Ann. Inst. Statist. Math.
Vol. 51, No. 4, 779–802 (1999)

CONSERVED QUANTITIES AND SYMMETRIES RELATED TO STOCHASTIC DYNAMICAL SYSTEMS*

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(Received May 26, 1997; revised March 30, 1998)

Abstract. The present article focuses on the three topics related to the notions of "conserved quantities" and "symmetries" in stochastic dynamical systems described by stochastic differential equations of Stratonovich type. The first topic is concerned with the relation between conserved quantities and symmetries in stochastic Hamilton dynamical systems, which is established in a way analogous to that in the deterministic Hamilton dynamical theory. In contrast with this, the second topic is devoted to investigate the procedures to derive conserved quantities from symmetries of stochastic dynamical systems without using either the Lagrangian or Hamiltonian structure. The results in these topics indicate that the notion of symmetries is useful for finding conserved quantities in various stochastic dynamical systems. As a further important application of symmetries, the third topic treats the similarity method to stochastic dynamical systems. That is, it is shown that the order of a stochastic system can be reduced, if the system admits symmetries. In each topic, some illustrative examples for stochastic dynamical systems and their conserved quantities and symmetries are given.

Key words and phrases: Stochastic dynamical systems, conserved quantities, symmetries, similarity method.

1. Introduction

The theory of conserved quantities (the first integrals) and symmetry (invariant under a transformation) for the dynamical systems described by ordinary differential equations is one of the most important subjects. Indeed, these notions are often useful for finding various conservation laws or solutions to the dynamical systems. Hence, it must be natural to formulate thus notions for stochastic dynamical systems described by stochastic differential equations, since such stochastic systems arise in investigation of a number of problems concerning random-phenomena treated in physics, engineering, economics, and so on. In consideration of these

* This work was supported by Grant-in-Aid for Scientific Research No. 06740172, No. 07740166 and No. 08640300 from the Ministry of Education, Science and Culture.

facts, the author has been proposed a formalism of conserved quantities and symmetries for stochastic systems described by stochastic differential equations of Stratonovich type; the two notions are formulated in a way analogous to that in the dynamical theory for the deterministic systems mentioned above (Misawa (1994*a*, 1994*b*)). These works, however, mainly treat the basic formalisms and several examples for the notions, and hence, they are not sufficient for the advanced studies on these notions, especially the notion of symmetry, as mentioned in the above introductory part.

On account of this, the present article focuses on the following three topics related to the usefulness of the notion of symmetry for analysis of stochastic dynamical systems:

The first topic is concerned with the connection between conserved quantities and symmetries, when the stochastic dynamical system have the "Hamiltonian structure". In the theory of deterministic Hamilton mechanics, the two notions are closely related to each other, and thereby conserved quantities are obtained from symmetries. Therefore, it may be important that such a relation is extended to that in our stochastic dynamical systems with the Hamiltonian structure. We examine it in Section 3 after a review together with a supplement of the author's preceding works which are given in Section 2 for consistency of this article.

In contrast with the first topic, the second topic is devoted to investigate the methods to derive conserved quantities from symmetries in stochastic systems without using either the Lagrangian or the Hamiltonian structure. In the case of deterministic systems, most of conservation laws are derived from symmetries together with either the Lagrangian or, as mentioned above, the Hamiltonian structure. However, we often meet the general stochastic systems which do not always have thus structures, so that it must be useful to formulate the procedure mentioned above. From this point of view, the author formulated such a method (Misawa (1994*b*)). As a continuation of this work, in Section 4, we propose the new procedures which correspond to the modifications of the author's previous result.

The results in the above two topics indicate that the notion of symmetry is useful for the conserved quantities in various stochastic dynamical systems. As a further important application of it, the last topic treats the similarity method to stochastic dynamical systems; it is shown that the order of a stochastic system can be reduced, if the system admits symmetries. Through this procedure, it may be easy to find the geometric structure or the solution to the stochastic dynamical system under study. Section 5 is devoted to this topic.

In each topic, we touch upon some illustrative examples for stochastic dynamical systems together with their conserved quantities and symmetries; for example, a stochastic cyclic Lotka-Volterra system of competing 3-species, the stochastic harmonic oscillator, Maruyama-Itoh's stochastic model related to Fisher-Wright model in population genetics, a stochastic version of the neo-classical optimal growth model in economics suggested by Samuelson, and so on.

Finally, let us stress that one may formulate the notions of conserved quantities and symmetries for stochastic systems in another way (e.g. Itoh (1993), Thieullen and Zambrini (1997)). We also touch upon this together with the other concluding remarks on this article in Section 6. Moreover, we note that a wide class of conservation law of completely integrable dynamical systems is given by using their Lax representations; such a study for stochastic dynamical systems includes in Nakamura (1994).

2. Conserved quantities and symmetries in stochastic dynamical systems

Let $(\Omega, \mathcal{F}, P)$ be a probability space, that is a triple where the sample space Ω is a set of all elementary events, $\mathcal{F}$ is the sigma algebra of the observable random events and P is a probability measure on Ω. In this article, we consider stochastic dynamical systems described by the following n-dimensional vector valued stochastic differential equations (e.g. Ikeda and Watanabe (1989), Arnold (1973)) of Stratonovich type on $(\Omega, \mathcal{F}, P)$, equipped with a non-decreasing family of sigma-algebras $\mathcal{F}_t \subset \mathcal{F}$, $t \geq t_0$:

$$(2.1) \qquad d\boldsymbol{x}_t = \boldsymbol{b}(\boldsymbol{x}_t, t)dt + \sum_{r=1}^{m} \boldsymbol{g}_r(\boldsymbol{x}_t, t) \circ dw_t^r, \qquad \boldsymbol{x}_{t_0} = \boldsymbol{c}, \quad t \in [t_0, T]$$

where $\boldsymbol{b} = (b^i)_{i=1}^n$ and $\boldsymbol{g}_r = (g_r^i)_{i=1}^n$ are n-dimensional smooth functions, respectively, $\boldsymbol{w}_t = (w_t^r)_{r=1}^m$ is an m-dimensional standard Wiener process and $\boldsymbol{c}$ is a deterministic constant n-vector. In what follows, we assume that equation (2.1) together with an initial condition $\boldsymbol{x}_{t_0} = \boldsymbol{c}$ and a time interval $[t_0, T]$ satisfy restrictions allowing the existence and uniqueness of solutions of stochastic differential equation. Note that equation (2.1) is rewritten in the form of the stochastic differential equation of Itô type as follows:

$$(2.1') \qquad d\boldsymbol{x}_t = \left(\boldsymbol{b}(\boldsymbol{x}_t, t) + \frac{1}{2} \sum_{r=1}^{m} \sum_{j=1}^{n} g_r^j \partial_j \boldsymbol{g}_r(\boldsymbol{x}_t, t) \right) dt + \sum_{r=1}^{m} \boldsymbol{g}_r(\boldsymbol{x}_t, t) dw_t^r,$$

where $\partial_j = \partial / \partial x^j$. Hence, through the relation, the results obtained for the system (2.1) in the followings may be applied to the corresponding system of Itô type.

Let I be a smooth function on $R^n \times R^1$, and $\boldsymbol{x}_t$ a diffusion process governed by equation (2.1). According to Misawa (1994*a*), a conserved quantity for (2.1) is defined as follows:

DEFINITION 2.1. We call a function I *a conserved quantity for a stochastic dynamical system* (2.1), if the function satisfies

$$(2.2) \qquad (\partial_t + X_0) I(\boldsymbol{x}, t) = 0, \qquad X_r I(\boldsymbol{x}, t) = 0 \quad (r = 1, 2, \ldots, m),$$

where ∂_t, X_0 and X_r $(r = 1, 2, \ldots, m)$ are differential operators defined by

$$(2.3) \qquad \partial_t = \partial / \partial t, \qquad X_0 = \sum_{i=1}^{n} b^i \partial_i, \qquad X_r = \sum_{i=1}^{n} g_r^i \partial_i.$$

Through the change of variables formula for (2.1) (Ikeda and Watanabe (1989), Arnold (1973)), we derive $dI(\boldsymbol{x}_t, t) = 0$ from (2.2); this means that "$I(\boldsymbol{x}_t, t) =$ constant" on the diffusion process $\boldsymbol{x}_t$ satisfying equation (2.1), and hence the above definition gives a stochastic version of that of conserved quantities in the deterministic dynamical theory.

Remark 2.1. Conversely, using the following lemma together with the change of variables formula mentioned above, we can verify that the equation $dI(\boldsymbol{x}_t, t) = 0$ implies that the equations (2.2) hold almost surely under a mild condition in the lemma (see equation (2.5)). We assume that the stochastic systems treated in this paper satisfy the condition:

LEMMA 2.1. *Suppose that $\boldsymbol{x}_t$ is a solution of* (2.1) *for any initial value* $\boldsymbol{c}$. *If the R^n-valued smooth functions $\boldsymbol{f}_\alpha = (f_\alpha^j)_{j=1}^n$, $(\alpha = 0, 1, \dots, m)$ on $R^n \times [t_0, T]$ satisfy*

$$\boldsymbol{f}_0(\boldsymbol{x}_t, t)dt + \sum_{r=1}^{m} \boldsymbol{f}_r(\boldsymbol{x}_t, t) \circ dw_t^r = \boldsymbol{0} \tag{2.4}$$

under the condition

$$E\left[\int_{t_0}^{T} \left(\sum_{r=1}^{m} |f_r^j(\boldsymbol{x}_t, t)|^2\right) dt\right] < +\infty \qquad (j = 1, \dots, n), \tag{2.5}$$

where $\boldsymbol{0}$ *is the n-dimensional null-vector and* $E[\cdot]$ *denotes the expectation with respect to P, then* $\boldsymbol{f}_\alpha(\boldsymbol{x}, t) = \boldsymbol{0}$ $(\alpha = 0, 1, \dots, m)$ *hold almost surely (a.s.).*

PROOF. Rewriting equation (2.4) in the form of Itô's stochastic integral equation (cf. equation (2.1$'$)), and using the uniqueness of Doob-Meyer's canonical decomposition of semi-martingales (Ikeda and Watanabe (1989)), we can prove that for any $t \in [t_0, T]$,

$$\int_{t_0}^{t} \left(\boldsymbol{f}_0(\boldsymbol{x}_t, t) + \frac{1}{2}\sum_{r=1}^{m}\sum_{k=1}^{n} f_r^k \partial_k \boldsymbol{f}_r(\boldsymbol{x}_t, t)\right) dt = \boldsymbol{K}, \tag{2.6}$$

$$\int_{t_0}^{t} \left(\sum_{r=1}^{m} \boldsymbol{f}_r(\boldsymbol{x}_t, t)\right) dw_t^r = \boldsymbol{0}, \tag{2.7}$$

where $\boldsymbol{K}$ is an n-dimensional constant random variable. From (2.7), we get

$$E\left[\int_{t_0}^{t} \left(\sum_{r=1}^{m} |f_r^j(\boldsymbol{x}_t, t)|^2\right) dt\right] = 0 \qquad (j = 1, \dots, n). \tag{2.8}$$

Hence, $f_r^j(\boldsymbol{x}_t, t) = 0$ holds for any $j(= 1, \dots, n)$ and $r(= 1, \dots, m)$ (a.s.). For one may choose the values of t and the initial value of $\boldsymbol{x}_t$ arbitrarily, the equations prove that $\boldsymbol{f}_r(\boldsymbol{x},t) = \mathbf{0}$ $(r = 1,\ldots,m)$ (a.s.). Because of the same reasons, the equation (2.6) together with this result indicate that $\boldsymbol{f}_0(\boldsymbol{x},t) = \mathbf{0}$ holds (a.s.).

Example 2.1. Let us consider the following stochastic dynamical systems:

$$d\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{pmatrix} x_t^1(x_t^3 - x_t^2) \\ x_t^2(x_t^1 - x_t^3) \\ x_t^3(x_t^2 - x_t^1) \end{pmatrix} dt + \sum_{r=1}^{3} \boldsymbol{g}_r(\boldsymbol{x}_t) \circ dw_t^r, \tag{2.9}$$

where $\boldsymbol{g}_r = (g_r^i)_{i=1}^3$ $(r = 1,2,3)$ are arbitrary three-dimensional smooth functions. This may be regarded as a cyclic Lotka-Volterra system of competing 3-species in a chaotic environment; the complicated interaction between the system and the external world is formally characterized by the random fluctuation $\sum_{r=1}^3 \boldsymbol{g}_r(\boldsymbol{x}_t) \circ dw_t^r$. (Compare this with the equation treated in Itoh (1993).) Suppose that the conserved quantity in the deterministic case, $I_1 = x^1 + x^2 + x^3$ or $I_2 = x^1 \cdot x^2 \cdot x^3$, is also conserved in our stochastic system. Then, Definition 2.1 asserts that $\boldsymbol{g}_r$ should satisfy the following conditions:

$$I_1 : \text{conserved quantity} \quad \text{if and only if} \quad \sum_{i=1}^{3} g_r^i = 0 \qquad (r = 1,2,3), \tag{2.10}$$

$$I_2 : \text{conserved quantity} \quad \text{if and only if} \quad \sum_{k,\ell,m=1;cyclic}^{3} g_r^k x^\ell x^m = 0 \qquad (r = 1,2,3). \tag{2.11}$$

For instance, we choose the function $(\boldsymbol{g}_r)_{r=1}^3 = (g_r^i)_{i,r=1}^3$ as

$$(g_r^i) = \begin{pmatrix} x^1(x^3 - x^2) & x^1(x^3 - x^2) & x^1(x^3 - x^2) \\ x^2(x^1 - x^3) & x^2(x^1 - x^3) & x^2(x^1 - x^3) \\ x^3(x^2 - x^1) & x^3(x^2 - x^1) & x^3(x^2 - x^1) \end{pmatrix} \tag{2.12}$$

or

$$(g_r^i) = \begin{pmatrix} x^1 & x^1 & -2x^1 \\ x^2 & -2x^2 & x^2 \\ -2x^3 & x^3 & x^3 \end{pmatrix}. \tag{2.13}$$

The facts (2.10) and (2.11) indicate that in the case of (2.12), I_1 and I_2 are conserved quantities and that in the case of (2.13), I_2 is a conserved quantity but I_1 is not.

Next, we proceed to the notion of symmetry for (2.1) (Misawa (1994*a*)). Let

$$\boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{x},t) \tag{2.14}$$

be a transformation from $R^n \times R^1$ to R^n. Through equation (2.14), an R^n-valued diffusion process $\boldsymbol{y}_t$ is determined from a solution $\boldsymbol{x}_t$ of equation (2.1) as $\boldsymbol{y}_t = \boldsymbol{\phi}(\boldsymbol{x}_t, t)$. Suppose that this transformation (2.14) satisfies

$$\text{(2.15)} \qquad \boldsymbol{b}(\boldsymbol{\phi}(\boldsymbol{x},t),t) = (\partial_t + X_0)\boldsymbol{\phi}(\boldsymbol{x},t), \qquad \boldsymbol{g}_r(\boldsymbol{\phi}(\boldsymbol{x},t),t) = X_r\boldsymbol{\phi}(\boldsymbol{x},t), \quad (r = 1,2,\ldots,m).$$

By the change of variables formula, we see that $\boldsymbol{y}_t$ is governed by

$$\text{(2.16)} \qquad d\boldsymbol{y}_t = \boldsymbol{b}(\boldsymbol{y}_t,t)dt + \sum_{r=1}^{m} \boldsymbol{g}_r(\boldsymbol{y}_t,t) \circ dw_t^r.$$

This means that a stochastic system (2.1) is invariant under (2.14). Hence, as in the case of deterministic systems, we formulate the notion of symmetry for (2.1) as follows:

DEFINITION 2.2. We call a transformation (2.14) satisfying (2.15) *a symmetry transformation for a stochastic dynamical system* (2.1).

Remark 2.2. Conversely, using Lemma 2.1 and the change of variables formula, we can prove that equation (2.16) derived from (2.1) through (2.14) implies that equations (2.15) hold almost surely.

On the basis of Definition 2.2, the notion of symmetry operators is formulated. Let

$$\text{(2.17)} \qquad Y = \sum_{i=1}^{n} f^i(\boldsymbol{x},t)\partial_i,$$

be a differential operator given by an R^n-valued smooth function $\boldsymbol{f} = (f^i)_{i=1}^n$, and

$$\text{(2.18)} \qquad \boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{x},t;a)$$

be a local one-parameter transformation generated by Y (e.g. Eisenhert (1961)), where a is a parameter on $I = (-a_0, a_0)$ and $\boldsymbol{\phi}(\boldsymbol{x},t;0) = \boldsymbol{x}$. Suppose that $\boldsymbol{\phi}(\boldsymbol{x},t;a)$, $\boldsymbol{b}(\boldsymbol{\phi}(\boldsymbol{x},t;a),t)$ and $\boldsymbol{g}_r(\boldsymbol{\phi}(\boldsymbol{x},t;a),t)$ $(r = 1,\ldots,m)$ are analytic with respect to a on I. Then, we get the following theorem and the definition of a *symmetry operator* (Misawa (1994*a*)):

THEOREM 2.1. *The one-parameter transformation* (2.18) *generated by* (2.17) *is a symmetry transformation of a stochastic system* (2.1), *if and only if the operator* Y *satisfies*

$$\text{(2.19)} \qquad [\partial_t + X_0, Y] = 0, \qquad [X_r, Y] = 0 \quad (r = 1,2,\ldots,m),$$

where $[\cdot,\cdot]$ *be the commutator.*

DEFINITION 2.3. We call a differential operator Y given by equation (2.17) *a symmetry operator* for a stochastic dynamical system (2.1), if it satisfies equations (2.19).

The equations (2.2) and (2.19) prove that the notion of symmetry operators generates new conserved quantities for stochastic dynamical systems (Misawa (1994*a*)):

THEOREM 2.2. *Suppose that* $I = I(\boldsymbol{x}, t)$ *is a conserved quantity and* Y *is a symmetry operator for equation* (2.1). *Then,* $YI(\boldsymbol{x}, t)$ *is also a conserved quantity for the system.*

Example 2.2. We give an illustrative example with respect to the symmetry for stochastic systems. Consider the following non-linear stochastic dynamical systems:

$$d\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{pmatrix} (x_t^1)^2 \\ x_t^1 \cdot x_t^2 \\ x_t^1 \cdot x_t^3 \end{pmatrix} dt + \begin{pmatrix} x_t^1 \cdot x_t^2 \\ (x_t^2)^2 \\ x_t^2 \cdot x_t^3 \end{pmatrix} \circ dw_t. \tag{2.20}$$

In this case, the operators (2.3), X_0 and X_1, are given by

$$\begin{aligned} X_0 &= (x_t^1)^2 \partial_1 + x_t^1 \cdot x_t^2 \partial_2 + x_t^1 \cdot x_t^3 \partial_3, \\ X_1 &= x_t^1 \cdot x_t^2 \partial_1 + (x_t^2)^2 \partial_2 + x_t^2 \cdot x_t^3 \partial_3, \end{aligned} \tag{2.21}$$

respectively. Then, we see that the operator $Y = x^2 \partial_3$ is a symmetry operator for equation (2.20). Moreover, Y generates the one-parameter transformation (2.18) as follows:

$$\phi^1(\boldsymbol{x}; a) = x^1, \qquad \phi^2(\boldsymbol{x}; a) = x^2, \qquad \phi^3(\boldsymbol{x}; a) = ax^2 + x^3. \tag{2.22}$$

Theorem 2.1 asserts that equation (2.22) gives an example of symmetry transformations. Moreover, we find out a conserved quantity $I = x^1/x^3$ for the system (2.20). Then, Theorem 2.2 proves that $YI = -x^1 \cdot x^2/(x^3)^2$ is also a conserved quantity.

3. Conserved quantities and symmetries in stochastic Hamilton systems

In this section, we are concerned with conserved quantities and symmetries defined in Section 2, when stochastic systems have such a "Hamiltonian structure" as that in Bismut (1981). Let us consider the following 2ℓ-dimensional stochastic system:

$$\begin{aligned} d\begin{pmatrix} \boldsymbol{x}_t^i \\ \boldsymbol{x}_t^{\ell+i} \end{pmatrix} &= \begin{pmatrix} \partial_{\ell+i} H_0(\boldsymbol{x}_t, t) \\ -\partial_i H_0(\boldsymbol{x}_t, t) \end{pmatrix} dt \\ &\quad + \sum_{r=1}^{m} \begin{pmatrix} \partial_{\ell+i} H_r(\boldsymbol{x}_t, t) \\ -\partial_i H_r(\boldsymbol{x}_t, t) \end{pmatrix} \circ dw_t^r \qquad (i = 1, 2, \ldots, \ell), \end{aligned} \tag{3.1}$$

where $H_\alpha(\boldsymbol{x},t)$ $(\alpha = 0,1,\dots,m)$ are smooth scalar functions on $R^n \times R^1$, and $\partial_j = \partial/\partial x^j$ $(j = 1,2,\dots,2\ell)$, respectively. Formally, this is rewritten in the form of a Hamilton dynamical system

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{x}_t^i \\ \boldsymbol{x}_t^{\ell+i} \end{pmatrix} = \begin{pmatrix} \partial_{\ell+i}\hat{H}(\boldsymbol{x}_t,t) \\ -\partial_i \hat{H}(\boldsymbol{x}_t,t), \end{pmatrix} \qquad (i = 1,\dots,\ell)$$

with a "randomized" Hamiltonian $\hat{H}$ given by $\hat{H} = H_0 + \sum_{r=1}^m H_r\gamma_t^r$, where $\gamma_t = (\gamma_t^r)_{r=1}^m$ is an m-dimensional Gaussian white noise (Arnold (1973)). One may regard this as an open Hamilton system within the external world; the random part in (3.1) characterizes the complicated interaction between the "deterministic" Hamilton system with the Hamiltonian H_0 and the chaotic environment. Hence, we call (3.1) and H_α $(\alpha = 0,1,\dots,m)$ an (ℓ-dimensional) *stochastic Hamilton dynamical system* and the *Hamiltonians*.

We now investigate the conditions that conserved quantities and symmetry operators for (3.1) should satisfy. First, by applying equation (2.2) to the system (3.1) and through a straightforwardly calculation, we obtain the conditions for conserved quantities as follows:

THEOREM 3.1. *A smooth function $I = I(\boldsymbol{x},t)$ is a conserved quantity for* (3.1), *if and only if*

$$\partial_t I + \{H_0, I\} = 0, \qquad \{H_r, I\} = 0, \quad (r = 1,2,\dots,m) \tag{3.2}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket *defined by* $\{I,J\} = \sum_{i=1}^{\ell}(\partial_{\ell+i}I\partial_i J - \partial_i I \partial_{\ell+i}J)$.

For simplicity, in what follows, we work with the system with the Hamiltonians $H_\alpha = H_\alpha(\boldsymbol{x})$ $(\alpha = 0,1,\dots,m)$. Then, we look into the symmetry operators for (3.1). Let us consider the following differential operator generated by a smooth function $I = I(\boldsymbol{x})$:

$$Y_I = \sum_{i=1}^{\ell}(\partial_{\ell+i}I)\partial_i - \sum_{i=1}^{\ell}(\partial_i I)\partial_{\ell+i}. \tag{3.3}$$

We see that the operators X_0 and X_r in (2.3) are rewritten for the system (3.1) in terms of (3.3) as $X_0 = Y_{H_0}$ and $X_r = Y_{H_r}$ $(r = 1,\dots,m)$. Then, applying Definition 2.3 to the system (3.1) and the operator (3.3), we find the following theorem:

THEOREM 3.2. *The operator Y_I is a symmetry operator for* (3.1), *if and only if*

$$[Y_I, Y_{H_\alpha}] = 0 \qquad (\alpha = 0,1,\dots,m). \tag{3.4}$$

Next, we go into the relations of conserved quantities and symmetry operators. Note here the following relations between the Poisson bracket and the operator (3.3) in ordinary dynamical theory (Abraham and Marsden (1978)):

$$[Y_I, Y_J] = Y_{\{I,J\}} \tag{3.5}$$

$$Y_I J = \{I, J\}, \tag{3.6}$$

where $J = J(\boldsymbol{x})$ is also a smooth function. Then, the equation (3.5) indicates that the left-hand side of equation (3.4) is put into

$$[Y_I, Y_{H_\alpha}] = Y_{\{I,H_\alpha\}} \qquad (\alpha = 0, 1, \ldots, m), \tag{3.7}$$

and thereby, Theorem 3.1 and Theorem 3.2 prove the next theorem:

THEOREM 3.3. *Suppose that* $I = I(\boldsymbol{x})$ *is a smooth function and* Y_I *is the differential operator generated by* I *through* (3.3). *Then* Y_I *is the symmetry operator for the system* (3.1) *with* $H_\alpha = H_\alpha(\boldsymbol{x})$ $(\alpha = 0, 1, \ldots, m)$, *if* I *is a conserved quantity for the system.*

Moreover, in terms of equation (3.6), Theorem 2.2 for our stochastic Hamilton dynamical systems (3.1) and the operator (3.3) is rewritten in the following form:

THEOREM 3.4. *Suppose that* Y_I *is a symmetry operator for* (3.1) *generated by a smooth function* $I = I(\boldsymbol{x})$. *If a smooth function* $J = J(\boldsymbol{x})$ *is a conserved quantity for* (3.1), $Y_I J = \{I, J\}$ *is so.*

Therefore, if I and J are conserved quantities, $\{I, J\}$ is so, because of Theorem 3.3 and Theorem 3.4. These assertions are just corresponding to those in classical Hamilton mechanics (e.g. Abraham and Marsden (1978)). Thus, in the stochastic Hamilton system, we can connect conserved quantities with symmetries in a way similar to that in classical mechanics.

Example 3.1. We give an illustrative example of the stochastic Hamilton system with the conserved quantities and a symmetry operator. Let us consider

$$d\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \\ x_t^4 \end{pmatrix} = \begin{pmatrix} x_t^3 \\ x_t^4 \\ -x_t^1 \\ -x_t^2 \end{pmatrix} (dt + c \circ dw_t), \tag{3.8}$$

where c is a constant. This corresponds to the system (3.1) having the Hamiltonians of the harmonic oscillator type, $H_0(\boldsymbol{x}) = H_1(\boldsymbol{x})/c = [\{(x^3)^2 + (x^4)^2\}/2] + [\{(x^1)^2 + (x^2)^2\}/2]$. Hence, we call the system (3.8) the (two-dimensional) stochastic harmonic oscillator system.

Remark 3.1. As in (3.1), we formally regard this as a stochastic system obtained by such a randomization of the deterministic Hamilton system with the

Hamiltonian H_0 as $H_0(1+c\gamma_t)$, where γ_t is a Gaussian white noise. On the other hand, Theorem 3.1 verifies that the Hamiltonian H_0 is also a conserved quantity for this stochastic system; this fact means that a solution of the above stochastic Hamilton system "randomly" moves on the orbit determined by a solution of the ordinary Hamilton system with H_0. Hence, we may regard the perturbation for H_0 by a white noise mentioned above as a randomized procedure which leaves the orbits of the original Hamilton system with H_0 invariant.

We will come back to our example. For the system (3.8), Theorem 3.1 indicates that $I(\boldsymbol{x}) = (x^1x^4 - x^2x^3)$ is a conserved quantity, and hence Theorem 3.3 asserts the operator generated by this function, $Y_I = -x^2\partial_1 + x^1\partial_2 - x^4\partial_3 + x^3\partial_4$, is a symmetry operator for the system (3.8). Moreover, by Theorem 3.4 with the operator, we find out the conserved quantities for the system,

$$I(\boldsymbol{x}) = I_1(\boldsymbol{x}) = \frac{1}{2}(x^1x^4 - x^2x^3), \qquad I_2(\boldsymbol{x}) = -\frac{1}{4}((x^1)^2 - (x^2)^2 + (x^3)^2 - (x^4)^2),$$
$$I_3(\boldsymbol{x}) = -\frac{1}{2}(x^3x^4 + x^1x^2),$$

which satisfy SO(3) algebraic relations $\{I_1, I_2\} = -I_3$, $\{I_2, I_3\} = -I_1$ and $\{I_3, I_1\} = -I_2$ (Eisenhert (1961)).

4. Derivation of conserved quantities from symmetries in stochastic dynamical systems

In this section, we focus on the procedure to derive conserved quantities from symmetries in stochastic dynamical systems. As mentioned in Section 1, we present two methods to derive conserved quantities from the symmetries of stochastic dynamical systems, without referring either Lagrangians or Hamiltonians. The first method corresponds to a stochastic version of Hojman's procedure to the second-order differential dynamical systems (Hojman (1992), Misawa (1994*b*)); through the method, we obtain a conserved quantity from symmetry, which is essentially of *divergence type*. The second method corresponds to a modification of the first procedure; it gives a conserved quantity of a non-divergence type from symmetry.

We start with the main result and the examples related to the first method. Note that the part of this presentation corresponds to a review of Misawa (1994*b*) with a supplement, but we need it for consistency of this section.

THEOREM 4.1. *For given stochastic dynamical system* (2.1), *assume that there exists a function* $\varphi = \varphi(\boldsymbol{x}, t)$ *satisfying*

$$\text{(4.1)} \qquad \operatorname{div} \boldsymbol{b} + (\partial_t + X_0)\varphi = 0, \qquad \operatorname{div} \boldsymbol{g}_r + X_r\varphi = 0 \quad (r = 1, 2, \ldots, m),$$

where X_0 *and* X_r *are given by equation* (2.3). *Then, the function* $\boldsymbol{f} = (f^i)_{i=1}^n$ *in a symmetry operator* $Y = \sum_{i=1}^n f^i(\boldsymbol{x}, t)\partial_i$ *for* (2.1) *yields the following conserved quantity*:

$$\text{(4.2)} \qquad I = \operatorname{div} \boldsymbol{f} + Y\varphi.$$

COROLLARY 4.1.1. *If* $\boldsymbol{b}$ *and* $\boldsymbol{g}_r$ $(r = 1, 2, \ldots, m)$ *in equation* (2.1) *satisfy*

$$\operatorname{div} \boldsymbol{b} = 0, \qquad \operatorname{div} \boldsymbol{g}_r = 0 \quad (r = 1, 2, \ldots, m). \tag{4.3}$$

Then, the function

$$I = \operatorname{div} \boldsymbol{f} \tag{4.4}$$

becomes a conserved quantity for (2.1).

The proofs of this theorem and the corollary are given in Misawa (1994*b*). In addition, Albeverio and Fei (1995) proposed a more general version of Theorem 4.1 by extending a class of symmetry operators for stochastic systems. Note that the conserved quantity given by (4.2) or (4.4) is essentially of "divergence type".

We here give two illustrative examples of conserved quantities obtainable by applying Theorem 4.1 and the corollary to stochastic dynamical systems.

Example 4.1. Let us consider again the stochastic system treated in Example 2.2:

$$d \begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{pmatrix} (x_t^1)^2 \\ x_t^1 \cdot x_t^2 \\ x_t^1 \cdot x_t^3 \end{pmatrix} dt + \begin{pmatrix} x_t^1 \cdot x_t^2 \\ (x_t^2)^2 \\ x_t^2 \cdot x_t^3 \end{pmatrix} \circ dw_t. \tag{4.5}$$

This admits $Y = (x^2 x^3 / x^1)\partial_3$ as a symmetry operator. Moreover, one may choose the function $\varphi(\boldsymbol{x}, t)$ satisfying equation (4.1) in Theorem 4.1 as $\varphi(\boldsymbol{x}, t) = -2\log|x^1| - 2\log|x^2|$. Then, applying Theorem 4.1 to the system, we get a conserved quantity $I_1 = x^1/x^2$. In the same manner, if we choose the function $\varphi(\boldsymbol{x}, t)$ as x^1/x^3, then Theorem 4.1 together with Y yield a conserved quantity $I_2 = -(x^2/x^3) + x^1/x^2$.

Example 4.2. Next we work with the stochastic Hamilton dynamical systems given by

$$d \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} = \begin{pmatrix} x_t^2 \\ -x_t^1 \end{pmatrix} \circ dw_t \tag{4.6}$$

with the Hamiltonians $H_0(\boldsymbol{x}) = 0$, $H_1(\boldsymbol{x}) = \frac{1}{2}\{(x^1)^2 + (x^2)^2\}$. Note that Theorem 3.1 indicates this system (4.4) has a conserved quantity $I(\boldsymbol{x}) = (x^1)^2 + (x^2)^2$. On the other hand, this system satisfies the conditions (4.3) and admits as symmetry operators $Y_1 = x^1\partial_1 + x^2\partial_2$ and $Y_2 = x^2\partial_1 - x^1\partial_2$. Therefore, Corollary 4.1.1 is applicable. However, it yields only trivial conserved quantities 0 and 2 from the above operators, respectively. We remark that for this example, the equations (4.1) are put into $(\partial_t + X_0)\varphi(\boldsymbol{x}, t) = 0$ and $X_r\varphi(\boldsymbol{x}, t) = 0$ $(r = 1, \ldots, n)$, and I given by (4.2) becomes $Y\varphi +$ *constant*; as a result, for the above system and the operators, Theorem 4.1 only reduces to Theorem 2.2.

Thus, for Example 4.2, Theorem 4.1 and the corollary do not play an essential role in finding a non-trivial conserved quantity. Hence we need to formulate the another procedure to derive conserved quantities from symmetry for stochastic systems. For this purpose, we further remark the following fact with respect to the system (4.6): The system satisfies not only equation (4.1) i.e. $\operatorname{div} \boldsymbol{b} = \operatorname{tr}(\nabla \boldsymbol{b}) = 0$ and $\operatorname{div} \boldsymbol{g}_r = \operatorname{tr}(\nabla \boldsymbol{g}_r) = 0$ $(r = 1, \ldots, m)$, where tr and ∇ denote trace and gradient, respectively, but also $(\nabla \boldsymbol{b}) + (\nabla \boldsymbol{b})^{\mathrm{T}} = \bigcirc$ and $(\nabla \boldsymbol{g}_r) + (\nabla \boldsymbol{g}_r)^{\mathrm{T}} = \bigcirc$ $(r = 1, 2, \ldots, m)$, where T denotes transpose. Then, on the basis of this observation, we may establish the following proposition:

PROPOSITION 4.1. *Suppose that the n-vector valued functions $\boldsymbol{b}$ and $\boldsymbol{g}_r$ $(r = 1, 2, \ldots, m)$ in a stochastic dynamical system* (2.1) *satisfy*

$$(\nabla \boldsymbol{b}) = -(\nabla \boldsymbol{b})^{\mathrm{T}}, \qquad (\nabla \boldsymbol{g}_r) = -(\nabla \boldsymbol{g}_r)^{\mathrm{T}} \quad (r = 1, 2, \ldots, m), \tag{4.7}$$

respectively. Then, any pairs of symmetry operators for (2.1), $Y = \sum_{i=1}^{n} f^i(\boldsymbol{x}, t)\partial_i$ *and* $Z = \sum_{i=1}^{n} h^i(\boldsymbol{x}, t)\partial_i$, *yield the following conserved quantity*:

$$I = \boldsymbol{f} \cdot \boldsymbol{h} = \sum_{i=1}^{n} f^i h^i. \tag{4.8}$$

PROOF. First, we assume that Y is a symmetric operator. Then, $I = \boldsymbol{f} \cdot \boldsymbol{h}$ is a conserved quantity of (2.1), if and only if the following equations hold:

$$(\partial_t + X_0)h^i = -\sum_{j=1}^{n} h^j \partial_i b^j, \qquad X_r h^i = -\sum_{j=1}^{n} h^j \partial_i g_r^j \qquad (i = 1, \ldots, n; r = 1, \ldots, m). \tag{4.9}$$

This is proved by (2.2) and the following equations which are derived from (2.19):

$$(\partial_t + X_0)f^i = \sum_{j=1}^{n} f^j \partial_j b^i, \qquad X_r f^i = \sum_{j=1}^{n} f^j \partial_j g_r^i \qquad (i = 1, \ldots, n; r = 1, \ldots, m). \tag{4.10}$$

The equations (4.9) are rewritten by the operator $Z = \sum_{i=1}^{n} h^i(\boldsymbol{x}, t)\partial_i$ in the following form:

$$[\partial_t + X_0, Z] = -\sum_{j=1}^{n} h^j \Phi_j^i \partial_i, \qquad [X_r, Z] = -\sum_{j=1}^{n} h^j {\Phi_r}_j^i \partial_i \qquad (r = 1, \ldots, m), \tag{4.11}$$

where $\Phi = (\Phi_j^i)$ and $\Phi_r = ({\Phi_r}_j^i)$ $(r = 1, 2, \ldots, m)$ are the $n \times n$ matrix-valued functions given by

$$\Phi = (\Phi_j^i) = (\nabla \boldsymbol{b}) + (\nabla \boldsymbol{b})^{\mathrm{T}}, \qquad \Phi_r = ({\Phi_r}_j^i) = (\nabla \boldsymbol{g}_r) + (\nabla \boldsymbol{g}_r)^{\mathrm{T}} \qquad (r = 1, \ldots, m), \tag{4.12}$$

respectively. Hence, if $\boldsymbol{b}$ and $\boldsymbol{g}_r$ satisfy the condition (4.7) and Z is a symmetric operator, then equations (4.11) are satisfied, and thereby completing the proof of Proposition 4.1.

Note that the conserved quantity obtainable from the above proposition is obviously not of divergence type. Moreover, if $\boldsymbol{b}$ and $\boldsymbol{g}_r$ ($r = 1, \ldots, m$) satisfy the condition (4.7), the conditions (4.3) in Theorem 4.1 are automatically satisfied. Hence we can also apply Theorem 4.1 to the stochastic system to which Proposition 4.1 is applicable.

Remark 4.1. By using Proposition 4.1, we easily see that if $Y = \sum_{i=1}^{n} f^i(\boldsymbol{x}, t)\partial_i$ is a symmetric operator, $I = \boldsymbol{f} \cdot \boldsymbol{f}$ is a conserved quantity under the condition (4.7).

Example 4.3. The previous Example 4.2 satisfies the conditions (4.5). Moreover, as stated, $Y_1 = x^1\partial_1 + x^2\partial_2$ and $Y_2 = x^2\partial_1 - x^1\partial_2$ are symmetry operators of the system. Hence, Remark 4.1 indicates that each operator yields a conserved quantity $I = \boldsymbol{f} \cdot \boldsymbol{f} = (x^1)^2 + (x^2)^2$, and this is just a result we want.

Example 4.4. Consider the stochastic harmonic oscillator treated in Example 3.1:

$$d\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \\ x_t^4 \end{pmatrix} = \begin{pmatrix} x_t^3 \\ x_t^4 \\ -x_t^1 \\ -x_t^2 \end{pmatrix} (dt + c \circ dw_t). \tag{4.13}$$

It is easy to examine that this system satisfies equation (4.7) (and hence equation (4.3)). Moreover, we see that the system has as symmetry operators

$$\begin{aligned} Y_1 &= -x^2\partial_1 + x^1\partial_2 - x^4\partial_3 + x^3\partial_4, & Y_2 &= x^3\partial_1 + x^4\partial_2 - x^1\partial_3 - x^2\partial_4, \\ Y_3 &= x^3\partial_1 - x^4\partial_2 - x^1\partial_3 + x^2\partial_4, & Y_4 &= x^4\partial_1 + x^3\partial_2 - x^2\partial_3 - x^1\partial_4. \end{aligned}$$

Thereby, Proposition 4.1 asserts that $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$, $(x^1)^2 - (x^2)^2 + (x^3)^2 - (x^4)^2$, $2(x^1x^4 - x^2x^3)$, and $2(x^1x^2 + x^3x^4)$, which are obtained through the scalar products of any pairs from among the coefficient vector-valued functions of the above operators, are conserved quantities of the system (4.13). In contrast with this, Corollary 4.1.1 indicates that each of these operators generates only 0.

Remark 4.2. In general, a stochastic Hamilton dynamical system (3.1) satisfies the conditions (4.3) in Corollary 4.1.1. Suppose that the system (3.1) has the Hamiltonians $H_\alpha(\boldsymbol{x})$ ($\alpha = 0, 1, \ldots, m$) and that the operator Y_I generated by the function I through (3.3) is a symmetry operator. For example, as stated in Theorem 3.3, if I is a conserved quantity for the system, Y_I becomes a symmetry operator of the system. Then, we apply (4.4) to the system with the symmetry operator. However, it only yields the trivial conservation "zero" through the operator because of (3.3). On the other hand, as shown in Example 4.2 and Example

4.3, if the system satisfies the conditions (4.7), Proposition 4.1 brings to us the essential conservations from such a operator. Thus, it appears that for stochastic Hamilton systems, Proposition 4.1 is more useful than Corollary 4.1.1 in finding conserved quantities from symmetry operators of equation (3.3) type.

Through the equations (4.11) and (4.12), we further find the following procedure to obtain the conserved quantities of non-divergence type from symmetries for (2.1); It is corresponds to a generalization of Proposition 4.1.

THEOREM 4.2. *Let* $Y = \sum_{i=1}^{n} f^i(\boldsymbol{x},t)\partial_i$ *be a symmetric operator of a stochastic dynamical system* (2.1). *Assume that there is some operator* $Z = \sum_{i=1}^{n} h^i(\boldsymbol{x},t)\partial_i$ *satisfying equation* (4.11) *together with equation* (4.12). *Then, the following function is a conserved quantity of the stochastic system under study*:

$$I = \boldsymbol{f} \cdot \boldsymbol{h}. \tag{4.14}$$

PROOF. The operators Y and Z in Theorem 4.2 indicates that the equation (4.10) (and hence (4.9)) holds for I given by (4.14), and thereby I becomes a conserved quantity for (2.1).

Note that the above theorem corresponds to a stochastic version of Theorem 2 in Mimura and Nôno (1995). In general, however, it may not be easy to find out the operators Z dealt with in Theorem 4.2. The following proposition gives a way to obtain such an operator.

PROPOSITION 4.2. *Suppose that a coefficient vector-valued function,* $\boldsymbol{h} = (h^i)_{i=1}^n$, *of a differential operator* $Z = \sum_{i=1}^{n} h^i(\boldsymbol{x},t)\partial_i$ *satisfies*

$$\operatorname{rot} \boldsymbol{h} = \boldsymbol{0}, \qquad \partial_t \boldsymbol{h} + \nabla(\boldsymbol{h}\cdot\boldsymbol{b}) = \boldsymbol{0}, \qquad \nabla(\boldsymbol{h}\cdot\boldsymbol{g}_r) = \boldsymbol{0} \quad (r = 1,\dots,m). \tag{4.15}$$

Then, the equations (4.11) *together with* (4.12), *hold for the operator* Z.

PROOF. We see that the equations (4.9) are rewritten in the following form:

$$\partial_t h^i + \partial_i(\boldsymbol{h}\cdot\boldsymbol{b}) + \sum_{j=1}^{n} b^j(\partial_j h^i - \partial_i h^j) = 0,$$

$$\partial_i(\boldsymbol{h}\cdot\boldsymbol{g}_r) + \sum_{j=1}^{n} g_r^j(\partial_j h^i - \partial_i h^j) = 0 \quad (i = 1,\dots,n; r = 1,\dots,m).$$

These equations turn out that the equations (4.7) hold for the function $\boldsymbol{h} = (h^i)_{i=1}^n$ satisfying the conditions (4.15), and thereby the above proposition is proved.

Example 4.5. Consider the non-linear stochastic system in Example 4.1 (and hence Example 2.2). The operator $Y = (x^2x^3/x^1)\partial_3$ was a symmetry one of this system. On the other hand, let Z be a differential operator with the coefficient vector-valued function $\boldsymbol{h} = (h^i)_{i=1}^3 = (1/x^1, 0, -1/x^3)^{\mathrm{T}}$ (T: transpose). Then, the function $\boldsymbol{h}$ satisfies equation (4.15), so that Theorem 4.2 is applicable for the operators Y and Z by virtue of Proposition 4.2. As a result, through the theorem, we obtain a conserved quantity $I = -x^2/x^1$.

5. The similarity method in stochastic dynamical systems

The notion of symmetries for dynamical systems plays an important role in the "similarity method" which is a powerful tool for solving differential equations (e.g. Bluman and Cole(1974); Bluman and Kumei (1989)) and difference equations (Maeda (1987)); the existence of symmetries for differential equations leads to a reduction in order in ordinary differential equations or difference equations and to a particular solution in partial differential equations. Hence, it must be relevant and important to formulate the similarity method for stochastic dynamical systems in the framework of the theory of symmetries mentioned in the preceding sections; this is the main theme of this section. For this purpose, we first prove a lemma with respect to the symmetry operators satisfying (2.19).

LEMMA 5.1. *The equations* (2.19) *are coordinate-free almost surely.*

PROOF. It proves this lemma that equations (2.19) are also satisfied by a symmetry operator for the stochastic system (2.1) under another coordinate system. Let $(y^i)_{i=1}^n$ be another local coordinate system in $U \subset R^n$, and suppose that equation (2.1) and the symmetry operator of the equation, $Y = \sum_{i=1}^n f^i(\boldsymbol{x}, t)\partial/\partial x^i$, are expressed in terms of these coordinates as

$$dy_t = \hat{\boldsymbol{b}}(\boldsymbol{y}_t, t)dt + \sum_{r=1}^m \hat{\boldsymbol{g}}_r(\boldsymbol{y}_t, t) \circ dw_t^r, \tag{5.1}$$

$$Y = \sum_{k=1}^n h^k(\boldsymbol{y}, t)\frac{\partial}{\partial y^k}. \tag{5.2}$$

Then, it is to be noted that the operators X_0 and X_r ($r = 1, \ldots, m$) given by equations (2.3) are expressed in terms of the local coordinates (y^i) as follows:

$$X_0 = \sum_{k=1}^n \hat{b}^k(\boldsymbol{y}, t)\frac{\partial}{\partial y^k}, \qquad X_r = \sum_{k=1}^n \hat{g}_r^k(\boldsymbol{y}, t)\frac{\partial}{\partial y^k}, \tag{5.3}$$

since equation (5.1) holds. It is easily examined that

$$h^k(\boldsymbol{y}, t) = \sum_{j=1}^n f^j(\boldsymbol{x}, t)\frac{\partial y^k}{\partial x^j} \qquad (k = 1, \ldots, n), \tag{5.4}$$

and further that the process $\boldsymbol{y}_t$ given by (5.1) is connected with the original process $\boldsymbol{x}_t$ satisfying (2.1) as follows:

$$\boldsymbol{y}_t = \boldsymbol{y}(\boldsymbol{x}_t). \tag{5.5}$$

Hence, by applying the change of variables formula to equations (5.5) and (2.1) (e.g. Ikeda and Watanabe (1989)), we can verify that $\boldsymbol{y}_t$ also satisfies

$$(5.6)\qquad d\boldsymbol{y}_t = \sum_{k=1}^{n} \frac{\partial y^i}{\partial x^k} \circ dx_t^k = \sum_{k=1}^{n} b^k \frac{\partial y^i}{\partial x^k} dt + \sum_{k=1}^{n}\sum_{r=1}^{m} g_r^k \frac{\partial y^i}{\partial x^k} \circ dw_t^r.$$

Rewriting equations (5.1) and (5.6) in the form of Itô type, we compare the martingale differential and the non-martingale differential of equation (5.1) with those of equation (5.6), respectively. Then, by virtue of the uniqueness of Doob-Meyer's canonical decomposition of semi-martingales (exactly, by Lemma 2.1; cf. Remark 2.1), we find

$$(5.7)\qquad \hat{b}^i(\boldsymbol{y},t) = \sum_{j=1}^{n} b^j(\boldsymbol{x},t)\frac{\partial y^i}{\partial x^j}, \qquad \hat{g}_r^i(\boldsymbol{y},t) = \sum_{j=1}^{n} g_r^j(\boldsymbol{x},t)\frac{\partial y^i}{\partial x^j} \quad \text{(a.s.)},$$
$$(i = 1,\ldots,n; r = 1,\ldots,m).$$

We are to show that equations (2.19) hold for the symmetry operator Y under the new coordinate system (y^i); especially, we are concerned with the first equation of (2.19). After a calculation by using (5.3) and (5.7) together with the chain rule, we obtain the following two equations for the operators Y and X_0 given by equations (5.2) and (5.3), respectively:

$$(5.8)\qquad \left(\frac{\partial}{\partial t} + X_0\right) h^i(\boldsymbol{y},t) = \left(\frac{\partial}{\partial t} + \sum_{k=1}^{n} \hat{b}^k(\boldsymbol{y},t)\frac{\partial}{\partial y^k}\right) h^i(\boldsymbol{y},t)$$
$$= \sum_{j=1}^{n}\left(\frac{\partial y^i}{\partial x^j}\right)\left\{\left(\frac{\partial}{\partial t} + \sum_{\ell=1}^{n} b^\ell \frac{\partial}{\partial x^\ell}\right) f^j\right\}$$
$$(i = 1,\ldots,n), \qquad \text{(a.s.)},$$

$$(5.9)\qquad Y\hat{b}^i(\boldsymbol{y},t) = \left(\sum_{k=1}^{n} h^k(\boldsymbol{y},t)\frac{\partial}{\partial y^k}\right)\hat{b}^i(\boldsymbol{y},t)$$
$$= \sum_{j=1}^{n}\left(\frac{\partial y^i}{\partial x^j}\right)\left\{\left(\sum_{\ell=1}^{n} f^\ell \frac{\partial}{\partial x^\ell}\right) b^j\right\} \qquad (i = 1,\ldots,n), \qquad \text{(a.s.)}.$$

Since Y is a symmetry operator for equation (2.1), the first equation in equations (2.19) (i.e. equations (4.8)) with respect to the original coordinates (x^i)

$$\left(\frac{\partial}{\partial t} + \sum_{\ell=1}^{n} b^\ell \frac{\partial}{\partial x^\ell}\right) f^j = \left(\sum_{\ell=1}^{n} f^\ell \frac{\partial}{\partial x^\ell}\right) b^j \qquad (j = 1,\ldots,n)$$

holds, and thereby the right-hand side of equation (5.8) is equal to that of equation (5.9) almost surely. Therefore, we finally see that the symmetry operator Y satisfies

$$(5.10)\qquad \left(\frac{\partial}{\partial t} + X_0\right) h^i(\boldsymbol{y},t) = Y\hat{b}^i(\boldsymbol{y},t) \qquad (i = 1,\ldots,n),$$

and hence, the first equation in equations (2.19) for the symmetry operator

$$[\partial_t + X_0, Y] = 0 \tag{5.10'}$$

also holds under the new coordinate system (y^i) almost surely. The remainder of equations in equations (2.19) under the coordinate system (y^i) can be proved in a similar manner to that of the proof on the first equation mentioned above, and thereby Lemma 5.1 is proved.

We proceed to the main theorem. In what follows, we assume that the new coordinate system introduced below covers a sufficiently large domain for a given system (2.1).

THEOREM 5.1. *Suppose that a stochastic dynamical system* (2.1) *admits s symmetry operators Y_α, such that,*

> *for any integer ℓ $(1 \le \ell \le s-1)$, $[Y_\alpha, Y_\beta] \equiv 0 \bmod(Y_1, \dots, Y_\ell)$ holds for arbitrary integers α and β $(1 \le \alpha, \beta \le \ell+1)$, and* $\mathrm{rank}(Y_1, \dots, Y_s) = s$.

Then there is a local coordinate system (y^i) in which the stochastic system is expressed almost surely as

$$dy_t^i = \hat{b}^i(y_t^{i+1}, \dots, y_t^n, t)dt + \sum_{r=1}^{m} \hat{g}_r^i(y_t^{i+1}, \dots, y_t^n, t) \circ dw_t^r \qquad (i = 1, \dots, s) \tag{5.11a}$$

$$dy_t^\alpha = \hat{b}^\alpha(y_t^{s+1}, \dots, y_t^n, t)dt + \sum_{r=1}^{m} \hat{g}_r^\alpha(y_t^{s+1}, \dots, y_t^n, t) \circ dw_t^r \qquad (\alpha = s+1, \dots, n). \tag{5.11b}$$

PROOF. In a way analogous to that in ordinary differential equations (e.g. Bluman and Kumei (1989)) or difference equations (Maeda (1987)), we may prove this theorem. That is, it follows by induction with respect to α that the symmetry operator Y_α are expressed as the following forms:

$$Y_1 = \frac{\partial}{\partial y^1}, \qquad Y_\alpha = \frac{\partial}{\partial y^\alpha} + \sum_{\beta=1}^{\alpha-1} h_\alpha^\beta(\boldsymbol{y}, t)\frac{\partial}{\partial y^\beta} \quad (\alpha = 2, \dots, s).$$

Then from the previous lemma, we see that the symmetry conditions (2.19) (i.e. (4.8)) for the new coordinate system (y^i) implies that the equation (5.1) must take the form (5.11a,b) under the coordinate.

The coordinates (y^i) are obtained by solving the differential equations

$$Y_\alpha y^i = \delta_\alpha^i \quad (\alpha = 1, \dots, \min(i, s); i = 1, \dots, n). \tag{5.12}$$

The equations (5.11a,b) indicate that the original stochastic dynamical system (2.1) is composed of an $(n-s)$-dimensional stochastic system (5.11b) and s remaining system (5.11a) in which each variable y_t^i is contained as a stochastic differential. Hence, if a solution of (5.11b) is obtained in some way, we can construct the whole solution step by step by the way of a stochastic integral procedure (e.g. Ikeda and Watanabe (1989)).

Remark 5.1. The condition in the above theorem means that the operators $\{Y_\alpha\}$ satisfy

$$[Y_\alpha, Y_\beta] = \sum_{\gamma=\alpha+1}^{s} C_{\alpha,\beta}^{\gamma} Y_\gamma \qquad (\alpha < \beta; C_{\alpha,\beta}^{\gamma} : \text{constant}),$$

that is, $\{Y_\alpha\}$ forms a completely integrable system (Kobayashi and Nomizu (1969)). Hence we may choose a specific local coordinate system with respect to the foliation. The condition does not require that $\{Y_\alpha\}$ spans a Lie algebra; it contains the case that they form a solvable or nilpotent Lie algebra. In particular, if s symmetry operators form a commutative Lie algebra, that is, they commute with one another, the equation (2.1) is expressed in a simpler form as follows:

$$(5.11\text{a}') \qquad dy_t^i = \hat{b}^i(y_t^{s+1}, \dots, y_t^n, t)dt + \sum_{r=1}^{m} \hat{g}_r^i(y_t^{s+1}, \dots, y_t^n, t) \circ dw_t^r \qquad (i = 1, \dots, s)$$

$$(5.11\text{b}') \qquad dy_t^\alpha = \hat{b}^\alpha(y_t^{s+1}, \dots, y_t^n, t)dt + \sum_{r=1}^{m} \hat{g}_r^\alpha(y_t^{s+1}, \dots, y_t^n, t) \circ dw_t^r \qquad (\alpha = s+1, \dots, n).$$

The coordinates are also given by solving the equations (5.12).

Now, by means of the similarity method, we give several examples of reduction and an analytic expression for solutions in stochastic dynamical systems.

Example 5.1. Let us consider again the stochastic system treated in Example 2.2:

$$(5.13) \qquad d\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{pmatrix} (x_t^1)^2 \\ x_t^1 \cdot x_t^2 \\ x_t^1 \cdot x_t^3 \end{pmatrix} dt + \begin{pmatrix} x_t^1 \cdot x_t^2 \\ (x_t^2)^2 \\ x_t^2 \cdot x_t^3 \end{pmatrix} \circ dw_t.$$

This system admits the following two symmetric operators:

$$(5.14) \qquad Y_1 = x^1 x^2 \frac{\partial}{\partial x^1} + (x^2)^2 \frac{\partial}{\partial x^2} + x^2 x^3 \frac{\partial}{\partial x^3}, \qquad Y_2 = x^3 \frac{\partial}{\partial x^3}.$$

Then, it is easy to examine that these operators commute with one another and $\text{rank}(Y_1, Y_2) = 2$, and hence Theorem 5.1 with Remark 5.1 indicate that the above system is reduced to an one-dimensional system. Indeed, we may choose the following new coordinate (y^i) by solving (5.12) for Y_1 and Y_2;

$$y^1 = -\frac{1}{x^2}, \qquad y^2 = \log x^3 - \log x^2, \qquad y^3 = \frac{x^1}{x^2}. \tag{5.15}$$

By the change of variables formula, equation (5.13) reduces to

$$d\begin{pmatrix} y_t^1 \\ y_t^2 \\ y_t^3 \end{pmatrix} = \begin{pmatrix} y_t^3 \\ 0 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \circ dw_t. \tag{5.16}$$

In the right-hand side of equation (5.16), y^3 is solely contained as the variables. Thus, the system (5.13) essentially becomes a one-dimensional system (5.16) under the new coordinates. Moreover, by a stochastic integral procedure, we find a solution of equation (5.16):

$$\begin{pmatrix} y_t^1 \\ y_t^2 \\ y_t^3 \end{pmatrix} = \begin{pmatrix} y_0^1 + y_0^3 t + w_t \\ y_0^2 \\ y_0^3 \end{pmatrix}, \tag{5.17}$$

where (y_0^1, y_0^2, y_0^3) are initial values of (y_t^1, y_t^2, y_t^3), respectively. From equation (5.17), an explicit form of the solution of equation (5.13) are derived through (5.15) as follows:

$$\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{pmatrix} x_0^1/(1 - x_0^1 t - x_0^2 w_t) \\ x_0^2/(1 - x_0^1 t - x_0^2 w_t) \\ x_0^3/(1 - x_0^1 t - x_0^2 w_t) \end{pmatrix}, \tag{5.18}$$

where (x_0^1, x_0^2, x_0^3) are initial values of (x_t^1, x_t^2, x_t^3) which are connected with (y_0^1, y_0^2, y_0^3) through (5.15).

Example 5.2. Next, we investigate a stochastic Fisher-Wright model of 3-species type in population genetics, which is given by Maruyama and Itoh (1991, 1997). That is, we consider the following stochastic dynamical systems:

$$d\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{x_t^1 x_t^2} & -\sqrt{x_t^1 x_t^3} \\ \sqrt{x_t^2 x_t^3} & -\sqrt{x_t^1 x_t^2} & 0 \\ -\sqrt{x_t^2 x_t^3} & 0 & \sqrt{x_t^1 x_t^3} \end{pmatrix} \circ d\begin{pmatrix} w_t^1 \\ w_t^2 \\ w_t^3 \end{pmatrix}, \tag{5.19}$$

where x^i's denote the frequencies of alleles; they are non-negative, and $\sum_{i=1}^3 x^i \neq 0$ (in fact, $= 1$). Then, we find out a symmetry operator for this system as $Y = (2x^1/r)\partial_1 + (2x^2/r)\partial_2 + (2x^3/r)\partial_3$, where $r = \sqrt{x^1 + x^2 + x^3}$ and $\partial_i = \partial/\partial x^i$. Therefore, Theorem 5.1 indicates that this system is reduced to a two-dimensional system. Indeed, one may introduce the "polar coordinate" (r, θ, φ) for (x^i) as

$$x^1 = r^2 \cos^2 \varphi \sin^2 \theta, \qquad x^2 = r^2 \sin^2 \varphi \sin^2 \theta, \qquad x^3 = r^2 \cos^2 \theta, \tag{5.20}$$

where $0 \leq \theta \leq \pi/2$, $0 \leq \varphi \leq \pi/2$ and $0 < r$, because the new coordinate satisfies (5.12) for Y. In terms of (r, θ, φ), we can rewrite the equation (5.19) in the following form:

$$(5.21) \qquad d\begin{pmatrix} r_t \\ \varphi_t \\ \theta_t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ \sin\varphi & 0 & -\cos\varphi \\ \frac{\cos\theta\cos\varphi}{\sin\theta} & -1 & \frac{\cos\theta\sin\varphi}{\sin\theta} \end{pmatrix} \circ d\begin{pmatrix} w_t^1 \\ w_t^2 \\ w_t^3 \end{pmatrix},$$

and this means that the system is a diffusion process on the sphere with the radius $r(=1)$. Thus, the system (5.19) is expressed as the 2-dimensional system (5.21). In addition, the Fokker-Planck's equation (e.g. Arnold (1973)) for (5.21) is given by

$$(5.22) \qquad \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}\left(\frac{\cos\theta}{8\sin\theta}\rho\right) + \frac{1}{8}\left(\frac{\partial^2}{\partial\theta^2}\rho + \frac{\partial^2}{\partial\varphi^2}\left(\frac{\rho}{\sin^2\theta}\right)\right).$$

It is easy to verify that $\rho = (2\sin\theta/\pi)$ is the stationary solution to (5.22), that is, stationary probability density function for the stochastic process (5.22) on the sphere with the radius $r = 1$. This result just corresponds to that in Maruyama-Itoh's works.

Example 5.3. As the final example, we work with a stochastic Hamilton dynamical system related to "the neoclassical optimal growth model" suggested by Samuelson (1972). In theoretical economics, Lagrangian and Hamiltonian dynamical formalisms are often treated in a way analogous to that in classical mechanics. The study on the neoclassical optimal growth models, of which various versions are proposed by many authors, may be the most popular example for such a topic (Satô and Ramachandran (1990)); in a typical formalism of such models, it is assumed that the economical growth is described by the maximization of the functional of the discount present value for "welfare function" which corresponds to the Lagrangian. Then, we note that "conservation law" for the dynamical model is also investigated with great interest in theoretical economics as well as classical mechanics. In particular, Samuelson (1972) formulated such an optimal problem near the stationary point, and further obtained the conserved quantity for the dynamical model, which was called *the local income-wealth conservation law*. In what follows, we first touch upon the model and its conservation law, and treat a stochastic version of it.

Samuelson's original model is represented by the maximization-problem for the following local Lagrangian L,

$$(5.23) \qquad L(k, \dot{k}, t) = e^{-\rho t}\left(-\frac{1}{2}\dot{k}^2 - a\dot{k}k - \frac{1}{2}k^2\right),$$

with the Euler-Lagrange equation

$$(5.24) \qquad \ddot{k} - \rho\dot{k} - (1 + a\rho)k = 0,$$

where k is the capital-labor ratio, $\rho \geq 0$ the fixed discount rate and a the constant satisfying $-1 < a < 1$. For simplicity, we here treat one-dimensional system.

By introducing momentum $p = \partial L/\partial \dot{k}$ and Hamiltonian $H = \dot{k}(\partial L/\partial \dot{k}) - L$ (e.g. Abraham and Marsden (1978)), we here rewrite equation (5.24) in the Hamiltonian formalism as follows:

$$\frac{d}{dt}\begin{pmatrix} k_t \\ p_t \end{pmatrix} = \begin{pmatrix} \partial H/\partial p_t \\ -\partial H/\partial k_t \end{pmatrix} = \begin{pmatrix} -e^{\rho t}p_t - ak_t \\ ap_t - e^{-\rho t}(1-a^2)k_t \end{pmatrix}. \tag{5.25}$$

Moreover, we define the new variables z^1 and z^2 by $z^1 = e^{-\rho t/2}k$ and $z^2 = e^{\rho t/2}p$, respectively. In terms of these variables, equation (5.25) is transformed to an autonomous Hamilton equation:

$$\frac{d}{dt}\begin{pmatrix} z_t^1 \\ z_t^2 \end{pmatrix} = \begin{pmatrix} \partial H/\partial z_t^2 \\ -\partial H/\partial z_t^1 \end{pmatrix} = \begin{pmatrix} -z_t^2 - (a+\frac{\rho}{2})z_t^1 \\ (a+\frac{\rho}{2})z_t^2 - (1-a^2)z_t^1 \end{pmatrix}, \tag{5.26}$$

where the new Hamiltonian is given by

$$H(z^1, z^2) = \frac{1}{2}(1-a^2)(z^1)^2 - \left(a + \frac{\rho}{2}\right) z^1 z^2 - \frac{1}{2}(z^2)^2. \tag{5.27}$$

Note that the Hamiltonian H is a conserved quantity for this system, and this fact just represents the local income-wealth conservation law derived by Samuelson (1972).

We are now to a stochastic version for the Hamilton system (5.26) with (5.27): we first randomize the above system in a way similar to that in Example 3.1; that is, we consider a stochastic Hamilton dynamical system (3.1) with $H_0 = H$ and $H_1 = cH$, where H is given by (5.27) and c is a constant. Then, the following stochastic equation describes it:

$$d\begin{pmatrix} z_t^1 \\ z_t^2 \end{pmatrix} = \begin{pmatrix} -z_t^2 - (a+\frac{\rho}{2})z_t^1 \\ (a+\frac{\rho}{2})z_t^2 - (1-a^2)z_t^1 \end{pmatrix} dt + c\begin{pmatrix} -z_t^2 - (a+\frac{\rho}{2})z_t^1 \\ (a+\frac{\rho}{2})z_t^2 - (1-a^2)z_t^1 \end{pmatrix} \circ dw_t. \tag{5.28}$$

As mentioned in Remark 3.1 in Example 3.1, the system is formally the Hamilton system of which Hamiltonian is randomized as $H(1 + c\gamma_t)$, and the original Hamiltonian (5.27) is still a conserved quantity for this stochastic system. These facts assert that the stochastic Hamilton system we treat here is regarded as a randomized Hamilton system which leaves the above income-wealth conservation law invariant; a solution of the system "randomly" moves on a solution orbit of the original Hamilton system (5.26) with (5.27).

For equation (5.28), we find out two symmetric operators Y_1 and Y_2, which commute with one another and satisfy $\mathrm{rank}(Y_1, Y_2) = 2$, as follows:

$$\begin{aligned} Y_1 &= \left(-z^2 - \left(a + \frac{\rho}{2}\right) z^1\right)\partial_1 + \left(\left(a + \frac{\rho}{2}\right) z^2 - (1-a^2)z^1\right)\partial_2 \\ Y_2 &= z^1\partial_1 + z^2\partial_2, \end{aligned} \tag{5.29}$$

where $\partial_i = \partial/\partial z^i$. Thereby the system (5.28) is reduced to a system which does not contain any variables in terms of new coordinate system. In fact, we choose the following new coordinate (y^i) by solving (5.12) for Y_1 and Y_2:

$$y^1 = \frac{1}{2r}\log\left|\frac{z^2 + \left(\frac{\rho}{2} + a - r\right)z^1}{z^2 + \left(\frac{\rho}{2} + a + r\right)z^1}\right|, \qquad y^2 = H(z^1, z^2) \tag{5.30}$$

where $r = \{(a+(\rho/2))^2+(1-a)^2\}^{1/2}$. Then the stochastic equation (5.28) reduces to

$$d\begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 0 \\ c \end{pmatrix} \circ dw_t. \tag{5.31}$$

Thus, the right-hand side of equation (5.31) contains only constants. Therefore, we easily get a solution of the system:

$$\begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix} = \begin{pmatrix} y_0^1 \\ y_0^2 + t + cw_t \end{pmatrix}, \tag{5.32}$$

where (y_0^1, y_0^2) are initial values of (y_t^1, y_t^2), respectively.

6. Concluding remarks

In the preceding sections, we investigate conserved quantities and symmetries in stochastic dynamical systems, and particularly, examine the relations between the two notions and the application of symmetries to the reduction of stochastic systems in detail. Finally, we give some concluding remarks for the related studies and the further topics.

In the present article, we formulate the notions of conserved quantities and symmetries for stochastic system in a way analogous to those in the deterministic dynamical systems; since the stochastic systems are treated, one may formulate thus notions for the system in another way. For example, Itoh (1993) and Thieullen and Zambrini (1997) are dealt with in the conserved quantities in the sense of "martingale". According to the context of their works, the conserved quantity is defined by a function $I(\boldsymbol{x}, t)$ satisfying that the stochastic process $I_t = I(\boldsymbol{x}_t, t)$ generated by the function I and a solution of (2.1), becomes a "martingale" with respect to P (it does not need to be a constant!); that is, $E[I_t \mid \mathcal{F}_s] = I_s$ $(s \le t)$ holds, where $E[\cdot \mid \cdot]$ denotes the conditional expectation under P. Apparently, our conserved quantity formulated in this paper satisfies this condition; it is also a conserved quantity in martingale sense. Therefore, in general, the conserved quantities in martingale sense are more easily found out than our ones. However, such a conserved quantity in martingale sense seems to be not similar to that in the usual deterministic dynamical systems, and hence we should investigate in detail how to use the quantity for analysis of stochastic dynamical systems in a future work.

As to the connection between conserved quantities in martingale sense and symmetries, Thieullen and Zambrini (1997) formulated the stochastic Noether's theorem in their framework of stochastic variational principle. They give the method to derive the conserved quantities in martingale sense from "symmetries of the functional" for the stochastic dynamical system under study. Therefore, it may be an interesting problem to formulate a procedure to derive conserved quantities in martingale sense from "symmetries of the stochastic differential equation" defined in our work. Conversely, it is also important to investigate both notions of conserved quantities and symmetries defined in this paper from viewpoint of variational principle.

Apart from the stochastic dynamical theory, it will be of certain interest to work with conserved quantities from viewpoint of stochastic numerical analysis. Recently, there has been increasing interest in stochastic numerical analysis to stochastic differential equations, and many important works have been proposed (e.g. Kloeden and Platen (1992)). Most of them treat the accuracy and stability of numerical solutions obtained through various numerical schemes for stochastic differential equations. On the other hand, it is well-known that studies focused on the numerical preservation of characters of dynamical systems, which are often given by the several conserved quantities, are very important in performing reliable numerical calculations to deterministic dynamical systems describing by ordinary differential equations (e.g. Greenspan (1984)). Despite this fact, however, the attempts related to such a topic in stochastic numerical analysis are very rare, and hence, it seems quite natural to investigate stochastic numerical schemes which leave the conserved quantities of stochastic systems numerically invariant. We will come back to the problem in future.

Acknowledgements

The author is very grateful to the referees for their invaluable comments and suggestions.

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