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SECOND ORDER EXPANSIONS FOR THE MOMENTS OF MINIMUM POINT OF AN UNBALANCED TWO-SIDED NORMAL RANDOM WALK*

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Abstract. In this paper, the second order expansions for the first two moments of the minimum point of an unbalanced two-sided normal random walk are obtained when the drift parameters approach zero. The basic technique is the uniform strong renewal theorem in the exponential family. The comparison with numerical values shows that the approximations are very accurate. It is shown, particularly, that the first moment is significantly different from its continuous Brownian motion analog while the second moments are the same in the first order. The results can be used to study properties of the maximum likelihood estimator for the change point.

Key words and phrases: Brownian motion, ladder height and ladder epoch, strong renewal theorem.

1. Introduction

Let $\{Z_i\}$ be independent and identically distributed normal variables with variance 1 and mean $\theta_0 < 0$ for $i < 0$ and $\theta > 0$ for $i > 0$. We define the following two-sided normal random walk:

$$W_n = \begin{cases} \sum_{i=1}^{n} Z_i & \text{for} \quad n > 0 \\ 0 & \text{for} \quad n = 0 \\ -\sum_{i=n}^{-1} Z_i & \text{for} \quad n < 0 \end{cases}$$

and denote the minimum point ν_0 of W_n as

$$W_{\nu_0} = \min_{-\infty < n < \infty} W_n.$$

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The behavior of ν_0 is of particular interest in the change point problem as it can be considered as the maximum likelihood estimator of the change point in the large sample case (Hinkley (1971) and Siegmund (1988)).

The goal of this paper is to give second order expansions for the first two moments of ν_0 when the drift parameters θ_0 and θ approach zero at the same order. These results can be used to study the bias and variance of the maximum likelihood estimator for the change point. The strong renewal theorem will be used extensively in our discussion. It is shown in particular that the first moment of ν_0 is fundamentally different from its continuous Brownian motion analog although the second moments are the same in the symmetric case.

For the convenience of presentation, the following standard notation will be used. Denote

$$m_0 = \inf_{-\infty<n<0} W_n = -\sup_{-\infty<n<0}(-W_n) = -M_0; \quad \text{and} \quad m_1 = \inf_{0<n<\infty} W_n,$$

and

$$m_0' = \min(0, m_0), \quad M_0' = \max(0, M_0) \quad \text{and} \quad m_1' = \min(0, m_1).$$

Further, we denote by ν_2 the minimum point of $S_n = W_n$ for $n \geq 0$, i.e. $W_{\nu_2} = m_1$, and

$$\begin{aligned} \tau_x &= \inf\{n > 0 : S_n \leq x\}, && \text{for} \quad x < 0; \\ \tau_x &= \inf\{n > 0 : S_n > x\}, && \text{for} \quad x > 0; \\ \tau_- &= \inf\{n > 0 : S_n \leq 0\}, && \text{and} \quad \tau_+ = \inf\{n > 0 : S_n > 0\}. \end{aligned}$$

We also denote by $R_x = S_{\tau_x} - x$ the overshoot at the crossing time at the boundary x, and $R_\infty = \lim_{x\to\infty}(S_{\tau_x} - x)$. Finally, for notational convenience, we denote by $P_\theta(\cdot)$ the probability measure associated with the random walk $\{S_n\}$ for $n \geq 0$, and $P(\cdot)$ the probability measure associated with $\{W_n\}$ for $-\infty < n < \infty$ when there is no confusion.

The rest of the paper is organized as follows. In Section 2, we first give some preliminary results related to strong renewal theorem about the crossing time τ_x, the overshoot R_x and their covariance. The second order expansions for $E\nu_0$, $E|\nu_0|$ and $E\nu_0^2$ will be given in Sections 3 and 4. Continuous analog in the continuous Brownian motion case is given in Section 5 along with some numerical comparisons. The results show that $E|\nu_0|$'s are significantly different while $E\nu_0^2$'s are the same in the first order.

2. Some preliminary results from renewal theory

Under the assumption of strong nonlattice, i.e.

$$\lim_{|t|\to\infty} \sup |E_\theta \exp(itZ_1)| < 1,$$

Stone (1965) showed that the convergence rate for the renewal function is exponentially fast. Siegmund (1979) further showed that in the exponential family, the result holds uniformly in a neighborhood of 0 if the baseline distribution function is strong nonlattice. This covers our case since the carrier distribution function has a continuous component.

The first result gives the convergence rate for the distribution function of the overshoot R_x given by Theorem 2.2 of Chang (1992).

THEOREM 1. *There exist positive* θ^*, r *and* C *such that*

$$|P_\theta(R_x < y) - P_\theta(R_\infty < y)| \le Ce^{-r(x+y)},$$

uniformly for $\theta \in [0, \theta^*]$ *and* $x, y \ge 0$.

The second result gives the convergence rate for the boundary crossing probability $P_\theta(\tau_{-x} < \infty)$ which can be proved by using Wald's likelihood ratio identity and Theorem 1. Related results are given by Carlsson (1983) and Klass (1983).

THEOREM 2. *There exist* $\theta^* > 0$ *and positive constants* C *and* r *such that uniformly for* $\theta \in [0, \theta^*]$,

$$|P_\theta(\tau_{-x} < \infty)/e^{-2\theta x} E_\theta e^{-2\theta R_\infty} - 1| \le Ce^{-rx}.$$

The third result is a modified version of Theorem 3.1 of Chang (1992) which generalizes a result of Lai and Siegmund (1979) about the limit of the covariance between the crossing time and the overshoot.

THEOREM 3. *There exist positive* θ^*, r *and* C *such that uniformly for* $\theta \in [0, \theta^*]$,

$$\left|\theta \operatorname{Cov}_\theta(\tau_x, R_x) - \int_0^\infty (E_\theta R_x - E_\theta R_\infty) P_\theta(\tau_{-x} = \infty) dx\right| \le Ce^{-rx}.$$

Related second order expansion can be seen in Alsmeyer (1988). In the next two sections, we shall use these results to derive the second order expansions for the first and second order moments for ν_0.

3. Second order expansion of $E(\nu_0)$ and $E|\nu_0|$

We first write

$$\begin{aligned} E(\nu_0) &= E[\nu_0; \nu_0 < 0] + E[\nu_0; \nu_0 > 0] \\ &= E[\nu_0; m_0 < m_1'] + E[\nu_0; m_1 < m_0']. \end{aligned}$$

As the two terms on the right hand side have similar structures, we shall only give the detailed derivation for $E[\nu_0; m_1 < m_0']$. From the strong Markov property of $S_n = \sum_{i=1}^n Z_i$ for $n > 0$, we can write

$$\begin{aligned} E[\nu_0; m_1 < m_0'] &= E[\nu_0; \tau_{m_0'} < \infty] \\ &= E[\tau_{m_0'}; \tau_{m_0'} < \infty] + E[\nu_2] P(\tau_{m_0'} < \infty), \end{aligned} \tag{3.1}$$

where we assume that $\tau_0 = \tau_-$ when $m_0' = 0$ as in the following discussion.

We first give some expansions related to the ladder time τ_-.

LEMMA 4. *For* $0 < \theta \to 0$,

$$P_\theta(\tau_- = \infty) = \sqrt{2}\theta e^{-\theta\rho}(1 + O(\theta^3)),$$
$$E_\theta[\tau_-; \tau_- < \infty] = \frac{1}{\sqrt{2}\theta} e^{-\theta\rho}\left(1 - \frac{\theta^2}{2} + O(\theta^3)\right),$$

where $\rho = E_0 R_\infty \approx 0.583$.

The result can be proved by using (10.26) of Siegmund (1985) or Theorem 4.2 of Chang (1992) combined with Wald's likelihood ratio identity.

Next three lemmas give the second order expansions for the three terms in (3.1) respectively.

LEMMA 5. *As* $0 < \theta \to 0$,

$$E_\theta(\nu_2) = \frac{E_\theta[\tau_-; \tau_- < \infty]}{P_\theta(\tau_- = \infty)} = \frac{1}{2\theta^2} - \frac{1}{4} + O(\theta).$$

PROOF. We only have to note that ν_2 is a geometric sum of iid random variables with the same distribution as $[\tau_-; \tau_- < \infty]$ and stopping probability $P_\theta(\tau_- = \infty)$.

LEMMA 6. *As* $\theta, |\theta_0| \to 0$ *at the same order*,

$$P(\tau_{m_0'} < \infty) = -\frac{\theta_0}{\theta - \theta_0} + \theta\theta_0 + o(\theta_0^2).$$

PROOF. We first write

$$\begin{aligned} P(\tau_{m_0'} < \infty) \\ = P_\theta(\tau_- < \infty)P_{\theta_0}(\tau_+ = \infty) + P(\tau_{-M_0} < \infty; M_0 > 0). \end{aligned} \tag{3.2}$$

The first term of (3.2) can be evaluated by using Lemma 4 and the symmetric property of the normal distribution. For the second term, by using Theorem 2 and (10.6) of Siegmund (1985), it can be shown that as $x \to \infty$, $0 < \theta \to 0$,

$$P_\theta(\tau_{-x} < \infty) = e^{-2\theta(x+\rho)}(1 + o(\theta^2) + O(e^{-rx})). \tag{3.3}$$

By using (3.3), we can write

$$\begin{aligned} - P(\tau_{-M_0} < \infty, M_0 > 0) \\ = \int_0^\infty P_\theta(\tau_{-x} < \infty) dP_{\theta_0}(\tau_x < \infty) \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-2\theta(x+\rho)} de^{2\theta_0(x+\rho)} \\
&+ \int_0^\infty (P_\theta(\tau_{-x} < \infty) - e^{-2\theta(x+\rho)}) de^{2\theta_0(x+\rho)} \\
&+ \int_0^\infty e^{-2\theta(x+\rho)} d(P_{\theta_0}(\tau_x < \infty) - e^{2\theta_0(x+\rho)}) \\
&+ \int_0^\infty (P_\theta(\tau_{-x} < \infty) - e^{-2\theta(x+\rho)}) d(P_{\theta_0}(\tau_x < \infty) - e^{2\theta_0(x+\rho)}).
\end{aligned}
$$

The first term of (3.4) can be obtained by routine calculation. For the fourth term, by using the Wald's likelihood ratio identity and the symmetric property of the normal random walk, we have

$$
\begin{aligned}
P_\theta(\tau_{-x} < \infty) - e^{-2\theta(x+\rho)} &= e^{-2\theta(x+\rho)}(E_\theta e^{-2\theta(R_x-\rho)} - 1) \\
&= -2\theta E_0(R_x - \rho) + o(\theta),
\end{aligned}
$$

as $\theta \to 0$. From Theorem 1, we have

$$
|E_\theta R_x - E_\theta R_\infty| \leq C e^{-rx},
$$

for some $C, r > 0$ uniformly for $\theta \in [0, \theta^*)$ for some $\theta^* > 0$. This implies that $E_0(R_x - \rho)$ is integrable as $E_0 R_\infty = \rho$. On the other hand, from the Appendix we can show

$$
\begin{aligned}
\text{(3.5)} \qquad \frac{d}{dx}(P_{\theta_0}(\tau_x < \infty) - e^{2\theta_0(x+\rho)}) &= 2\theta_0(E_0 R_0 f_{R_x}(0) - 1) + o(\theta_0) \\
&= 2\theta_0 \frac{d}{dx} E_0 R_x + o(\theta_0),
\end{aligned}
$$

where $f_{R_x}(y)$ is the density function of R_x. Thus, the fourth term of (3.4) equals to

$$
\begin{aligned}
&-4\theta\theta_0 \int_0^\infty E_0(R_x - \rho) dE_0(R_x - \rho) + o(\theta_0^2) \\
&= 2\theta\theta_0(E_0 R_0 - \rho)^2 + o(\theta_0^2) \\
&= 2\theta\theta_0 \left(\frac{1}{\sqrt{2}} - \rho\right)^2 + o(\theta_0^2),
\end{aligned}
$$

since $E_0 R_0 = 1/\sqrt{2}$.

Similarly, the second term of (3.4) is equal to

$$
-4\theta\theta_0 \int_0^\infty E_0(R_x - \rho) dx + o(\theta_0^2).
$$

The third term of (3.4) can be rewritten by integration by part as

$$
-e^{-2\theta\rho}(P_{\theta_0}(\tau_+ < \infty) - e^{2\theta_0\rho}) - \int_0^\infty (P_{\theta_0}(\tau_x < \infty) - e^{2\theta_0(x+\rho)}) de^{-2\theta(x+\rho)}.
$$

The second term on the right hand side of the above equation is approximately equal to

$$4\theta_0\theta\int_0^\infty E_0(R_x-\rho)dx+o(\theta^2),$$

which is canceled with the second term of (3.4). The proof is completed by combining the above results and some simplifications.

LEMMA 7. *As* $|\theta_0|$ *and* $\theta\to 0$ *at the same order,*

$$E[\tau_{m_0'};\tau_{m_0'}<\infty]=-\frac{\theta_0}{2\theta(\theta-\theta_0)^2}-\frac{\theta_0}{2\theta}+o(1).$$

PROOF. Since the proof is similar to that of Lemma 6, we only give the main steps. Write

$$\begin{aligned}E[\tau_{m_0'},\tau_{m_0'}<\infty]&\\ &=E_\theta[\tau_-;\tau_-<\infty]P_{\theta_0}(\tau_+=\infty)\\ &\quad-\int_0^\infty E_\theta(\tau_{-x};\tau_{-x}<\infty)dP_{\theta_0}(\tau_x<\infty).\end{aligned}\tag{3.6}$$

The first term of (3.6) can be evaluated by using Lemma 4. By using Theorem 2 and Wald's likelihood ratio identity again, we can write the second term of (3.6) as

$$\begin{aligned}&\int_0^\infty E_\theta[\tau_{-x};\tau_{-x}<\infty]dP_{\theta_0}(\tau_x<\infty)\\ &=\int_0^\infty E_{-\theta}[\tau_{-x}e^{-2\theta(x+R_x)}]dP_\theta(\tau_x<\infty)\\ &=\int_0^\infty E_{-\theta}(\tau_{-x})e^{-2\theta(x+\rho)}de^{2\theta_0(x+\rho)}\\ &\quad+\int_0^\infty E_{-\theta}(\tau_{-x})e^{-2\theta(x+\rho)}d(P_{\theta_0}(\tau_x<\infty)-e^{2\theta_0(x+\rho)})\\ &\quad+\int_0^\infty E_{-\theta}[\tau_{-x}(e^{-2\theta(R_x-\rho)}-1)e^{-2\theta(x+\rho)}]de^{2\theta_0(x+\rho)}\\ &\quad+\int_0^\infty E_{-\theta}[\tau_{-x}(e^{-2\theta(R_x-\rho)}-1)]\\ &\qquad\cdot e^{-2\theta(x+\rho)}d(P_{\theta_0}(\tau_x<\infty)-e^{2\theta_0(x+\rho)}).\end{aligned}\tag{3.7}$$

The first term of (3.7) is equal to

$$\begin{aligned}&\frac{2\theta_0}{\theta}e^{-2(\theta-\theta_0)\rho}\int_0^\infty(x+E_\theta R_x)e^{-2(\theta-\theta_0)x}dx\\ &=\frac{2\theta_0}{\theta}e^{-2(\theta-\theta_0)\rho}\left(\frac{1}{4(\theta-\theta_0)^2}+\frac{\rho}{2(\theta-\theta_0)}+\int_0^\infty(E_0R_x-\rho)dx+o(1)\right).\end{aligned}$$

By using (3.5), the second term of (3.7) is equal to

$$\frac{2\theta_0}{\theta}\int_0^\infty (x+E_0R_x)d(E_0(R_x-\rho))+o(1)$$
$$=\frac{2\theta_0}{\theta}\left(-\int_0^\infty (E_0R_x-\rho)dx-\frac{1}{2}\left(\frac{1}{2}-\rho^2\right)+o(1)\right).$$

For the third term of (3.7), we first note from Theorems 1 and 3 that as $\theta\to 0$ by the dominated convergence theorem,

$$\theta\,\mathrm{Cov}_\theta(\tau_x;R_x)=2\theta\int_0^\infty (x+E_0R_x)(E_0R_x-E_0R_\infty)dx+o(\theta)+O(e^{-rx}).$$

Thus, as $\theta\to 0$, the third term of (3.7) is equals to

$$4\theta\theta_0\int_0^\infty E_\theta(\tau_x(R_x-\rho))E^{-2(\theta-\theta_0)(x+\rho)}dx+o(\theta^2)$$
$$=4\theta\theta_0\left[\int_0^\infty \mathrm{Cov}_\theta(\tau_x,T_x)e^{-2(\theta-\theta_0)(x+\rho)}dx\right.$$
$$\left.+\int_0^\infty E_\theta(\tau_x)(E_\theta R_x-\rho)e^{-2(\theta-\theta_0)(x+\rho)}dx+o(1)\right]$$
$$=O(\theta).$$

Similarly, one can prove that the fourth term of (3.7) is $O(\theta^2)$. The proof is completed by combining the above approximations.

Combining the results of Lemmas 4 and 5–7, we get

THEOREM 8. *As $|\theta_0|$ and θ approach zero at the same order,*

$$E[\nu_0]=-\frac{\theta^2-\theta_0^2}{2\theta^2\theta_0^2}+\frac{1}{4}\frac{\theta+\theta_0}{\theta-\theta_0}+o(1).$$

In particular, if $\theta=-\theta_0\to 0$,

$$E|\nu_0|=\frac{3}{8\theta^2}-\frac{1}{4}+o(1).$$

4. Second order expansion for $E(\nu_0^2)$

In this section, we further explore the techniques used in the last section and derive the second order expansion for $E(\nu_0^2)$. As the techniques are basically the same, we only provide the details in the balanced case by assuming that $\theta=-\theta_0=\delta/2\to 0$. For notational convenience, we shall drop the subscript for the parameter when there is no confusion. Same notations will be used. First, from the strong Markov property of W_n, we have

$$\begin{aligned}
E[\nu_0^2] &= E[\nu_0^2; \nu_0 < 0] + E[\nu_0^2; \nu_0 > 0] \\
&= 2E[\nu_0^2; m_0 < m_1'] \\
&= 2E[(\tau_{m_0'} + \nu_2)^2; \tau_{m_0'} < \infty] \\
&= 2[E[\tau_{m_0'}^2; \tau_{m_0'} < \infty] + 2E[\tau_{m_0'}\nu_2; \tau_{m_0'} < \infty] + E[\nu_2^2; \tau_{m_0'} < \infty]] \\
&= 2[E[\tau_{m_0'}^2; \tau_{m_0'} < \infty] + 2E[\nu_2]E[\tau_{m_0'}; \tau_{m_0'} < \infty] + E[\nu_2^2]P(\tau_{m_0'} < \infty)].
\end{aligned}$$

Therefore, there are five terms to be evaluated separately. Approximations for $E(\nu_2)$, $P(\tau_{m_0'} < \infty)$ and $E[\tau_{m_0'}; \tau_{m_0'} < \infty]$ have been given in Section 3. The expansions for the other two terms will be given after three lemmas. The first lemma comes from Chow *et al.* (1965) and Theorems 5.2 and 5.3 of Gut (1988).

LEMMA 9. *For any stopping time N adapted to S_n such that $EN^2 < \infty$,*

$$EN^2 = \frac{4}{\delta^2}\left[\frac{2}{\delta}ES_N + \delta E(NS_N) - ES_N^2\right].$$

The second lemma is given in Lemma 10.27 of Siegmund (1985) and further extended in Theorem 4.3 of Chang (1992).

LEMMA 10. *As $\delta \to 0$,*

$$\frac{\delta}{2}E_\theta[\tau_+ S_{\tau_+}] \to \frac{1}{2}E_0 S_{\tau_+}^2 = \frac{\rho}{\sqrt{2}};$$

$$E_\theta S_{\tau_+} = \frac{1}{\sqrt{2}}e^{(\delta/2)\rho} + O(\delta^3),$$

where $\rho = 0.583$.

LEMMA 11.

$$E_\theta[\tau_-^2; \tau_- < \infty] = \frac{4\sqrt{2}}{\delta^3}e^{(\delta/2)\rho}\left(1 - \frac{\delta}{\sqrt{2}}\right) + O\left(\frac{1}{\delta}\right).$$

PROOF. From Wald's likelihood ratio identity and symmetry, we have

$$\begin{aligned}
E[\tau_-^2; \tau_- < \infty] &= E[\tau_+^2 e^{-\delta S_{\tau_+}}] = E[\tau_+^2] - \delta E[\tau_+^2 S_{\tau_+}] + O\left(\frac{1}{\delta}\right) \\
&= E[\tau_+^2] - \delta E(\tau_+^2)ES_{\tau_+} - \delta\,\mathrm{Cov}(\tau_+^2, S_{\tau_+}) + O\left(\frac{1}{\delta}\right) \\
&= E[\tau_+^2] - \delta E(\tau_+^2)ES_{\tau_+} + O\left(\frac{1}{\delta}\right),
\end{aligned}$$

as the covariance between S_{τ_+} and τ_+^2 is of smaller order. From Lemma 9, we have

$$E[\tau_+^2] = \frac{4}{\delta^2}\left[\frac{2}{\delta}ES_{\tau_+} + \delta E(\tau_+ S_{\tau_+}) - ES_{\tau_+}^2\right]$$
$$= \frac{4}{\delta^2}\left[\frac{2}{\delta}\frac{1}{\sqrt{2}}(e^{(\delta/2)\rho} + o(\delta^2)) + o(1)\right] = \frac{4\sqrt{2}}{\delta^3}[e^{(\delta/2)\rho} + o(\delta)].$$

Thus,

$$E[\tau_-^2; \tau_- < \infty] = \frac{4\sqrt{2}}{\delta^3}e^{(\delta/2)\rho}\left(1 - \frac{\delta}{\sqrt{2}}\right) + O\left(\frac{1}{\delta}\right).$$

This completes the proof.

By using the same argument as in the proof for Lemma 5, we have

LEMMA 12.

$$E_\theta[\nu_2^2] = \frac{E_\theta[\tau_-^2; \tau_- < \infty]}{P_\theta(\tau_- = \infty)} + 2\left(\frac{E_\theta[\tau_-; \tau_- < \infty]}{P_\theta(\tau_- = \infty)}\right)^2$$
$$= \frac{8}{\delta^4} + \frac{8}{\delta^4}e^{-\delta(1/\sqrt{2}-\rho)} + O\left(\frac{1}{\delta^2}\right).$$

LEMMA 13. *As* $\theta \to 0$,

$$E_\theta[\tau_{m_0'}^2; \tau_{m_0'} < \infty] = \frac{3}{16\theta^4} + O\left(\frac{1}{\delta^2}\right).$$

PROOF. We first write

$$E[\tau_{m_0'}^2; \tau_{m_0'} < \infty]$$
$$= E[\tau_-^2; \tau_- < \infty]P(\tau_- = \infty) - \int_0^\infty E[\tau_{-x}^2; \tau_{-x} < \infty]dP(\tau_{-x} < \infty).$$

The first term can be shown at the order $O(\frac{1}{\delta^2})$. By using (3.2) and Wald's likelihood ratio identity again, we can write the second term as

$$(4.1)\quad \int_0^\infty E[\tau_x^2 e^{-\delta S_{\tau_x}}]dP(\tau_{-x} < \infty)$$
$$= \int_0^\infty E[\tau_x^2]e^{-\delta(x+\rho)}de^{-\delta(x+\rho)}$$
$$+ \int_0^\infty E(\tau_x^2)e^{-\delta(x+\rho)}d(P(\tau_{-x} < \infty) - e^{-\delta(x+\rho)})$$
$$+ \int_0^\infty E[\tau_x^2(e^{-\delta(R_x-\rho)} - 1)]e^{-\delta(x+\rho)}de^{-\delta(x+\rho)}$$
$$+ \int_0^\infty E[\tau_x^2(e^{-\delta(R_x-\rho)} - 1)]e^{-\delta(x+\rho)}d(P(\tau_{-x} < \infty) - e^{-\delta(x+\rho)}).$$

We only give the detailed evaluation for the first term of (4.1) and the other three terms can be shown of lower orders by using the same technique as in the proofs for Lemmas 6 and 7. From Lemma 9, we have

$$\begin{aligned}
E[\tau_x^2] &= \frac{4}{\delta^2}\left[(x+ER_x)^2 + \delta\operatorname{cov}(\tau_x, R_x) - \operatorname{var}(R_x) + \frac{2}{\delta}(x+ER_x)\right] \\
&= \frac{4}{\delta^2}\left[x^2 + 2xER_x + (ER_x)^2 + \frac{2}{\delta}(x+ER_x) + \delta\operatorname{cov}(\tau_x, R_x) - \operatorname{var}(R_x)\right] \\
&= \frac{4}{\delta^2}\left[x^2 + 2x\rho + \frac{2}{\delta}(x+\rho) + \cdots\right],
\end{aligned}$$

where the neglected terms are of lower orders. Substituting the above expansion into the first term of (4.1), we get

$$\begin{aligned}
&\int_0^\infty E[\tau_x^2]e^{-\delta(x+\rho)}de^{-\delta(x+\rho)} \\
&= -\frac{4}{\delta}e^{-2\delta\rho}\int_0^\infty \left[x^2 + 2x\rho + \frac{2}{\delta}(x+\rho) + \cdots\right]e^{-2\delta x}dx \\
&= -\frac{4}{\delta}e^{-2\delta\rho}\left(\frac{2}{(2\delta)^3} + \frac{2\rho}{(2\delta)^2} + \frac{2}{\delta}\left(\frac{1}{(2\delta)^2} + \frac{2\rho}{2\delta^2}\right) + \cdots\right) \\
&= -\frac{3}{\delta^4} + O\left(\frac{1}{\delta^2}\right),
\end{aligned}$$

which completes the proof.

Combining the results of Lemmas 5–7 and 12–13, we get

THEOREM 14. *As* $\delta \to 0$,

$$E(\nu_0^2) = \frac{26}{\delta^4} - \frac{2}{\delta^3} + O\left(\frac{1}{\delta^2}\right).$$

5. Continuous analog and numerical comparison

In this section, the notations may be different from the last two sections. Denote ν_0 as the maximum point of the unbalanced two-sided Brownian motion

$$W_{1t}I_{[t<0]} + W_{2t}I_{[t>0]}$$

where W_{1t} has drift θ_0 and W_{2t} has drift $-\theta$. Write

$$M = \max_{-\infty<t<0} W_{1t}; \qquad \text{and} \qquad M' = \max_{0<t<\infty} W_{2t}.$$

Then,

$$E[\nu_0] = E[\nu_0; M' > M] + E[\nu_0; M > M'].$$

As the two terms on the right hand side has the same structure, only the first term will be evaluated.

Define

$$\tau_x = \inf\{t > 0 : W_{2t} \geq x\}.$$

By using the strong Markov property of W_{2t}, it follows that

$$E[\nu_0; M' > M] = E[\tau_M; \tau_M < \infty] + E[\nu_2]P(\tau_M < \infty),$$

where ν_2 is the maximum point of W_{2t}.

First, by noting that $P_\theta(\tau_x < \infty) = e^{-2\theta x}$ and $P_{\theta_0}(M > x) = e^{2\theta_0 x}$, we get

$$P(\tau_M < \infty) = -\frac{\theta_0}{\theta - \theta_0}.$$

Second, we note that

$$\begin{aligned} P_\theta(\nu_2 > t) &= P_\theta\left(\max_{0<s<t} W_{2t} < \max_{t<s<\infty} W_{2t}\right) \\ &= P_\theta\left(\max_{0<s<t}(B_{1s} + \theta s) < \max_{0<s<\infty}(B_{2s} - \theta s)\right), \end{aligned}$$

where B_{1s} and B_{2s} are two independent standard Brownian motions.

Since

$$P_\theta\left(\max_{0<s<\infty}(B_{2s} - \theta_1 s) > x\right) = e^{-2\theta x},$$

and

$$P_\theta\left(\max_{0<s<t}(B_{1s} + \theta s) > x\right) = 1 - \Phi\left(-\frac{x}{\sqrt{t}} - \theta\sqrt{t}\right) + e^{2\theta x}\Phi\left(-\frac{x}{\sqrt{t}} - \theta\sqrt{t}\right),$$

which is the inverse Gaussian distribution. A straightforward calculation gives

$$E_\theta[\nu_2] = 1.5/(2\theta)^2.$$

Finally, by using Wald's likelihood ratio identity, we get

$$\begin{aligned} E[\tau_M; \tau_M < \infty] &= E_\theta[\tau_M e^{-2\theta W_{2\tau_M}}] \\ &= E\left[e^{-2\theta M}\frac{M}{\theta}\right] \\ &= \frac{-\theta_0}{2\theta(\theta - \theta_0)^2}. \end{aligned}$$

Combining the above results, we get the expression for $E[\nu_0; M' > M]$. $E[\nu_0; M > M']$ can be evaluated similarly. The result is summarized as follows:

THEOREM 15.

$$E[\nu_0] = \frac{-3(\theta^3 + \theta_0^3)}{8\theta_0^2\theta^2(\theta - \theta_0)} + \frac{\theta + \theta_0}{2\theta_0\theta(\theta - \theta_0)}.$$

Table 1. Comparison between three approximations.

$\theta_0 \mid \Delta$		0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
−0.4	Num.	−1.71	−2.30	−2.56	−2.69	−2.81	−2.83	−2.84	−2.84	−2.84	−2.84	−2.84
	Approx.	−1.69	−2.26	−2.52	−2.65	−2.73	−2.78	−2.81	−2.83	−2.85	−2.86	−2.87
	Cont.	−1.41	−1.89	−2.10	−2.21	−2.28	−2.31	−2.34	−2.35	−2.37	−2.37	−2.37
−0.5	Num.	0.00	−0.97	−1.35	−1.53	−1.64	−1.69	−1.72	−1.74	−1.75	−1.75	−1.75
	Approx.	0.00	−0.94	−1.31	−1.49	−1.59	−1.65	−1.69	−1.72	−1.73	−1.75	−1.75
	Cont.	0.00	−0.79	−1.12	−1.28	−1.36	−1.42	−1.45	−1.47	−1.49	−1.50	−1.51
−0.6	Num.		0.0	−0.59	−0.86	−0.99	−1.07	−1.11	−1.14	−1.15	−1.15	−1.15
	Approx.		0.0	−0.57	−0.83	−0.96	−1.03	−1.08	−1.12	−1.13	−1.14	−1.15
	Cont.		0.0	−0.49	−0.72	−0.84	−0.91	−0.95	−0.98	−1.00	−1.02	−1.03
−0.7	Num.			0.0	−0.39	−0.58	−0.68	−0.73	−0.77	−0.79	−0.80	−0.80
	Approx.			0.0	−0.38	−0.55	−0.65	−0.71	−0.74	−0.77	−0.78	−0.79
	Cont.			0.0	−0.33	−0.49	−0.58	−0.64	−0.68	−0.70	−0.72	−0.74
−0.8	Num.				0.0	−0.26	−0.40	−0.48	−0.52	−0.55	−0.57	−0.58
	Approx.				0.0	−0.25	−0.38	−0.46	−0.50	−0.53	−0.55	−0.56
	Cont.				0.0	−0.23	−0.35	−0.42	−0.47	−0.50	−0.53	−0.54
−0.9	Num.					0.0	−0.19	−0.30	−0.35	−0.38	−0.40	−0.41
	Approx.					0.0	−0.18	−0.28	−0.33	−0.37	−0.39	−0.40
	Cont.					0.0	−0.17	−0.26	−0.32	−0.36	−0.38	−0.40
−1.0	Num.						0.0	−0.14	−0.21	−0.26	−0.29	−0.30
	Approx.						0.0	−0.13	−0.20	−0.25	−0.27	−0.29
	Cont.						0.0	−0.12	−0.20	−0.25	−0.27	−0.30

$\Delta = (\theta - \theta_0)/2$ as defined in Hinkley (1971).

In particular, if $\theta = -\theta_0$,

$$E|\nu_0| = \frac{1}{8\theta^2}.$$

One can see that even at the first order the continuous analog gives totally different result from the discrete case. However, the second moment in the discrete time case is the same at the first order as the exact result in the continuous time case which is $\frac{26}{\delta^4}$ (Ibragimov and Khasminskii (1981)).

To show the accuracy of the second order expansion given in the discrete time case, Table 1 gives some numerical comparison among the second order expansion, the continuous analog and the numerical values provided by Hinkley (1971). One can see that the second order approximation is generally good.

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Appendix

PROOF OF (3.5). We first write

$$\begin{aligned}\frac{d}{dx}(P_{\theta_0}(\tau_x < \infty) - e^{2\theta_0(x+\rho)}) &= \frac{d}{dx}(e^{2\theta_0(x+\rho)}(E_{-\theta_0}e^{2\theta_0(R_x-\rho)-1})) \\ &= 2\theta_0 e^{2\theta_0(x+\rho)}(E_{-\theta_0}e^{2\theta_0(R_x-\rho)} - 1) \\ &\quad + 2\theta_0 e^{2\theta_0 x}\frac{d}{dx}E_{-\theta_0}e^{2\theta_0 R_x}.\end{aligned}$$

Conditional on $\{R_x < \Delta x\}$ or $\{R_x \geq \Delta x\}$, we have

$$\begin{aligned}E_{-\theta_0}e^{2\theta_0 R_{x+\Delta x}} &= E_{-\theta_0}[e^{2\theta_0 R_{x+\Delta x}}; R_x > \Delta x] + E_{-\theta_0}[e^{2\theta_0 R_{x+\Delta x}}; R_x < \Delta x] \\ &= E_{-\theta_0}[e^{2\theta_0(R_x-\Delta x)}; R_x > \Delta x] + E_{-\theta_0}[e^{2\theta_0 R'_{(\Delta x - R_x)}}; R_x < \Delta x] \\ &= E_{-\theta_0}[e^{2\theta_0 R_x}; R_x > \Delta x] - 2\theta_0 \Delta x E_{-\theta_0}[e^{2\theta_0 R_x}] \\ &\quad + E_{-\theta_0}e^{2\theta_0 R_0}P_{-\theta_0}(R_x \leq \Delta x) + o(\Delta x)\end{aligned}$$

where R'_x is the overshoot at the boundary x for another independent copy of $\{S_n\}$. Similarly, we can write

$$\begin{aligned}E_{-\theta_0}e^{2\theta_0 R_x} &= E_{-\theta_0}[e^{2\theta_0 R_x}; R_x > \Delta x] + E_{-\theta_0}[e^{2\theta_0 R_x}; R_x < \Delta x] \\ &= E_{-\theta_0}[e^{2\theta_0 R_x}; R_x > \Delta x] + P_{-\theta_0}(R_x \leq \Delta x) + o(\Delta x).\end{aligned}$$

We thus have

$$\frac{d}{dx}E_{-\theta_0}e^{2\theta_0 R_x} = -2\theta_0 E_{-\theta_0}e^{2\theta_0 R_x} + f_{R_x}(0)(E_{-\theta_0}e^{2\theta_0 R_0} - 1),$$

where $f_{R_x}(y)$ is the density function of R_x. As $\theta_0 \to 0$, we have

$$\begin{aligned}\frac{d}{dx}E_{-\theta_0}e^{2\theta_0 R_x} &= 2\theta_0((E_0R_0)f_{R_x}(0)-1)+o(\theta_0)\\ &= 2\theta_0\frac{d}{dx}E_0(R_x-\rho)+o(\theta_0).\end{aligned}$$

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