f_i}{\eta n_i (1 - a_i)}, \tag{4.63}$$

then from (4.62), we obtain

$$\sum_{i=1}^{L} (a_i^* - a_i)^2 \cdot \frac{1 - \eta f_i}{\eta n_i (1 - a_i)} \le 0. \tag{4.64}$$

Thus, $a_i^* = a_i$, $i = 1, \ldots, L$. Analogous to (1^0), it can be seen that $\omega_{0s_i}^* = \omega_{0s_i}$, $i = 1, \ldots, L$. This shows that T is an admissible estimator.

Case 2. Assume that $\eta = 0$. We only consider the case where $0 < a_i < 1$, $i = 1, \ldots, L$. Suppose that $E\|T^* - \bar{Y}\|^2 \le E\|T - \bar{Y}\|^2$. That is, equation (4.56) is true. If we let $\lambda = \frac{n_i(1-a_i)}{a_i} \cdot \delta$, and multiply δ on both sides of (4.56) and let $\delta \to 0^+$, then we get

$$\sum_{i=1}^{L} \left[\frac{a_i^{*2}}{n_i} + (a_i^* - 1)^2 \cdot \frac{a_i}{n_i(1 - a_i)} \right] \le \sum_{i=1}^{L} \left[\frac{a_i^2}{n_i} + (a_i - 1)^2 \cdot \frac{a_i}{n_i(1 - a_i)} \right], \tag{4.65}$$

which holds if and only if

$$\sum_{i=1}^{L} \frac{(a_i^* - a_i)^2}{n_i(1-a_i)} \leq 0. \tag{4.66}$$

Therefore, $a_i^* = a_i$, $i = 1, \ldots, L$. So, as in part (1^0) of Case 1, we have $\omega_{0s_i}^* = \omega_{0s_i}$, $i = 1, \ldots, L$. That is, T is an admissible estimator. Similar are the cases for $a_i = 0$ and $a_i = 1$ and the details are omitted.

Case 3. Assume that $\eta = 1$. It can be established in a fashion similar to Case 1 that T is an admissible estimator. The details are available on request from the authors.

This completes the proof of Theorem 2.

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