Ann. Inst. Statist. Math.
Vol. 51, No. 2, 369–382 (1999)

LIMIT THEOREMS FOR STOPPED FUNCTIONALS OF MARKOV RENEWAL PROCESSES

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(Received June 30, 1997; revised February 23, 1998)

Abstract. Some results for stopped random walks are extended to the Markov renewal setup where the random walk is driven by a Harris recurrent Markov chain. Some interesting applications are given; for example, a generalization of the alternating renewal process.

Key words and phrases: Markov renewal process, semi-Markov process, Harris recurrent, stopped sums, m-dependence, Anscombe's theorem, law of large numbers, central limit theorem, law of the iterated logarithm.

1. Introduction

The origin of the present paper is a problem from the theory of chromatography (Gut and Ahlberg (1981)) and subsequent results emerging from that problem (Gut and Janson (1983) and Gut (1988), in particular Chapter IV). We begin with a very short description of the concept chromatography; some further details are given in Section 6 (and in Gut and Ahlberg (1981), Gut and Janson (1983) and Gut (1988), Section IV.3).

The basis for chromatographic separation is the distribution of a sample of molecules between a *stationary* phase and a *mobile* phase which percolates through the stationary bed. The sample is injected onto the column and is transported along the column oscillating between the phases, causing separation of the compounds because of their different patterns of behaviour. The main results in the source(s) cited above are limiting results for the relative time spent in the mobile phase and, hence, assuming that the velocity is constant in that phase, results for the (longitudinal) distance a molecule has travelled during a given time interval.

The main feature of the present paper is that the results allow us to split the mobile phase into several layers, which makes the model more realistic. Another application that is briefly mentioned is replacement policies within the theory of reliability, for which our results permit more general "components" than those who just work or do not in that we allow for different states of deterioration.

Section 2 is devoted to some preliminaries on Markov renewal theory, after which Section 3 contains the statements of our two main results. Proofs are given in Sections 4 and 5, respectively. The applications are given in Section 6. A final section contains some remarks.

2. Preliminaries on Markov renewal theory

We begin by describing the standard setup of Markov renewal theory; for further details, see e.g. Alsmeyer (1994). Let $(\mathcal{S},\mathfrak{S})$, $(\mathcal{Y},\mathfrak{Y})$ be measurable spaces with countably generated σ-fields $\mathfrak{S}$, $\mathfrak{Y}$, respectively, $\mathfrak{B}$ the Borel σ-field on $\mathbb{R}$ and $\mathbb{P}:\mathcal{S}\times(\mathfrak{S}\otimes\mathfrak{B}\otimes\mathfrak{Y})\to[0,1]$ a transition kernel. Let further $(M_n,X_n,Y_n)_{n\geq 0}$ be an associated Markov chain, defined on a probability space $(\Omega,\mathfrak{A},P)$, with state space $\mathcal{S}\times\mathbb{R}\times\mathfrak{Y}$, that is

$$
\begin{aligned}
(2.1)\qquad & P(M_{n+1}\in A, X_{n+1}\in B, Y_{n+1}\in C\mid M_n,X_n,Y_n)\\
&\quad = \mathbb{P}(M_n, A\times B\times C)\qquad \text{a.s.}
\end{aligned}
$$

for all $n\geq 0$ and $A\in\mathfrak{S}$, $B\in\mathfrak{B}$, $C\in\mathfrak{Y}$. Thus $(M_{n+1},X_{n+1},Y_{n+1})$ depends on the past only through M_n. It is easily seen that $(M_n)_{n\geq 0}$ forms a Markov chain with state space $\mathcal{S}$ and transition kernel $\mathbb{P}^*(s,A)\stackrel{\text{def}}{=}\mathbb{P}(s,A\times\mathbb{R}\times\mathcal{Y})$. Given $(M_j)_{j\geq 0}$, the (X_n,Y_n), $n\geq 0$, are conditionally independent with

$$
(2.2)\qquad P(X_n\in B, Y_n\in C\mid (M_j)_{j\geq 0}) = Q(M_{n-1},M_n,B\times C)\qquad \text{a.s.}
$$

for all $n\geq 1$, $B\in\mathfrak{B}$, $C\in\mathfrak{Y}$ and a kernel $Q:\mathcal{S}^2\times(\mathfrak{B}\otimes\mathfrak{Y})\to[0,1]$. Let throughout a canonical model be given with probability measures $P_{s,x,y}$, $s\in\mathcal{S}$, $x\in\mathbb{R}$, $y\in\mathbb{R}$, on $(\Omega,\mathfrak{A})$ such that $P_{x,y}(M_0=s,X_0=x,Y_0=y)=1$. If λ denotes any distribution on $\mathcal{S}\times\mathbb{R}\times\mathfrak{Y}$, put $P_\lambda(\cdot)=\int_{\mathcal{S}\times\mathbb{R}\times\mathcal{Y}}P_{s,x,y}(\cdot)\lambda(ds,dx,dy)$ in which case (M_0,X_0,Y_0) has initial distribution λ under P_λ. Expectation under P_λ is denoted by E_λ. For a distribution ζ on $\mathcal{S}$ only, we write P_ζ and E_ζ for probabilities and expectations that are independent of the initial distribution of (X_0,Y_0). Finally, P and E are used for probabilities and expectations, respectively, that do not depend on the initial distribution at all.

Markov renewal theory deals with certain asymptotic properties of the *Markov random walk* $(M_n,S_n)_{n\geq 0}$, where $S_n=X_0+\cdots+X_n$ for $n\geq 0$, and related processes. In the following we only deal with the most interesting, proper renewal case, i.e. we assume that the X_j are positive; more formally:

ASSUMPTION 1. $\mathbb{P}(x,\mathcal{S}\times(0,\infty))=1$ for all $x\in\mathcal{S}$.

Now, let $N(t)=\sup\{n\geq 0: S_n\leq t\}$. We can then define

$$
(2.3)\qquad Z_0=M_0\qquad\text{and}\qquad Z_t=M_{N(t)}\qquad\text{for}\quad t>0,
$$

which is a pure jump process starting at M_0 and moving from M_{n-1} to M_n at S_n. Consequently, X_n denotes its sojourn time at M_{n-1}. Under these conditions we call $(M_n, S_n)_{n\geq 0}$ a *Markov renewal process* and $(Z_t)_{t\geq 0}$ the associated *semi-Markov process* (*SMP*) with embedded chain $(M_n)_{n\geq 0}$ and jump epochs $(S_n)_{n\geq 0}$. The additional sequence $(Y_n)_{n\geq 0}$ is a sequence of *marks* or *rewards*, which in most applications are real-valued, i.e. $(\mathcal{Y}, \mathfrak{Y}) = (\mathbb{R}, \mathfrak{B})$.

Two further assumptions will always be valid throughout this article.

ASSUMPTION 2. $(M_n, S_n)_{n\geq 0}$ is *non-explosive*, which means that $N(t) < \infty$ P_x-a.s. for all $t \geq 0$ and $x \in \mathcal{S}$.

ASSUMPTION 3. $(M_n)_{n\geq 0}$ is *Harris recurrent*, that is, if $\mathbb{P}^*_n(x, \cdot)$ denotes the n-step transition kernel of $(M_n)_{n\geq 0}$, then there exists a set $\mathfrak{R} \in \mathfrak{S}$, some $r \geq 1$, $\alpha > 0$ and a probability measure φ on $\mathfrak{R}$ such that $P_x(M_n \in \mathfrak{R} \text{ i.o.}) = 1$ for all $x \in \mathcal{S}$ and

$$\mathbb{P}^*_r(x, A) \geq \alpha\varphi(A) \quad \text{for all} \quad x \in \mathfrak{R} \text{ and } A \in \mathfrak{S}. \tag{2.4}$$

The Harris recurrence of $(M_n)_{n\geq 0}$ induces a regenerative structure on the full sequence $(M_n, X_n, Y_n)_{n\geq 0}$, possibly after redefining it on an enlarged probability space. Namely, there exists a filtration $\mathcal{F}_n$ and a strictly increasing sequence $0 = \sigma_0, \sigma_1, \sigma_2, \ldots$ of (under each $P_{s,x,y}$) a.s. finite $\mathcal{F}_n$-times, called *regeneration times*, such that:

(2.5) Under each $P_{s,x,y}$, $\tau_n \stackrel{\text{def}}{=} \sigma_n - \sigma_{n-1}$ are independent for $n \geq 1$ and even identically distributed for $n \geq 2$ with distribution $\zeta = P_\varphi(\sigma_1 \in \cdot)$ and φ as in (2.4). Hence $(\sigma_n)_{n\geq 0}$ forms an ordinary zero-delayed renewal process under P_φ.

(2.6) $(M_n, X_n, Y_n)_{n\geq 0}$ is Markov-adapted to $(\mathcal{F}_n)_{n\geq 0}$, i.e. $\sigma((M_k, X_k, Y_k)_{0\leq k\leq n}) \subset \mathcal{F}_n$ for each $n \geq 0$ and $P((M_{n+1}, X_{n+1}, Y_{n+1}) \in \cdot \mid \mathcal{F}_n) = \mathbb{P}(M_n, \cdot)$ a.s.

(2.7) Under each $P_{s,x,y}$, the *cycles* or *blocks* $C_n \stackrel{\text{def}}{=} (M_k, X_k, Y_k)_{\sigma_n\leq k<\sigma_{n+1}}$ are one-dependent for $n \geq 0$ and stationary for $n \geq 2$ with distribution $P_\varphi(C_1 \in \cdot)$. Consequently, stationarity of the full cycle sequence holds under P_η, with η defined as $\eta = P_\varphi((M_{\sigma_1}, X_{\sigma_1}, Y_{\sigma_1}) \in \cdot)$.

The construction of $(\sigma_n)_{n\geq 0}$, which has become standard by now, will not be described here. It only involves the chain $(M_n)_{n\geq 0}$ and additional coin tossing events. We refer the reader to Athreya and Ney (1978) and Asmussen (1987). Notice, however, that the cycles $C_0, C_1, \ldots$ are in general *not* independent, only one-dependent, as stated in (2.7). For exceptional cases see Athreya *et al.* (1978) and Alsmeyer (1994). Notice further that the given dependence structure of $(M_n, X_n, Y_n)_{n\geq 0}$ implies stationarity of the cycles C_n only for $n \geq 2$ under arbitrary $P_{s,x,y}$, while for the cycles

$$C^M_n \stackrel{\text{def}}{=} (M_k)_{\sigma_n\leq k<\sigma_{n+1}} \tag{2.8}$$

of the driving chain this is true for all $n \geq 1$.

Next, with η as in (2.7), we define

$$\xi = E_\eta \left(\sum_{j=0}^{\sigma_1 - 1} \mathbf{1}_{\{(M_j, X_j, Y_j) \in \cdot\}} \right), \tag{2.9}$$

which is the unique (up to a multiplicative constant) σ-finite stationary measure of the chain $(M_n, X_n, Y_n)_{n\geq 0}$. Its first marginal $\xi_M = \xi(\cdot \times [0,\infty) \times \mathcal{Y})$ then clearly defines a stationary measure of $(M_n)_{n\geq 0}$ and satisfies

$$\xi_M = E_\varphi \left(\sum_{j=0}^{\sigma_1 - 1} \mathbf{1}_{\{M_j \in \cdot\}} \right). \tag{2.10}$$

If the stationary mean cycle length $E_\eta \sigma_1 = E_\varphi \sigma_1$ is finite, $(M_n, X_n, Y_n)_{n\geq 0}$ as well as $(M_n)_{n\geq 0}$ are called ergodic, and $\xi^* \stackrel{\text{def}}{=} \xi / E_\eta \sigma_1$, $\xi_M^* \stackrel{\text{def}}{=} \xi_M / E_\eta \sigma_1$, respectively, denote their unique stationary distributions.

With $\mathcal{M} = \mathcal{M}(\mathcal{S} \times \mathbb{R} \times \mathcal{Y}, \mathfrak{S} \otimes \mathfrak{B} \otimes \mathfrak{Y})$ denoting the space of real-valued, measurable functions on $\mathcal{S} \times \mathbb{R} \times \mathcal{Y}$ we next introduce the operators $\Sigma_n : \mathcal{M} \to \mathcal{M}(\Omega, \mathfrak{A})$, defined through

$$\Sigma_n f = \sum_{j=0}^{n} f(M_j, X_j, Y_j), \qquad n \geq 0. \tag{2.11}$$

Our purpose is to prove a number of limit theorems for $\Sigma_{N(t)} f$ and $\Sigma_{T(t)} f$, as $t \to \infty$, under suitable assumptions on f and $(M_n, X_n, Y_n)_{n\geq 0}$. Here $T(t) = N(t) + 1 = \inf\{n \geq 0 : S_n > t\}$. In case where $\mathcal{S}$ consists of one state only, the chain $(M_n)_{n\geq 0}$ thus being deterministic, $(S_n)_{n\geq 0}$ reduces to an ordinary renewal process with $(Y_n)_{n\geq 0}$ being an associated sequence of general rewards. Limit results for $\Sigma_{N(t)} f$ and $\Sigma_{T(t)} f$ have then been derived in Gut and Janson (1983), see also Gut (1988), Section IV.2, and in Alsmeyer (1988) for this case.

It is tempting to conjecture that extensions to the present situation are easily achieved by utilizing the regenerative structure of the sequence $(M_n, X_n, Y_n)_{n\geq 0}$. However, this is complicated by the one-dependence of the occurring cycles $C_0, C_1, C_2, \ldots$. Renewal theorems for such sequences are largely due to Janson (1983) and will be a major tool for proving the subsequent results.

3. Results

For $n \geq 0$ we define

$$A_n = S_{\sigma_{n+1}-1} - S_{\sigma_n}, \qquad \left(\text{i.e. } \sum_{j=1}^{n-1} A_j = S_{\sigma_n - 1} \right), \tag{3.1}$$

and, given any $f \in \mathcal{M}$,

$$(3.2)\qquad \begin{aligned} B_n &= \sum_{j=\sigma_n}^{\sigma_{n+1}-1} f(M_j, X_j, Y_j), \qquad \left(\text{i.e. } \sum_{j=1}^{n-1} B_j = \Sigma_{\sigma_n - 1} f\right), \\ B_n^* &= \sum_{j=\sigma_n}^{\sigma_{n+1}-1} |f(M_j, X_j, Y_j)|. \end{aligned}$$

Notice that $(A_n, B_n, B_n^*)_{n\geq 0}$ forms a stationary one-dependent sequence under P_η. Define $\mu_A = E_\eta A_0 = E_\eta S_{\sigma_1 - 1}$ and $\mu_B = E_\eta B_0$ provided the latter quantity exists. It follows that

$$(3.3)\qquad \begin{aligned} \mu_A &= \int_{\mathcal{S}} \int_{[0,\infty)} \int_{\mathcal{Y}} x \xi(ds, dx, dy) = E_\xi X_0, \\ \mu_B &= \int_{\mathcal{S}} \int_{[0,\infty)} \int_{\mathcal{Y}} f(s, x, y) \xi(ds, dx, dy) = E_\xi f(M_0, X_0, Y_0), \end{aligned}$$

where ξ is the stationary measure in (2.9). Finally, put $\mu_{B/A} \stackrel{\text{def}}{=} \frac{\mu_B}{\mu_A}$ which thus is an integral ratio and obviously invariant under replacement of ξ by any multiple of it; in particular,

$$(3.4)\qquad \mu_{B/A} = \frac{E_{\xi^*} f(M_0, X_0, Y_0)}{E_{\xi^*} X_0}$$

if $(M_n)_{n\geq 0}$ and, hence, $(M_n, X_n, Y_n)_{n\geq 0}$ is ergodic.

THEOREM 1. *Let $f \in \mathcal{M}$ be such that $\mu_B < \infty$.*

(i) *Then, as $t \to \infty$,*

$$\frac{\Sigma_{T(t)} f}{t} \to \mu_{B/A} \qquad a.s.$$

the limit being 0 if $\mu_A = \infty$.

(ii) *If $E_\eta |A_0|^p < \infty$ and $E_\eta |B_0^*|^p < \infty$ for some $p \in (1,2)$, then, as $t \to \infty$,*

$$\frac{\Sigma_{T(t)} f - \mu_{B/A} t}{t^{1/p}} \to 0 \qquad a.s.$$

(iii) *If $E_\eta A_0^2 < \infty$, $E_\eta B_0^{*2} < \infty$ and*

$$\gamma^2 \stackrel{\text{def}}{=} \mathrm{Var}_\eta(\mu_A B_0 - \mu_B A_0) + 2\,\mathrm{Cov}_\eta(\mu_A B_0 - \mu_B A_0, \mu_A B_1 - \mu_B A_1) > 0,$$

then, as $t \to \infty$,

$$\frac{\Sigma_{T(t)} f - \mu_{B/A} t}{\gamma \mu_A^{-3/2} t^{1/2}} \xrightarrow{d} N(0,1).$$

(iv) *Under the assumptions of* (iii) *we have*

$$\limsup_{t\to\infty}\left(\liminf_{t\to\infty}\right)\frac{\Sigma_{T(t)}f-\mu_{B/A}t}{\sqrt{2t\log\log t}}=\overset{+}{(-)}\frac{\gamma}{\mu_A^{3/2}}\qquad a.s.$$

(v) *All four assertions hold equally true for* $\Sigma_{N(t)}f$.

Remark 3.1. Note that γ^2 also equals $\lim_{n\to\infty}\frac{1}{n}\operatorname{Var}_\eta(\mu_A W_n-\mu_B U_n)$ where $U_n\overset{\text{def}}{=}A_0+\cdots+A_{n-1}$ and $W_n\overset{\text{def}}{=}B_0+\cdots+B_{n-1}$ for $n\ge 1$, see Janson (1983).

Remark 3.2. In the special case when $\mathcal{S}$ is degenerate and $f(M_j,X_j,Y_j)=Y_j$, $\Sigma_{T(t)}f$ reduces to the quantity $V_{\tau(t)}$ in Gut and Janson (1983) and our results to those of Gut and Janson (1983). The results there, however, are valid for random walks $(S_n)_{n\ge0}$ on $\mathbb{R}$, such that the mean of the increments is positive, i.e. without Assumption 1.

In our second result we establish asymptotics for the mean and variance of $\Sigma_{T(t)}f$ as $t\to\infty$. The proofs consist essentially of a proper blend of the corresponding proofs in Gut and Janson (1983) (see also Gut (1988), Section IV.2) and Janson (1983). An important ingredient are asymptotics for $\tau(t)$ from Janson (1983).

THEOREM 2. *Let* $f\in\mathcal{M}$ *be such that* $\mu_B<\infty$.
(i) *Then, as* $t\to\infty$,

$$E_\eta\Sigma_{T(t)}f=\mu_{B/A}t+o(t).$$

(ii) *Suppose that* $E_\eta A_0^2<\infty$, $E_\eta B_0^{*2}<\infty$ *and set* $\gamma_A^2\overset{\text{def}}{=}\operatorname{Var}_\eta A_0+2\operatorname{Cov}_\eta(A_0,A_1)$, $\gamma_B^2\overset{\text{def}}{=}\operatorname{Var}_\eta B_0+2\operatorname{Cov}_\eta(B_0,B_1)$, $\Delta^2\overset{\text{def}}{=}\operatorname{Cov}_\eta(A_0,B_0)+\operatorname{Cov}_\eta(A_1,B_0)+\operatorname{Cov}_\eta(A_0,B_1)$. *Note that, for* γ^2 *as defined above, we have*

$$\gamma^2=\mu_A^2\gamma_B^2+\mu_B^2\gamma_A^2-2\Delta\mu_A\mu_B.$$

Then, as $t\to\infty$,

$$E_\eta\Sigma_{T(t)}f=\mu_{B/A}t+o(t^{1/2})$$
$$\operatorname{Var}_\eta\Sigma_{T(t)}f=\frac{\mu_A^2\gamma_B^2+\mu_B^2\gamma_A^2-2\Delta\mu_A\mu_B}{\mu_A^3}t+o(t)=\frac{\gamma^2}{\mu_A^3}t+o(t).$$

(iii) *All assertions hold equally true for* $\Sigma_{N(t)}f$.

Remark 3.3. In the special case where $\mathcal{S}$ is degenerate $(A_n)_{n\ge0}$ and $(B_n)_{n\ge0}$ are independent, and $\gamma_A^2=\operatorname{Var}_\eta A_0$, $\gamma_B^2=\operatorname{Var}_\eta B_0$ and $\Delta=\operatorname{Cov}_\eta(A_0,B_0)$. If, in addition, $f(M_j,X_j,Y_j)=Y_j$, then $\Sigma_{T(t)}f$, again, reduces to the quantity $V_{\tau(t)}$ in Gut and Janson (1983) and our results (essentially) to those of Gut and Janson (1983), Theorem 3, see also Gut (1988), Theorem IV.2.4 (recall Remark 3.2).

Second order approximations for this situation have been obtained in Alsmeyer (1988).

Remark 3.4. As stated Theorem 2 only holds under P_η as defined in (2.7); η the stationary cycle debut distribution. In order to make it hold under arbitrary P_λ the first cycle variable B_0^* has to be sufficiently integrable under this measure.

4. Proof of Theorem 1

Since the first block variables A_0 and B_0 are fixed random variables not depending on t, it follows immediately that it is sufficient to prove the theorem under P_η, under which $(A_n, B_n, B_n^*)_{n\geq 0}$ forms a stationary one-dependent sequence.

(i) Define $\tau(t) = \inf\{n \geq 1 : S_{\sigma_n - 1} > t\} = \inf\{n \geq 1 : A_0 + \cdots + A_{n-1} > t\}$ for $t \geq 0$ and observe that

$$W_{\tau(t)-1} - B^*_{\tau(t)-1} \leq \Sigma_{T(t)} f \leq W_{\tau(t)-1} + B^*_{\tau(t)-1}, \tag{4.1}$$

where $W_0 = 0$ and $W_n = B_0 + \cdots + B_{n-1}$ for $n \geq 1$. Now, by the stationarity and one-dependence of $(A_n, B_n, B_n^*)_{n\geq 0}$ under P_η together with the assumption that $\mu_B < \infty$ (which also gives $\mu_{B^*} < \infty$) we have the following analogs to the i.i.d. case:

$$\frac{S_{\sigma_n - 1}}{n} \to \mu_A, \quad \frac{W_n}{n} \to \mu_B, \quad \frac{B^*_{n-1}}{n} \to 0 \quad \text{and} \tag{4.2}$$
$$\frac{\tau(t)}{t} \to \frac{1}{\mu_A} \quad P_\eta\text{-a.s.},$$

see Janson (1983), Theorem 2.1(i) for the final convergence. Consequently, by introducing the overshoot $R_t \stackrel{\text{def}}{=} W_{\tau(t)} - \Sigma_{T(t)} f$ we obtain, using (4.1), $\tau(t) \uparrow \infty$ a.s. and (4.2), that

$$|R_t| = |W_{\tau(t)} - \Sigma_{T(t)} f| \leq B^*_{\tau(t)-1} = o(t) \quad P_\eta\text{-a.s.} \quad \text{as} \quad t \to \infty, \tag{4.3}$$

and, finally, that

$$\frac{\Sigma_{T(t)} f}{t} = \frac{W_{\tau(t)}}{\tau(t)} \cdot \frac{\tau(t)}{t} - \frac{R_t}{t} \to \mu_{B/A} \quad P_\eta\text{-a.s.},$$

which is the desired result.

(ii) If $E_\eta |A_0|^p < \infty$ and $E_\eta |B_0^*|^p < \infty$, the Marcinkiewicz law of large numbers yields

$$\frac{W_n - \mu_B n}{n^{1/p}} \to 0, \quad \frac{B^*_{n-1} - \mu_B}{n^{1/p}} \to 0, \quad \frac{\tau(t) - t/\mu_A}{t^{1/p}} \to 0 \quad P_\eta\text{-a.s.},$$

see Janson (1983), Theorem 2.1(iii) for the final convergence. It follows that

$$\frac{\Sigma_{T(t)} f - \mu_{B/A} t}{t^{1/p}} = \frac{W_{\tau(t)} - \mu_B \tau(t)}{\tau(t)^{1/p}} \cdot \frac{\tau(t)^{1/p}}{t^{1/p}} - \frac{R_t}{t^{1/p}}$$
$$+ \frac{\mu_B(\tau(t) - t/\mu_A)}{t^{1/p}} \to 0 \quad P_\eta\text{-a.s.}$$

(iii) Here $E_\eta A_0^2 < \infty$ and $E_\eta (B_0^*)^2 < \infty$. Since $(B_n - \mu_{B/A} A_n)_{n\geq 0}$ forms a centered, stationary one-dependent sequence under P_η, the central limit theorem for its associated random walk $(W_n - \mu_{B/A} S_{\sigma_n - 1})_{n\geq 0}$ gives

$$\frac{W_n - \mu_{B/A} S_{\sigma_n - 1}}{\gamma n^{1/2}/\mu_A} \xrightarrow{d} N(0,1), \tag{4.4}$$

with γ^2 as defined in Theorem 1(iii). Furthermore, by stationarity and the assumption that $E_\eta A_0^2 < \infty$, we have

$$\sum_{n\geq 1} P_\eta(A_{n-1} > n^{1/2}) = \sum_{n\geq 1} P_\eta(A_0 > n^{1/2}) < \infty, \tag{4.5}$$

whence $n^{-1/2} A_{n-1} \to 0$ P_η-a.s. Combined with $t^{-1}\tau(t) \to \mu_A^{-1}$ P_η-a.s., this further yields

$$0 \leq \frac{S_{\sigma_{\tau(t)}-1} - t}{t^{1/2}} \leq \frac{A_{\tau(t)-1}}{t^{1/2}} \to 0 \qquad P_\eta\text{-a.s.} \tag{4.6}$$

Since

$$\frac{\Sigma_{T(t)} f - \mu_{B/A} t}{\gamma \mu_A^{-3/2} t^{1/2}} = \frac{W_{\tau(t)} - \mu_{B/A} S_{\sigma_{\tau(t)}-1}}{\gamma \mu_A^{-3/2} t^{1/2}} + \frac{\mu_{B/A}(S_{\sigma_{\tau(t)}-1} - t)}{\gamma \mu_A^{-3/2} t^{1/2}}, \tag{4.7}$$

Anscombe's theorem applied to the first term in the right-hand side of (4.7), and (4.6) to the second one, finally yield the desired conclusion for $\Sigma_{T(t)} f$. As for Anscombe's theorem for sums of i.i.d. sequences, see Gut (1988), Theorem I.3.1. A version for m-dependent random variables is easily obtained by noting that the so called Anscombe condition (see Anscombe (1952)) is equivalent to the validity of Kolmogorov's inequality, which in turn holds for m-dependent sequences.

(iv) The proof follows the pattern of (iii). The essential modifications are that for Anscombe's theorem we now exploit Horváth (1986), Theorem 2.3 (cf. also Example 3 there) in order to conclude that

$$\limsup_{t\to\infty} \left(\liminf_{t\to\infty}\right) \frac{W_{\tau(t)} - \mu_{B/A} S_{\sigma_{\tau(t)}-1}}{\sqrt{2t \log\log t}} = \overset{+}{(-)} \frac{\gamma}{\mu_A^{3/2}} \qquad P_\eta\text{-a.s.}, \tag{4.8}$$

and that

$$\frac{\mu_{B/A}(S_{\sigma_{\tau(t)}-1} - t)}{\sqrt{t \log\log t}} \to 0 \qquad P_\eta\text{-a.s.} \tag{4.9}$$

to conclude that the overshoot is asymptoticallay negligible (cf. (4.6)). A decomposition like (4.7) finishes the proof.

(v) The analogous results for $\Sigma_{N(t)} f$ are obtained via the relation

$$|\Sigma_{T(t)} f - \Sigma_{N(t)} f| = |f(M_{T(t)}, X_{T(t)}, Y_{T(t)})| \leq B^*_{\tau(t)-1}, \tag{4.10}$$

and the fact that, by arguments like those that produced (4.5) and (4.6), we conclude that, for any $p \geq 1$, the assumption $E_\eta |B_0^*|^p < \infty$ implies that $t^{-1/p} B^*_{\tau(t)-1} \to 0$ P_η-a.s. as $t \to \infty$.

5. Proof of Theorem 2

(i) We first observe (Janson (1983), Lemma 2.1) that $E_\eta \tau(t) < \infty$, (Janson (1983), Theorem 2.3) that $W_{\tau(t)}$ throughout has moments of sufficiently high order for our arguments to hold, (Janson (1983), Theorem 2.2(i)) that

$$E_\eta \tau(t) = \frac{t}{\mu_A} + o(t) \quad \text{as} \quad t \to \infty, \tag{5.1}$$

and (Janson (1983), Theorem 1.1) that

$$E_\eta W_{\tau(t)+1} = (E\tau(t) + 1) \cdot \mu_B. \tag{5.2}$$

The first conclusion now follows from the fact that

$$|\Sigma_{T(t)} f - W_{\tau(t)+1}| \le B^*_{\tau(t)-1} + |B_{\tau(t)}| \le B^*_{\tau(t)-1} + B^*_{\tau(t)} \tag{5.3}$$

and Janson (1983), Theorem 2.2(iii), Corollary 1.1(ii), together with (5.1).

(ii) The first point is that under the present assumptions we have (Janson (1983), Theorem 2.2)

$$\begin{aligned} E_\eta \tau(t) &= \frac{t}{\mu_A} + o(t^{1/2}) \quad \text{as} \quad t \to \infty \\ \operatorname{Var}_\eta(\tau(t) + 1) &= \frac{\gamma_A^2}{\mu_A^3} t + o(t) \quad \text{as} \quad t \to \infty. \end{aligned} \tag{5.4}$$

Moreover, by Janson (1983), Corollary 1.1(vii) and (i) above we have, as $t \to \infty$,

$$\begin{aligned} E_\eta (W_{\tau(t)+1} - (\tau(t) + 1)\mu_B)^2 &= (E_\eta \tau(t) + 1) \cdot \gamma_B^2 + o(t^{1/2}) \\ &= \frac{\gamma_B^2}{\mu_A} t + o(t^{1/2}), \end{aligned} \tag{5.5}$$

and, by setting $U_n = A_0 + \cdots + A_{n-1}$, for $n \ge 1$,

$$\begin{aligned} E_\eta (U_{\tau(t)+1} - (\tau(t) + 1)\mu_A)^2 &= (E_\eta \tau(t) + 1) \cdot \gamma_A^2 + o(t^{1/2}) \\ &= \frac{\gamma_A^2}{\mu_A} t + o(t^{1/2}). \end{aligned} \tag{5.6}$$

By using the relation $xy = \frac{1}{4}((x + y)^2 - (x - y)^2)$ and (5.2) (cf. Gut and Janson (1983)) it follows that

$$\begin{aligned} &\operatorname{Cov}_\eta (U_{\tau(t)+1} - (\tau(t) + 1)\mu_A, W_{\tau(t)+1} - (\tau(t) + 1)\mu_B) \\ &\quad = (E_\eta \tau(t) + 1) \cdot \Delta + o(t^{1/2}) = \frac{\Delta}{\mu_A} t + o(t^{1/2}) \quad \text{as} \quad t \to \infty. \end{aligned} \tag{5.7}$$

Next we observe that, by Janson (1983), Theorem 2.2(ii) and Corollary 1.1(i), together with (i) above, we have

$$\begin{aligned} \operatorname{Var}_\eta U_{\tau(t)+1} &= \operatorname{Var}_\eta (U_{\tau(t)+1} - t) \le E_\eta (U_{\tau(t)+1} - t)^2 \\ &\le 2E_\eta (U_{\tau(t)} - t)^2 + 2E_\eta A^2_{\tau(t)+1} = o(t) \quad \text{as} \quad t \to \infty. \end{aligned} \tag{5.8}$$

An application of Cauchy-Schwarz' inequality and (5.5) therefore shows that

$$
\begin{aligned}
(5.9) \qquad & |\operatorname{Cov}_\eta(U_{\tau(t)+1}, W_{\tau(t)+1} - (\tau(t)+1)\mu_B)| \\
&\le \sqrt{\operatorname{Var}_\eta U_{\tau(t)+1} \cdot \operatorname{Var}_\eta(W_{\tau(t)+1} - (\tau(t)+1)\mu_B)} \\
&= o(t) \quad \text{as} \quad t \to \infty,
\end{aligned}
$$

which, together with (5.7), yields

$$
\begin{aligned}
(5.10) \qquad & \mu_A \operatorname{Cov}_\eta(\tau(t)+1, W_{\tau(t)+1} - (\tau(t)+1)\mu_B) \\
&= -(E_\eta\tau(t)+1)\cdot\Delta + o(t) = -\frac{\Delta}{\mu_A}t + o(t) \quad \text{as} \quad t \to \infty,
\end{aligned}
$$

and, hence (recall (5.4)),

$$
\begin{aligned}
(5.11) \qquad & \operatorname{Cov}_\eta(\tau(t)+1, W_{\tau(t)+1}) \\
&= \mu_B \operatorname{Var}_\eta(\tau(t)+1) - \frac{1}{\mu_A}(E_\eta\tau(t)+1)\cdot\Delta + o(t) \\
&= \mu_B\left(\frac{\gamma_A^2}{\mu_A^3}t + o(t)\right) - \frac{\Delta}{\mu_A^2}t + o(t) \\
&= \frac{\mu_B\gamma_A^2 - \Delta\mu_A}{\mu_A^3}t + o(t) \quad \text{as} \quad t \to \infty.
\end{aligned}
$$

Putting things together we thus obtain (cf. Gut and Janson (1983), formula (3.9) for the first equality sign),

$$
\begin{aligned}
(5.12) \qquad \operatorname{Var}_\eta W_{\tau(t)+1} &= (E_\eta\tau(t)+1)\gamma_B^2 - \mu_B^2 \operatorname{Var}_\eta(\tau(t)+1) \\
&\quad + 2\mu_B \operatorname{Cov}_\eta(\tau(t)+1, W_{\tau(t)+1}) \\
&= \left(\frac{1}{\mu_A}\gamma_B^2 - \mu_B^2\frac{\gamma_A^2}{\mu_A^3} + 2\mu_B\cdot\frac{\mu_B\gamma_A^2 - \Delta\mu_A}{\mu_A^3}\right)t + o(t) \\
&= \frac{\mu_A^2\gamma_B^2 + \mu_B^2\gamma_A^2 - 2\Delta\mu_A\mu_B}{\mu_A^3}t + o(t) \quad \text{as} \quad t \to \infty.
\end{aligned}
$$

A final appeal to (5.3) in conjuction with Janson (1983), Corollary 1.1(i) and (5.4) now shows that

$$
(5.13) \qquad E_\eta(\Sigma_{T(t)}f - W_{\tau(t)+1})^2 = o(t) \quad \text{as} \quad t \to \infty
$$

and the desired conclusion follows.

(iii) This part follows from (4.10), Janson (1983), Corollary 1.1(i) and the first part of the theorem.

The proof of the theorem thus is complete.

6. Applications

As mentioned in the introduction, the basis for chromatographic separation is the distribution of a sample of molecules between a stationary phase and a mobile phase and that the separation between the compounds is caused by their different molecular behaviours as they travel along the column. The essential idea of the present paper is that the results above allow us to consider a more general model (the results were, in reality, motivated by the idea), in that the mobile phase now can be divided into several layers, in the sense that the probability of being sorbed into the stationary phase within a small time interval is larger for a molecule that is "closer" to the stationary phase than for a molecule more "in the center" of the mobile phase. Let us, for simplicity, divide the mobile phase into two different layers, thus introducing a Markov chain with three states, 0, 1 and 2, respectively, corresponding to the stationary phase, one part of the mobile phase that is "near" the stationary phase and "the central part" of the mobile phase. The corresponding transition matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ p & 0 & q \\ 0 & 1 & 0 \end{pmatrix},$$

where $0 < p = 1 - q < 1$. (The extreme case $p = 1$ corresponds to the case with only two phases.) Further, let $(X_n^{(0)})_{n\geq 0}$, $(X_n^{(1)})_{n\geq 0}$, and $(X_n^{(2)})_{n\geq 0}$ be the random variables corresponding to the durations in the states, and assume that all durations are independent, and i.i.d. within each sequence with means μ_i and variances σ_i^2, $i = 0, 1, 2$, respectively. All random variables without indices are generic.

Suppose that we wish to compute the relative time spent in state 0 during the interval $(0, t]$ starting at 0. An identification shows that X_k then equals one of $X_k^{(0)}$, $X_k^{(1)}$ or $X_k^{(2)}$, for all k, depending on the state of the Markov chain, and, for all j, $f(M_j, X_j, Y_j)$ equals $X_j^{(0)}$ when $X_j = X_j^{(0)}$ and 0 otherwise. Furthermore, the regeneration times are the times at which a transition to state 0 occurs. An A-block has the form

$$X^{(0)} + X^{(1)} + \sum_{k=0}^{L} (X_k^{(2)} + X_k^{(1)}), \tag{6.1}$$

where L has a geometric distribution with $EL = q/p$, and where, in particular, L is independent of the sequences of duration times. Namely, the first visit in state 0 is followed by a visit in state 1, after which there are a geometric number of "state 2+ state 1" visits before state 0 is visited again. A B-block is of the form $X^{(0)}$. Note, in particular, that $(X_k)_{k\geq 0}$ are neither independent nor identically distributed. The A- and B-blocks, however, are equally distributed (within each sequence); in this example the blocks are, in fact, independent (within each sequence).

It follows that

$$\mu_A = \mu_0 + \mu_1 + \frac{q}{p}(\mu_2 + \mu_1) = \mu_0 + \frac{\mu_1}{p} + \frac{q\mu_2}{p}, \quad \text{and} \quad \mu_B = \mu_0, \tag{6.2}$$

and, hence, that

$$\mu_{B/A} = \left(1 + \frac{\mu_1}{p\mu_0} + \frac{q\mu_2}{p\mu_0}\right)^{-1}. \tag{6.3}$$

Furthermore, since by independence

$$\mathrm{Cov}_\eta(\mu_A B_0 - \mu_B A_0, \mu_A B_1 - \mu_B A_1) = 0, \tag{6.4}$$

it follows that

$$\begin{aligned} \gamma^2 &= \mathrm{Var}\left((\mu_B - \mu_A)X^{(0)} + \mu_B X^{(1)} + \mu_B \sum_{k=0}^{L}(X^{(2)} + X^{(1)})\right) \\ &= (\mu_B - \mu_A)^2\sigma_0^2 + \mu_B^2\sigma_1^2 + \mu_B^2\{EL(\sigma_1^2 + \sigma_2^2) + (\mu_1^2 + \mu_2^2)^2 \,\mathrm{Var}\, L\} \\ &= \left(\frac{\mu_1}{p} + \frac{q\mu_2}{p}\right)^2 \sigma_0^2 + \mu_0^2\sigma_1^2 + \mu_0^2\left\{\frac{q}{p}(\sigma_1^2 + \sigma_2^2) + (\mu_1 + \mu_2)^2\frac{q}{p^2}\right\} \\ &= \frac{\sigma_0^2}{p^2}(\mu_1 + q\mu_2)^2 + \frac{q\mu_0^2}{p^2}(\mu_1 + \mu_2)^2 + \frac{\mu_0^2}{p}(\sigma_1^2 + q\sigma_2^2), \end{aligned} \tag{6.5}$$

and, hence, that

$$\frac{\gamma^2}{\mu_A^3} = \frac{\frac{\sigma_0^2}{p^2}(\mu_1 + q\mu_2)^2 + \frac{q\mu_0^2}{p^2}(\mu_1 + \mu_2)^2 + \frac{\mu_0^2}{p}(\sigma_1^2 + q\sigma_2^2)}{\left(\mu_0 + \frac{\mu_1}{p} + \frac{q\mu_2}{p}\right)^3}. \tag{6.6}$$

In particular, for the boundary case $p = 1$, the above expressions reduce to the known ones; cf. e.g. Gut (1988), Section IV.3.3. For example, γ^2 then becomes $\sigma_0^2\mu_1^2 + \sigma_1^2\mu_0^2$.

In the chromatographic problem one assumes that the durations are exponential. With $\mu_i = \lambda_i^{-1}$ for $i = 1, 2, 3$, we then have

$$\mu_A = \frac{1}{\lambda_0} + \frac{1}{p\lambda_1} + \frac{q}{p\lambda_2}, \qquad \mu_B = \frac{1}{\lambda_0}, \qquad \mu_{B/A} = \left(1 + \frac{\lambda_0}{p\lambda_1} + \frac{q\lambda_0}{p\lambda_2}\right)^{-1},$$

$$\gamma^2 = \frac{1}{p^2\lambda_0^2}\left(\frac{1}{\lambda_1} + \frac{q}{\lambda_2}\right)^2 + \frac{q}{p^2\lambda_0^2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^2 + \frac{1}{p\lambda_0^2}\left(\frac{1}{\lambda_1^2} + \frac{q}{\lambda_2^2}\right).$$

In the degenerate case ($p = 1$) the formulas, once again, reduce to the known ones. To finish off it suffices to observe that we need analogs of Gut (1988), formulas (3.8) and (3.9), page 120. However, our results provide asymptotics for $\Sigma_{T(t)} f$, as $t \to \infty$ and the discrepancy between that quantity and the time spent in state 0 is dominated (in absolute value) by the last B-block (recall (4.1) above), and the negligibility of this block has already been established in Sections 3 and 5. Having identified all parameters involved, it thus follows, for example, that the time spent in state 0 is asymptotically $N(\mu_{B/A}t, \frac{\gamma^2}{\mu_A^3}t)$-distributed as $t \to \infty$, where $\mu_{B/A}$ and $\frac{\gamma^2}{\mu_A^3}$ are as given above.

In the chromatographic example we assume that the (longitudinal) velocities are v_1 and v_2 in phases 1 and 2, respectively, and, as before, that the durations are exponential. An A-block then, again, corresponds to $X^{(0)} + X^{(1)} + \sum_{k=0}^{L}(X_k^{(2)} + X_k^{(1)})$, and a B-block to $v_1 X^{(1)} + \sum_{k=0}^{L}(v_2 X_k^{(2)} + v_1 X_k^{(1)})$. Thus,

$$\mu_A = \frac{1}{\lambda_0} + \frac{1}{p\lambda_1} + \frac{q}{p\lambda_2}, \qquad \mu_B = \frac{v_1}{p\lambda_1} + \frac{qv_2}{p\lambda_2}, \qquad \mu_{B/A} = \frac{\frac{v_1}{p\lambda_1} + \frac{qv_2}{p\lambda_2}}{\frac{1}{\lambda_0} + \frac{1}{p\lambda_1} + \frac{q}{p\lambda_2}},$$

and, invoking independence (cf. (6.4)), computations analogous to those above yield

$$\gamma^2 = \frac{2}{p^2\lambda_0^2}\left(\frac{v_1^2}{\lambda_1^2} + \frac{qv_2^2}{\lambda_2^2} + \frac{qv_1v_2}{\lambda_1\lambda_2}\right) + \frac{q(2+q)(v_2 - v_1)^2}{p^2\lambda_1^2\lambda_2^2} + \frac{4qv_1(v_2 - v_1)}{p^2\lambda_0\lambda_1^2}.$$

By arguments like those above it is now obvious how to formulate asymptotics for the (longitudinal) distance travelled by a molecule at time t as $t \to \infty$.

Let us also briefly mention replacement policies. Some conclusions are discussed in Gut and Janson (1983) (see also Gut (1988), Section IV.3). The remarks below are also inspired by Kao (1973).

Suppose we have $d \geq 3$ different states, $0, 1, 2, \ldots, d-1$, where states 0 and $d-1$ correspond to the device being perfect and dead, respectively, and the intermediate states correspond to the successive states of deterioration, with the transition matrix $\mathbb{P} = (p_{ij})$ describing the evolution of the deterioration. (The case $d = 2$ would correspond to the classical case.) A typical example would be a device that consists of $(d-1)$ smaller components, and the states denote the number of defect ones. Our results can now be used to find asymptotics for the relative time spent in a given state, or, associating costs to the states, for the total cost at some large time point.

One further example is shock models, where the analogous extension is discussed in Boshuizen and Gouweleeuw (1993), p. 839. For the case of independent shocks, see also Gut (1990).

7. Complements

7.1 *Finiteness of moments*

Our main results above are devoted to the classical limit theorems and to asymptotics of mean and variance. One can of course do more. For example, by combining (4.1) and Janson (1983), Theorem 1.3 and Corollary 1.1 one can easily show that for any p, $1 \leq p < \infty$, we have

$$E_\eta \tau(t)^p < \infty, \; E_\eta|A_0|^p < \infty, \; E_\eta|B_0^*|^p < \infty \Longrightarrow E_\eta|\Sigma_{T(t)} f|^p < \infty.$$

Results of this kind on uniform integrability are also obtainable.

7.2 *On the assumptions*

The moment assumptions in the theorems are in terms of moments of the *blocks*. It would be more pleasant to phrase them in terms of moment assumptions on X_0 and $f(M_0, X_0, Y_0)$ under the stationary measure ξ of (2.9). Since the blocks consist of stopped sums we have no general solution to this problem without additional information (except for the first moment, cf. (3.3)); if, for example, the sequences are martingales or sums of i.i.d. random variables one may apply known inequalities for such objects. The generalization of the alternating renewal process in Section 5 is one such example.

7.3 *Extensions*

The results can easily be extended to the perturbed case analogous to Gut (1992), where stopped perturbed random walks are treated.

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