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A BERRY-ESSÉEN THEOREM FOR SERIAL RANK STATISTICS

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Abstract. Berry-Esséen bounds of the optimal $O(n^{-1/2})$ order are obtained, under the null hypothesis of randomness, for serial linear rank statistics, of the form $\sum a_1(R_t)a_2(R_{t-k})$. Such statistics play an essential role in distribution-free methods for time-series analysis, where they provide nonparametric analogues to classical (Gaussian) correlogram-based methods. Berry-Esséen inequalities are established under mild conditions on the score-generating functions, allowing for normal (van der Waerden) scores. They extend to the serial case the earlier result of Does (1982, *Ann. Probab.*, **10**, 982–991) on (*nonserial*) linear rank statistics, and to the context of nonparametric rank-based statistics the *parametric* results of Taniguchi (1991, *Higher Order Asymptotics for Time Series Analysis*, Springer, New York) on quadratic forms of Gaussian stationary processes.

Key words and phrases: Berry-Esséen bounds, serial rank statistics, time series.

1. Introduction

Denote by $H_0^{(n)}$ the hypothesis under which $\boldsymbol{X}^{(n)} = (X_1^{(n)}, \dots, X_n^{(n)})$ is an n-tuple of independent and identically distributed random variables, with continuous distribution function F and probability density f. Let $\boldsymbol{X}_{(\cdot)}^{(n)} = (X_{(1)}^{(n)}, \dots, X_{(n)}^{(n)})$ and $\boldsymbol{R}^{(n)} = (R_1^{(n)}, \dots, R_n^{(n)})$ be the corresponding *order statistic* and *vector of ranks*, respectively. Linear rank statistics, of the form

$$T_0^{(n)} = n^{-1/2} \sum_{t=1}^{n} c_t^{(n)} a^{(n)}(R_t^{(n)}), \tag{1.1}$$

where $a^{(n)}(1), \dots, a^{(n)}(n)$ and $c_1^{(n)}, \dots, c_n^{(n)}$, $n \in \mathbb{N}$ respectively denote a collection of scores and a triangular array of *regression constants*, have been studied extensively. In particular, Berry-Esséen bounds have been obtained for the distribution function of (1.1) by several authors, among which Jurečková and Puri (1975), von Bahr (1976), Hušková (1977, 1979), Ho and Chen (1978). The most general result—providing a bound of the *optimal* $O(n^{-1/2})$ order under mild conditions on the scores which are satisfied by the *van der Waerden* or *normal* scores—has been derived by Does (1982). The proof of this result itself relies on earlier work by Albers, Bickel and van Zwet (1976), and a bound on the characteristic function of (1.1) which is due to van Zwet (1980).

Nonserial linear rank statistics of the form (1.1) are known to provide *locally asymptotically optimal* tests for all classical testing problems arising in general linear models with independent observations (i.e., two-sample location problems, analysis of variance, regression analysis, etc.): see Hájek and Šidák (1967), Puri and Sen (1971, 1985). In the statistical analysis of time series and other stochastic processes, the observations are no longer independent, and more general rank-based statistics, taking into account the serial dependence structure of the data, are needed: the *serial* rank statistics.

Serial rank statistics actually have a long history, and can be traced back as far as Fisher (1926) or Wald and Wolfowitz (1943). A systematic study of the class of *linear serial rank statistics*, of the form

$$T^{(n)} = (n-K)^{-1/2} \sum_{t=K+1}^{n} a^{(n)}(R_{t-1}^{(n)}, \ldots, R_{t-K}^{(n)}) \tag{1.2}$$

has been initiated in Hallin *et al.* (1985); the *scores* $a^{(n)}(\cdots)$ here depend on the ranks of $K+1$ successive observations. In Hallin *et al.* (1987) and Hallin and Puri (1994), it is shown that *simple* linear serial rank statistics, of the form (the scores $a_1^{(n)}(\cdot)$ and $a_2^{(n)}(\cdot)$ are defined as in (1.1))

$$T_K^{(n)} = (n-K)^{-1/2} \sum_{t=K+1}^{n} a_1^{(n)}(R_t^{(n)}) a_2^{(n)}(R_{t-K}^{(n)}) \tag{1.3}$$

provide *locally asymptotically optimal* tests in the general context of linear models with ARMA error terms—and thus constitute a rank-based, nonparametric and non-Gaussian alternative correlogram-based methods; see Hallin and Puri (1992) for a nontechnical survey.

A prominent role in this context is played by the so-called *van der Waerden* or *normal* scores

$$a^{(n)}(i) = \Phi^{-1}(i/(n+1)), \qquad i = 1, \ldots, n. \tag{1.4}$$

It has been shown indeed (Chernoff and Savage (1958) for the nonserial case; Hallin (1994) for the serial one) that the ARE of the corresponding tests (based on (1.1) or (1.3)) with respect to their normal-theory counterparts (t- and F-tests, correlogram-based methods) is always larger than or equal to one, with equality under Gaussian assumptions only. It is thus very important that all results concerning statistics of the form (1.1) or (1.3) be valid for a class of scores which includes (1.4).

In this paper, we prove a Berry-Esséen result for the distribution function under $H_0^{(n)}$ of (a standardized version of) $T_K^{(n)}$ given in (1.3). Our bound is of the *optimal* $O(n^{-1/2})$ order, and the assumptions we are making (roughly, the same as in Does (1982)) are satisfied by the *van der Waerden* scores (1.4). Our derivation is based on a serial extension (Hallin and Rifi (1996)) of van Zwet (1980)'s result on characteristic functions. The parametric analogue of our result was obtained, for Gaussian stationary processes, by Taniguchi ((1986), see also (1991)).

2. Notation and main result

Closed-form expressions for the exact mean $\mu_K^{(n)}$ and variance $(\sigma_K^{(n)})^2$ (under $H_0^{(n)}$) of $T_K^{(n)}$ are easily obtained from combinatorial arguments: letting $S_{pq}^{(n)} = \sum_{i=1}^n [a_1^{(n)}(i)]^p [a_2^{(n)}(i)]^q$, we have, for $n \geq K+1$,

$$\mu_K^{(n)} = [n(n-1)(n-K)^{-1/2}]^{-1}(S_{10}^{(n)}S_{01}^{(n)} - S_{11}^{(n)}) \tag{2.1}$$

and, with $[x]^+ = \max(x,0)$, $x \in \mathbb{R}$,

$$\begin{aligned}(\sigma_K^{(n)})^2 = {} & [n(n-1)]^{-1}[S_{20}^{(n)}S_{02}^{(n)} - S_{22}^{(n)}] \\ & + 2\frac{(n-2K)^+}{n-K}[n(n-1)(n-2)]^{-1} \\ & \times [S_{10}^{(n)}S_{01}^{(n)}S_{11}^{(n)} - S_{21}^{(n)}S_{01}^{(n)} - S_{12}^{(n)}S_{10}^{(n)} - (S_{11}^{(n)})^2 + 2S_{22}^{(n)}] \\ & + \frac{(n-K)(n-K-1) - 2(n-2K)^+}{n-K} \\ & \times [n(n-1)(n-2)(n-3)]^{-1} \\ & \times [(S_{10}^{(n)}S_{01}^{(n)})^2 + 2(S_{11}^{(n)})^2 + S_{20}^{(n)}S_{02}^{(n)} - 6S_{22}^{(n)} \\ & \quad - 4S_{11}^{(n)}S_{10}^{(n)}S_{01}^{(n)} - S_{20}^{(n)}(S_{01}^{(n)})^2 - S_{02}^{(n)}(S_{10}^{(n)})^2 \\ & \quad + 4S_{21}^{(n)}S_{01}^{(n)} + 4S_{12}^{(n)}S_{10}^{(n)}] \\ & - (\mu_K^{(n)})^2.\end{aligned} \tag{2.2}$$

Then, for $n > K+1$ (the case of degenerate scores $a^{(n)}(i) = a^{(n)}$, $i = 1, \ldots, n$ is tacitly excluded), $T_*^{(n)} = (T_K^{(n)} - \mu_K^{(n)})/\sigma_K^{(n)}$ is exactly standardized under $H_0^{(n)}$.

As usual, we are assuming that the scores $a_j^{(n)}(\cdot)$, $j = 1, 2$, are derived from *score-generating functions* $J_j : (0,1) \to \mathbb{R}$, $j = 1, 2$ in either of the following two ways:

$$a_j^{(n)}(i) = E[J_j(U_{(i)}^{(n)})] \quad \textit{(exact scores)} \tag{2.3}$$

$$a_j^{(n)}(i) = J_j(i/(n+1)) \quad \textit{(approximate scores)}, \tag{2.4}$$

where $U_{(i)}^{(n)}$ denotes the i-th order statistic in a n-tuple $U_1^{(n)}, \ldots, U_n^{(n)}$ of independent variables uniformly distributed over $[0,1]$.

The following assumptions are made throughout.

ASSUMPTION (A1) The score-generating functions J_j have integrable fourth powers:

$$\int_0^1 J_j^4(u)du < \infty, \qquad j = 1, 2.$$

Without loss of generality (in view of the fact that $T_K^{(n)}$ can be exactly standardized), we then also assume that

$$\int_0^1 J_j(u)du = 0 \quad \text{and} \quad \int_0^1 J_j^2(u)du = 1, \qquad j = 1, 2. \tag{2.5}$$

ASSUMPTION (A2) The score-generating functions J_j, $j = 1, 2$ are continuously differentiable on $(0, 1)$, and there exist strictly positive constants M and $\alpha < 5/4$ such that, for all $u \in (0, 1)$

$$|J_j'(u)| \leq M[u(1-u)]^{-\alpha}, \qquad j = 1, 2. \tag{2.6}$$

Note that, letting $\delta = \frac{5}{4} - \alpha$, (2.6) takes the more convenient form

$$|J_j'(u)| \leq M[u(1-u)]^{\delta - 5/4}, \qquad 0 < \delta < \frac{1}{4}, \qquad j = 1, 2. \tag{2.7}$$

ASSUMPTION (A3) The score-generating functions J_1 and J_2 are *concordant*, i.e. for all $(u, v) \in (0, 1)^2$,

$$J_1(u) \leq J_1(v) \Longleftrightarrow J_2(u) \leq J_2(v). \tag{2.8}$$

ASSUMPTION (A_r) Let $0 < r \in \mathbb{R}$. We say that the function $h : (0, 1) \to \mathbb{R}$ satisfies assumption (A_r) if h is twice continuously differentiable, and

$$\limsup_{u \to 0 \text{ or } 1} [u(1-u)|h''(u)/h'(u)|] < 1 + \frac{1}{r}.$$

Assumptions (A2) and (A_r) are the same as in Does (1982). Assumption (A1) is slightly stronger (Does (1982) only requires $\int |J(u)|^3 du < \infty$); this is motivated by the fact that products of scores are to be handled. The concordance assumption (A3), of course, being specific to the serial context, is new.

We now may stay the main result of this paper (hereafter referred to as the "main" theorem); here and in the sequel, unless otherwise specified, all $O(\cdot)$ and $o(\cdot)$ quantities are to be understood as $n \to \infty$.

THEOREM 2.1. *Let $T_K^{(n)}$ be given in (1.3), with exact (2.3) or approximate (2.4) scores associated with score-generating functions J_1 and J_2 satisfying assumption (A1), (A2), (A3) and (A_r) with $r = 1$. Denote by $F_*^{(n)}$ the distribution function under $H_0^{(n)}$ of the standardized version $T_*^{(n)}$ of $T_K^{(n)}$. Then*

$$\sup_{x \in \mathbb{R}} |F_*^{(n)}(x) - \Phi(x)| = O(n^{-1/2}). \tag{2.9}$$

3. Some preliminary lemmas

Some preparation is required before turning to the proof of the main theorem. For simplicity, let $\lambda_i = i/(n+1)$, $i = 1, \ldots, n$. The notation $U_{(i)}^{(n)}$ is used with the same meaning as in (2.3).

LEMMA 3.1. (i) *Let* $J : (0,1) \to \mathbb{R}$ *satisfy* (A2). *Then, there exists* $\delta \in (0, 1/4)$ *such that*

$$\sum_{i=p}^{q} E[J(U_{(i)}^{(n)}) - J(\lambda_i)]^2 = O(\lambda_p^{2\delta-1/2} + (1-\lambda_q)^{2\delta-1/2}) \tag{3.1}$$

and

$$\sum_{i=1}^{n} \mathrm{Var}(J(U_{(i)}^{(n)})) = O(n^{-2\delta+1/2}), \tag{3.2}$$

uniformly in $1 \le p \le q \le n$.

(ii) *If* J *also satisfies* (A_r) *with* $r = 1$, *then*

$$\sum_{i=p}^{q} \{E[J(U_{(i)}^{(n)})] - J(\lambda_i)\}^2 = O(n^{-1}\lambda_p^{2\delta-3/2} + n^{-1}(1-\lambda_q)^{2\delta-1/2}), \tag{3.3}$$

$$\sum_{i=p}^{q} \{E[J(U_{(i)}^{(n)}) - a^{(n)}(i)]^2\}^{1/2} = O(n^{1/2}\lambda_p^{\delta+1/4} + n^{1/2}(1-\lambda_q)^{\delta+1/4}), \tag{3.4}$$

$$\sum_{i=p}^{q} \{E[J(U_{(i)}^{(n)}) - a^{(n)}(i)]^4\}^{1/2} = O(\lambda_p^{2\delta-1/2} + (1-\lambda_q)^{2\delta-1/2}), \tag{3.5}$$

$$\sum_{i=p}^{q} \{E[J(U_{(i)}^{(n)}) - a^{(n)}(i)]^4\}^{1/4} = O(n^{1/2}\lambda_p^{\delta+1/4} + n^{1/2}(1-\lambda_q)^{\delta+1/4}), \tag{3.6}$$

and

$$\sum_{i=p}^{q} |J(\lambda_i) - E[J(U_{(i)}^{(n)})]| = O(\lambda_p^{\delta-1/4} + (1-\lambda_q)^{\delta-1/4}), \tag{3.7}$$

still uniformly in $1 \le p \le q \le n$.

(iii) *If* J *moreover also satisfies* (A1), *then, as* $n \to \infty$,

$$\sum_{i=1}^{n} \left[J(\lambda_i) - n^{-1}\sum_{j=1}^{n} J(\lambda_j) \right]^2 = n + O(n^{-2\delta+1/2}) \tag{3.8}$$

and

$$\sum_{i=p}^{q} E\{J(U_{(i)}^{(n)}) - E[J(U_{(i)}^{(n)})]\}^2 = O(\lambda_p^{2\delta-1/2} + (1-\lambda_q)^{2\delta-1/2}), \tag{3.9}$$

still uniformly in $1 \le p \le q \le n$.

PROOF. Relations (3.1), (3.2), (3.3) and (3.8) are taken from Does (1982). The remaining ones can be established along the same lines from lemmas A2.3, A2.4 and A2.6 in Albers, Bickel and van Zwet (1976).

LEMMA 3.2. *Let $T_K^{(n)}$ be given in* (1.3), *with either* (2.3) *or* (2.4), *and score-generating functions J_1 and J_2 satisfying* (A1), (A2) *and* (A_r) *for $r = 1$. Then (for $\mu_K^{(n)}$ and $\sigma_K^{(n)}$ defined in* (2.1) *and* (2.2), *respectively*)

$$\mu_K^{(n)} = O(n^{-1/2}) \quad \textit{and} \quad (\sigma_K^{(n)})^2 = 1 + O(n^{2\delta-1/2}). \tag{3.10}$$

PROOF. See Hallin and Rifi (1995).

LEMMA 3.3. *Let $\varphi_*^{(n)}$ denote the characteristic function under $H_0^{(n)}$ of the standardized version $T_*^{(n)}$ of $T_K^{(n)}$. Then, under the assumptions of the main theorem, there exists a constant $c > 0$ such that*

$$\int_{\log n \le |u| \le n^{1/2} c\sigma_K^{(n)}} \left| \frac{\varphi_*^{(n)}(u)}{u} \right| du = o(n^{-1/2}). \tag{3.11}$$

PROOF. Letting $\varphi^{(n)}(u) = E\{\exp[iu(T_K^{(n)} - \mu_K^{(n)})]\}$, we have $\varphi_*^{(n)}(u) = \varphi^{(n)}(u/\sigma_K^{(n)})$, $u \in \mathbb{R}$; (3.11) then readily follows from Lemma 3.2 if Proposition 2.2 in Hallin and Rifi (1996) holds. It is thus sufficient to show that the two assumptions underlying this latter result are satisfied. One of these two assumptions is the concordance assumption (A3). The second one requires the existence of two positive constants b and B such that, letting $\bar{a}_j^{(n)} = n^{-1}\sum_{i=1}^n a_j^{(n)}(i)$,

$$\sum_{i=1}^n |a_j^{(n)}(i) - \bar{a}_j^{(n)}| \ge bn \quad \text{and} \quad \sum_{i=1}^n (a_j^{(n)}(i) - \bar{a}_j^{(n)})^2 \le Bn, \tag{3.12}$$

for $j = 1, 2$ and all $n \in \mathbb{N}$. Now, a sufficient condition for (3.12) to hold is the existence of two positive constants c and C such that

$$\sum_{i=1}^n (a_j^{(n)}(i) - \bar{a}_j^{(n)})^2 \ge cn \quad \text{and} \quad \sum_{i=1}^n (a_j^{(n)}(i) - \bar{a}_j^{(n)})^4 \le Cn, \tag{3.13}$$

$j = 1, 2$, $n \in \mathbb{N}$. Indeed, we may assume without loss of generality that $\bar{a}_j^{(n)} = 0$; (3.13) then implies

$$\frac{1}{n}\sum_{i=1}^n [(a_j^{(n)}(i)]^2 \le \left(\frac{1}{n}\sum_{i=1}^n [a_j^{(n)}(i)]^4\right)^{1/2} \le C^{1/2}, \quad j = 1, 2,$$

and

$$
\begin{aligned}
c \le \frac{1}{n}\sum_{i=1}^{n}[a_j^{(n)}(i)]^2 &\le \left[\frac{1}{n}\sum_{i=1}^{n}|a_j^{(n)}(i)|\right]^{1/2}\left[\frac{1}{n}\sum_{i=1}^{n}|a_j^{(n)}(i)|^3\right]^{1/2}\\
&\le \left[\frac{1}{n}\sum_{i=1}^{n}|a_j^{(n)}(i)|\right]^{1/2}\left[\frac{1}{n}\sum_{i=1}^{n}|a_j^{(n)}(i)|^4\right]^{3/8}\\
&\le \left[\frac{1}{n}\sum_{i=1}^{n}|a_j^{(n)}(i)|\right]^{1/2} C^{3/8},
\end{aligned}
$$

so that (3.12) holds for $b = c^2C^{-3/4}$ and $B = C^{1/2}$.

Next, let us show that (3.13) is satisfied for exact and approximate scores, under the assumptions of the main theorem. For exact scores, it follows from (A1), (2.5) and (3.2) that

$$
n\bar{a}_j^{(n)} = \sum_{i=1}^{n} E[J_j(U_{(i)}^{(n)})] = \sum_{i=1}^{n} E[J_j(U_i^{(n)})] = n\int_0^1 J(u)du = 0, \tag{3.14}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{n}[a_j^{(n)}(i)]^2 &= \sum_{i=1}^{n} E[J_j^2(U_{(i)}^{(n)})] - \sum_{i=1}^{n}\mathrm{Var}(J_j(U_{(i)}^{(n)}))\\
&= n + O(n^{-2\delta-1/2}),
\end{aligned} \tag{3.15}
$$

whence the first part of (3.13). The second part is a consequence of (A1) and Lemma A2.6 in Albers, Bickel and van Zwet (1976), which imply

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}[(a_j^{(n)}(i)]^4 &= \frac{1}{n}\sum_{i=1}^{n}[EJ_j(U_{(i)}^{(n)})]^4 = O\left(\int_0^1 J_j^4(u)du\right)\\
&= O(1).
\end{aligned}
$$

In the approximate score case, (3.3) and the triangular inequality imply the first part of (3.13). On the other hand, from (3.7) and (3.14),

$$
\begin{aligned}
\left|\sum_{i=1}^{n} a_j^{(n)}(i)\right| &= \left|\sum_{i=1}^{n} J_j\left(\frac{i}{n+1}\right)\right|\\
&= \left|\sum_{i=1}^{n}\left[J_j\left(\frac{i}{n+1}\right) - EJ_j(U_{(i)}^{(n)})\right]\right|\\
&\le \sum_{i=1}^{n}\left|J_j\left(\frac{i}{n+1}\right) - EJ_j(U_{(i)}^{(n)})\right| = O(n^{-\delta+1/4}),
\end{aligned} \tag{3.16}
$$

so that $\bar{a}_j^{(n)} = O(n^{-\delta-3/4})$. Applying again Lemma A2.6 of Albers, Bickel and van Zwet (1976), we obtain

$$
\frac{1}{n}\sum_{i=1}^{n}[a_j^{(n)}(i)]^4 = \frac{1}{n}\sum_{i=1}^{n} J_j^4\left(\frac{i}{n+1}\right) = O\left(\int_0^1 J_j^4(u)du\right) = O(1); \tag{3.17}
$$

(3.13) then follows from noting that

$$\frac{1}{n}\sum_{i=1}^{n}(a_j^{(n)}(i)-\bar{a}_j^{(n)})^4 \le 8\left\{\frac{1}{n}\sum_{i=1}^{n}(a_j^{(n)}(i))^4+(\bar{a}_j^{(n)})^4\right\}.$$

4. Proofs

The proof of the main result (inequality (2.9)) consists of a series of eight lemmas. Lemmas 4.1 through 4.3 show that if

$$\int_{|u|\le \log n} |E[(S_K^{(n)}-T_K^{(n)})\exp(iuT_K^{(n)})]|du = O(n^{-1/2}) \tag{4.1}$$

holds under $H_0^{(n)}$, then (2.9) also holds. Lemma 4.4 provides a sufficient condition for (4.1) to hold in the exact score case, and Lemma 4.5 shows this sufficient condition is satisfied, which concludes the proof for exact scores. The approximate score case is more complicated, and an indirect approach is required: Lemmas 4.6 through 4.8 establish that the effect of substituting approximate *scores* for the exact ones in (4.1) is *small*, so that the approximate score case follows from the exact score one.

Lemma 4.1 can be interpreted as reinforcing the *Hájek projection* result for serial linear rank statistics proved in Hallin *et al.* (1985). Associated with $T_K^{(n)}$ given in (1.3), with either (2.3) or (2.4), let

$$S_K^{(n)} = (n-K)^{-1/2}\sum_{t=K+1}^{n} J_1(U_t^{(n)})J_2(U_{t-K}^{(n)}),$$

with $U_t^{(n)} = F(X_t^{(n)})$. Under $H_0^{(n)}$, $U_1^{(n)},\dots,U_n^{(n)}$ of course are independent and uniformly distributed over $[0,1]$; if moreover assumption (A1) and (2.5) hold, $S_K^{(n)}$ is exactly standardized. We then have the following result.

LEMMA 4.1. *Under $H_0^{(n)}$ and assumptions* (A1) *and* (A2), *as $n\to\infty$,*

$$E[(T_K^{(n)}-S_K^{(n)})^2] = O(n^{-2\delta-1/2}), \tag{4.2}$$

with $\delta\in(0,1/4)$ given in (A2).

PROOF. For $t\ge K+1$, let

$$Z_{1;t} = a_1^{(n)}(R_t^{(n)})[a_2^{(n)}(R_{t-K}^{(n)}) - J_2(U_{t-K}^{(n)})]$$

and

$$Z_{2;t} = [a_1^{(n)}(R_t^{(n)}) - J_1(U_t^{(n)})]J_2(U_{t-K}^{(n)}).$$

Clearly,

$$(4.3)\quad E[(T_K^{(n)} - S_K^{(n)})^2] \leq 2(n-K)^{-1}\left\{E\left[\left(\sum_{t=K+1}^{n} Z_{1;t}\right)^2\right] + E\left[\left(\sum_{t=K+1}^{n} Z_{2;t}\right)^2\right]\right\}.$$

On the other hand,

$$(4.4)\quad E\left[\left(\sum_{t=K+1}^{n} Z_{1;t}\right)^2\right] \leq (2K+1)\sum_{t=K+1}^{n} E(Z_{1;t}^2) + 2\sum_{s=K+1}^{n-K-1}\sum_{t=s+K+1}^{n} |E[Z_{1;t}Z_{1;s}]|.$$

In the approximate score case, (3.1) and (3.7) imply

$$(4.5)\quad \left|\sum_{i=1}^{n}[a_1^{(n)}(i)]^2 - n\right| = \sum_{i=1}^{n}\left[J_1^2\left(\frac{i}{n+1}\right) - EJ_1^2(U_{(i)}^{(n)})\right] \leq \sum_{i=1}^{n} E\left[J_1\left(\frac{i}{n+1}\right) - J_1(U_{(i)}^{(n)})\right]^2 + 2\sum_{i=1}^{n}\left|J_1\left(\frac{i}{n+1}\right)\right|\left|J_1\left(\frac{i}{n+1}\right) - EJ_1(U_{(i)}^{(n)})\right| = O(n^{-2\delta+1/2}).$$

Hence, for all $K+1 \leq t \leq n$,

$$E[Z_{1;t}^2] = E\{[a_1^{(n)}(R_t^{(n)})]^2[a_2^{(n)}(R_{t-K}^{(n)}) - J_2(U_{t-K}^{(n)})]^2\} \times [n(n-1)]^{-1}\left\{\sum_{t_1=1}^{n}[a_1^{(n)}(t_1)]^2\right\}\left\{\sum_{t_2=1}^{n} E[a_2^{(n)}(t_2) - J_2(U_{(t_2)}^{(n)})]^2\right\}.$$

In view of (3.15), the same inequality also holds for exact scores. The first term in the right hand side of (4.4) accordingly is $O(n^{-1/2-2\delta})$.

Turning to the second term, we have, for $K+1 \leq s < t-K \leq n-K$,

$$(4.6)\quad E(Z_{1;t}Z_{1;s}) = E\{a_1^{(n)}(R_t^{(n)})a_1^{(n)}(R_s^{(n)}) \cdot [a_2^{(n)}(R_{t-K}^{(n)}) - J_2(U_{t-K}^{(n)})][a_2^{(n)}(R_{s-K}^{(n)}) - J_2(U_{s-K}^{(n)})]\}$$
$$= [n(n-1)(n-2)(n-3)]^{-1}\sum_{1\leq t_1}\sum_{\neq t_2}\sum_{\neq t_3}\sum_{\neq t_4 \leq n} a_1^{(n)}(t_1)a_1^{(n)}(t_2) \times E\{[a_2^{(n)}(t_3) - J_2(U_{(t_3)}^{(n)})][a_2^{(n)}(t_4) - J_2(U_{(t_4)}^{(n)})]\}$$
$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$$

with

$$\Sigma_1 = [n(n-1)(n-2)(n-3)]^{-1}\left[\sum_{t_1=1}^{n} a_1^{(n)}(t_1)\right]\left[\sum_{t_2=1}^{n} a_1^{(n)}(t_2)\right]$$
$$\times \sum\sum_{1\le t_1\ne t_2\le n} E\{[a_2^{(n)}(t_1) - J_2(U_{(t_1)}^{(n)})][a_2^{(n)}(t_2) - J_2(U_{(t_2)}^{(n)})]\},$$

$$\Sigma_2 = [n(n-1)(n-2)(n-3)]^{-1}\left[\sum_{t_1=1}^{n} a_1^{(n)}(t_1)\right]$$
$$\times \sum\sum_{1\le t_1\ne t_2\le n} a_1^{(n)}(t_1)E\{[a_2^{(n)}(t_1) - J_2(U_{(t_1)}^{(n)})][a_2^{(n)}(t_2) - J_2(U_{(t_2)}^{(n)})]\},$$

$$\Sigma_3 = -[n(n-1)(n-2)(n-3)]^{-1}\sum\sum\sum_{1\le t_1\ne t_2\ne t_3\le n}[a_1^{(n)}(t_1)]^2$$
$$\times E\{[a_2^{(n)}(t_2) - J_2(U_{(t_2)}^{(n)})][a_2^{(n)}(t_3) - J_2(U_{(t_3)}^{(n)})]\}$$

and

$$\Sigma_4 = -2[n(n-1)(n-2)(n-3)]^{-1}\sum\sum\sum_{1\le t_1\ne t_2\ne t_3\le n} a_1^{(n)}(t_1)a_1^{(n)}(t_2)$$
$$\times E\{[a_2^{(n)}(t_2) - J_2(U_{(t_2)}^{(n)})][a_2^{(n)}(t_3) - J_2(U_{(t_3)}^{(n)})]\}.$$

In the exact score case, (3.14) directly implies $\sum_1 = 0$. In the approximate score case, from (3.16) and (3.7) (with $p = 1$, $q = n$), we have

$$\left|\sum_{i=1}^{n} a_1^{(n)}(i)\right| = O(n^{-\delta+1/4}). \tag{4.7}$$

It follows that

$$\left[n^{-1}\sum_{i=1}^{n} J_1\left(\frac{i}{n+1}\right)\right]^2 = O(n^{-2\delta-3/2}). \tag{4.8}$$

From this and the Cauchy-Schwarz inequality,

$$\sum\sum_{1\le t_1\ne t_2\le n} |E\{[a_2^{(n)}(t_1) - J_2(U_{(t_1)}^{(n)})][a_2^{(n)}(t_2) - J_2(U_{(t_2)}^{(n)})]\}| \tag{4.9}$$
$$\le \left\{\sum_{i=1}^{n}[E[a_2^{(n)}(i) - J_2(U_{(i)}^{(n)})]^2]^{1/2}\right\}^2 = O(n^{-2\delta+1/2}).$$

Substituting (4.7), (4.8) and (4.9) into Σ_1 yields $|\Sigma_1| = O(n^{-3-4\delta})$. The remaining terms Σ_2, Σ_3 and Σ_4 can be treated similarly, and one finally obtains that $E(Z_{1;t}Z_{1;s}) = O(n^{-2-2\delta})$, uniformly in s and t, $K+1 \le s < t \le n-K$. Going back to (4.4) we thus have that $E\{[\sum_{t=K+1}^{n} Z_{1;t}]^2\} = O(n^{-2\delta+1/2})$. It can be shown along the same lines that $E\{[\sum_{t=K+1}^{n} Z_{2;t}]^2\} = O(n^{-2\delta+1/2})$, which completes the proof of Lemma 4.1. □

Next, denote by $\Phi^{(n)}$ the distribution function of a normal variable with mean $\mu_K^{(n)}/\sigma_K^{(n)}$ and variance $(\sigma_K^{(n)})^{-2}$.

LEMMA 4.2. *If, under the assumptions of the main theorem,*

$$\int_{|u|\leq \log n} |E[(S_K^{(n)} - T_K^{(n)}) \exp(iuT_*^{(n)})]|\,du = O(n^{-1/2}) \tag{4.10}$$

under $H_0^{(n)}$ as $n \to \infty$, then (2.9) holds.

PROOF. It follows from *Esséen's smoothing lemma* (Esséen (1945); see, e.g., Feller (1971), p. 538) that, under the assumptions made,

$$\begin{aligned}
(4.11)\quad \pi \sup_{x\in\mathbb{R}} |F_*^{(n)}(x) - \Phi^{(n)}(x)| \\
&\leq \int_{-n^{1/2}c\sigma_K^{(n)}}^{n^{1/2}c\sigma_K^{(n)}} \left| \frac{\varphi_*^{(n)}(u) - \exp\left[-\frac{1}{2}(u/\sigma_K^{(n)})^2 + iu(\mu_K^{(n)}/\sigma_K^{(n)})\right]}{u} \right| du \\
&\quad + O(n^{-1/2}) \\
&\leq \int_{|u|\leq \log n} \left| \frac{\varphi_*^{(n)}(u) - \exp\left[-\frac{1}{2}(u/\sigma_K^{(n)})^2 + iu(\mu_K^{(n)}/\sigma_K^{(n)})\right]}{u} \right| du \\
&\quad + O(n^{-1/2}) \\
&\quad + \int_{\log n\leq |u|\leq cn^{1/2}\sigma_K^{(n)}} \left| \frac{\varphi_*^{(n)}(u)}{u} \right| du \\
&\quad + \int_{\log n\leq |u|\leq cn^{1/2}\sigma_K^{(n)}} \left| \frac{\exp\left[-\frac{1}{2}(u/\sigma_K^{(n)})^2 + iu(\mu_K^{(n)}/\sigma_K^{(n)})\right]}{u} \right| du \\
&\quad + O(n^{-1/2}) \\
&= I_1 + I_2 + I_3 + O(n^{-1/2}), \qquad \text{say.}
\end{aligned}$$

Lemma 3.3 above implies that I_2 is $O(n^{-1/2})$. Similarly, I_3 is easily shown to be $O(n^{-1/2})$. As for I_1, an elementary Taylor expansion yields

$$\begin{aligned}
&E[\exp(iu(S_K^{(n)} - \mu_K^{(n)})/\sigma_K^{(n)})] \\
&\quad = E[e^{iuT_*^{(n)}}] + iu(\sigma_K^{(n)})^{-1}E[S_K^{(n)} - T_K^{(n)}]e^{iuT_*^{(n)}} \\
&\qquad + [u^2(\sigma_K^{(n)})^{-2}]O(E[S_K^{(n)} - T_K^{(n)}]^2).
\end{aligned}$$

Accordingly,

$$(4.12)\quad I_1 = \int_{|u|\leq \log n} |u|^{-1}|E[\exp(iuS_K^{(n)}/\sigma_K^{(n)})] - \exp(-u^2/2(\sigma_K^{(n)})^2)|du$$
$$+ (\sigma_K^{(n)})^{-1} \int_{|u|<\log n} |E[S_K^{(n)} - T_K^{(n)}] \exp(iuT_*^{(n)})|du$$
$$+ [(\sigma_K^{(n)})^{-2}(\log n)]O(E[S_K^{(n)} - T_K^{(n)}]^2).$$

Since $S_K^{(n)}$ is a sum of K-dependent random variables, the first integral term in the right hand side of (4.12) is $O(n^{-1/2})$ (see Shergin (1979), p. 794). The second term, under (4.10), is also $O(n^{-1/2})$. And Lemmas 4.1 and 3.2 imply that the third term still is $O(n^{-1/2})$. It thus follows that

$$\sup_{x\in\mathbb{R}} |F_*^{(n)}(x) - \Phi^{(n)}(x)| = O(n^{-1/2});$$

(2.9) then readily follows, since

$$|\Phi^{(n)}(x) - \Phi(x)| = |\Phi(\sigma_K^{(n)}x - \mu_K^{(n)}) - \Phi(x)|$$
$$\leq |\sigma_K^{(n)}(x-1) - \mu_K^{(n)}||[\Phi'(\sigma_K^{(n)}x - \mu_K^{(n)}) + \Phi'(x)],$$

a quantity which, due to (3.10) and the fact that $x\Phi'(x)$ is uniformly bounded, again is $O(n^{-1/2})$. □

LEMMA 4.3. *Lemma 4.2 still holds if $T_K^{(n)}$ is substituted for $T_*^{(n)}$ in* (4.10).

PROOF. It readily follows from Lemmas 3.2 and 4.1 that the effect of substituting $T_K^{(n)}$ for $T_*^{(n)}$ in (4.10) is $o(n^{-1/2})$. □

From this stage on, we need to distinguish between the exact and approximate score cases: let $T_e^{(n)}$ and $T_a^{(n)}$ denote the exact and approximate score versions of $T_K^{(n)}$, respectively. Similarly, the notation $a_1^e(\cdot)$, $a_2^e(\cdot)$, $a_1^a(\cdot)$, $a_2^a(\cdot)$, $Z_{1;t}^e$, $Z_{2;t}^e$, $Z_{1;t}^a$ and $Z_{2;t}^a$ will be used, in an obvious fashion, instead of $a_1^{(n)}(\cdot), \ldots, Z_{1;t}^{(n)}$ and $Z_{2;t}^{(n)}$. But, the general notation $T_K^{(n)}$, $a_1^{(n)}(\cdot)$, etc. still will be used in statements which are valid for both the exact and approximate versions.

LEMMA 4.4. *Let the score-generating functions J_1 and J_2 satisfy assumptions* (A1), (A2) *and* (A_r) *with $r = 1$. Then, under $H_0^{(n)}$,*

$$(4.13)\qquad \int_{|u|\leq \log n} \left|(n-K)^{-1/2} \sum_{t=2K+1}^{n-K} E[Z_{2;t}^e \exp(iuT_a^{(n)})]\right| du = O(n^{-1/2})$$

implies

$$(4.14)\qquad \int_{|u|\leq \log n} |E[(S_K^{(n)} - T_e^{(n)}) \exp(iuT_e^{(n)})]|du = O(n^{-1/2}).$$

PROOF. In view of Lemma 4.1, $\exp(iuT_a^{(n)})$ can be substituted for $\exp(iuT_e^{(n)})$ in (4.14). Thus, on behalf of inequality (4.3), it is sufficient to show that (4.13) implies

$$\int_{|u|\leq \log n} \left|(n-K)^{-1/2} \sum_{t=K+1}^{n} E[Z_{1;t}^e \exp(iuT_a^{(n)})]\right| du = O(n^{-1/2}) \tag{4.15}$$

and

$$\int_{|u|\leq \log n} \left|(n-K)^{-1/2} \sum_{t=K+1}^{n} E[Z_{2;t}^e \exp(iuT_a^{(n)})]\right| du = O(n^{-1/2}). \tag{4.16}$$

Since, for all $t \geq K+1$,

$$\begin{aligned} &E[Z_{1;t}^e \exp(iuT_a^{(n)})] \mid \boldsymbol{R}^{(n)}] \\ &\quad = a_1^e(R_t^{(n)})[a_2^e(R_{t-K}^{(n)}) - E[J_2(U_{t-K}^{(n)}) \mid \boldsymbol{R}^{(n)}]] \exp(iuT_a^{(n)}) = 0, \end{aligned}$$

(4.15) holds irrespectively of (4.13). Considering (4.16), the same arguments as in (4.4) yield

$$\begin{aligned} \left|(n-K)^{-1/2} \sum_{t=K+1}^{2K} E[Z_{2;t}^e \exp(iuT_a^{(n)})]\right| &\leq \left|(n-K)^{-1/2} \sum_{t=K+1}^{2K} \{E[Z_{2;t}^e]^2\}^{1/2}\right| \\ &= O(n^{-\delta-3/4}) \end{aligned}$$

and

$$\left|(n-K)^{-1/2} \sum_{t=n-K+1}^{2K} E[Z_{2;t}^e \exp(iuT_a^{(n)})]\right| = O(n^{-\delta-3/4}).$$

Lemma 4.4 follows. □

LEMMA 4.5. *Under $H_0^{(n)}$ and the assumptions of the main theorem,* (4.13) *holds, as $n \to \infty$.*

PROOF. Let $\Omega_t^{(n)} = (R_t^{(n)}, R_{t-K}^{(n)})$: $\Omega_t^{(n)}$ thus takes its values in $\mathbb{N}_*^2 = \{\omega = (\omega_1, \omega_2) \in \mathbb{N}^2 \mid \omega_1 \neq \omega_2\}$. Denote by $E_{\Omega;t}[\cdot]$ the conditional expectation $E[\cdot \mid \Omega_t^{(n)}]$, and let $E_{\omega;t}^{(n)}[\cdot] = E[\cdot \mid \Omega_t^{(n)} = \omega]$, $n \geq \max(\omega_1, \omega_2)$. For given n and ω, denote by $1 \leq l_1 < l_2 < \cdots < l_{n-2} \leq n$ the $(n-2)$ integers in $\{1, \ldots, n\} \backslash \{\omega_1, \omega_2\}$; put $l_0 = 0$ and $l_{n-1} = n+1$; let $q_{\omega;t}^{(n)} : [0,1] \to [0,1]$ be a monotonically increasing, twice continuously differentiable function such that $q_{\omega;t}^{(n)}(\frac{i}{n-1}) = l_i/(n+1)$, $i = 0, \ldots, n-1$, with $q_{\omega;t}^{(n)\prime}(u) = dq_{\omega;t}^{(n)}/du$ satisfying $\frac{1}{3} \leq q_{\omega;t}^{(n)\prime}(u) \leq 3$, $u \in [0,1]$. Finally, consider the array of score-generating functions

$$J_{1;\omega;t}^{(n)} = J_1 \circ q_{\omega;t}^{(n)} \quad \text{and} \quad J_{2;\omega;t}^{(n)} = J_2 \circ q_{\omega;t}^{(n)},$$

$\max(\omega_1, \omega_2) \leq n$, $K+1 \leq t \leq n$. It is easily checked that

$$\text{(4.17a)} \quad \mu^{(n)}_{j;\omega;t} = \int_0^1 J^{(n)}_{j;\omega;t}(u)du = O(n^{-3/4}), \qquad j = 1, 2$$

$$\text{(4.17b)} \quad (\sigma^{(n)}_{j;\omega;t})^2 = \int_0^1 [J^{(n)}_{j;\omega;t}(u)]^2 du - (\mu^{(n)}_{j;\omega;t})^2 = 1 - O(n^{-1/2}), \qquad j = 1, 2$$

and

$$\text{(4.17c)} \quad \int_0^1 [J^{(n)}_{j;\omega;t}(u)]^4 du = O(1), \qquad j = 1, 2,$$

where the $O(\cdot)$ quantities are uniform in t and ω, i.e. (4.17a), for instance, is to be interpreted as

$$\text{(4.17a}'\text{)} \qquad \limsup_{n\to\infty} \left[n^{3/4} \max_{K+1\leq t\leq n} \max_{1\leq\omega_1,\omega_2\leq n} |\mu^{(n)}_{j;\omega;t}| \right] < \infty.$$

After adequate standardization, the score-generating functions $J^{(n)}_{j;\omega;t}$ thus all satisfy assumptions (A1), (A2) and (A_r) with $r = 1$.

For all given t between $2K+1$ and $n-K$, $T^{(n)}_a$ can be decomposed into $T^{(n)}_a = T^{(n)a}_{1;t} + T^{(n)a}_{2;t}$, with

$$T^{(n)a}_{1;t} = (n-K)^{-1/2} \sum_{\substack{s=K+1 \\ s\neq t}}^{n} J_1\left(\frac{R^{(n)}_s}{n+1}\right) J_2\left(\frac{R^{(n)}_{s-K}}{n+1}\right)$$

and

$$T^{(n)a}_{2;t} = (n-K)^{-1/2} J_1\left(\frac{R^{(n)}_t}{n+1}\right) J_2\left(\frac{R^{(n)}_{t-K}}{n+1}\right).$$

Conditionally upon $\Omega^{(n)}_t = \omega$,

$$T^{(n)a}_{2;t} = (n-K)^{-1/2} J_1\left(\frac{\omega_1}{n+1}\right) J_2\left(\frac{\omega_2}{n+1}\right), \qquad J_1\left(\frac{R^{(n)}_{t-K}}{n+1}\right) = J_1\left(\frac{\omega_1}{n+1}\right)$$

and

$$J_2\left(\frac{R^{(n)}_t}{n+1}\right) = J_2\left(\frac{\omega_1}{n+1}\right)$$

are fixed, and

$$T^{(n)a}_{1;t} = (n-K)^{-1/2} \left\{ \sum_{\substack{s=K+1 \\ s\notin\{t,t\pm K\}}}^{n} J_1\left(\frac{R^{(n)}_s}{n+1}\right) J_2\left(\frac{R^{(n)}_{s-K}}{n+1}\right) + J_1(\omega_2) J_2\left(\frac{R^{(n)}_{t-2K}}{n+1}\right) + J_1\left(\frac{R^{(n)}_{t+K}}{n+1}\right) J_2(\omega_1) \right\}.$$

Letting $V_i = U_{l_i}^{(n)}$, $i = 1, \ldots, n-2$, with ranks $\boldsymbol{Q}^{(n)} = (Q_1^{(n)}, \ldots, Q_{n-2}^{(n)})$, we thus have, *in distribution*,

$$(4.18) \qquad T_{1;t}^{(n)a} \stackrel{\mathcal{L}}{=} (n-K)^{-1/2} \left\{ \sum_{s=K+1}^{n-3} J_{1;\omega;t}^{(n)}\left(\frac{Q_s^{(n)}}{n+1}\right) J_{2;\omega;t}^{(n)}\left(\frac{Q_{s-K}^{(n)}}{n+1}\right) + J_1\left(\frac{\omega_2}{n+1}\right) J_{2;\omega;t}^{(n)}\left(\frac{Q_{n-4}^{(n)}}{n+1}\right) + J_{1;\omega;t}^{(n)}\left(\frac{Q_{n-3}^{(n)}}{n+1}\right) J_2\left(\frac{\omega_1}{n+1}\right) \right\} = \bar{T}_{1;t}^{(n)a}, \quad \text{say},$$

where $\boldsymbol{Q}^{(n)}$ is (conditionally) uniformly distributed over the $(n-2)!$ possible permutations of $\{1, \ldots, n-2\}$. Associated with $T_{1;t}^{(n)a}$, consider the sum of K-dependent variables

$$S_{1;t}^{(n)} = (n-K)^{-1/2} \left\{ \sum_{s=K+1}^{n-3} J_1(V_s) J_2(V_{s-K}) + J_1\left(\frac{\omega_2}{n+1}\right) J_2(V_{n-4}) + J_1(V_{n-3}) J_2\left(\frac{\omega_1}{n+1}\right) \right\}.$$

The same method of proof as in Lemma 4.1 can be used to show that

$$(4.19) \qquad E_{\omega;t}^{(n)}[(\bar{T}_{1;t}^{(n)a} - S_{1;t}^{(n)})^2] = \left[1 + n^{-1}\left(J_1^2\left(\frac{\omega_2}{n+1}\right) + J_2^2\left(\frac{\omega_1}{n+1}\right)\right)\right] O(n^{-2\delta-1/2}),$$

where the $O(\cdot)$ quantity is uniform in t and ω (same acception as in (4.17a)). In turn, (4.19) entails

$$(4.20) \qquad |E_{\omega;t}^{(n)}[\exp(iuT_{1;t}^{(n)a})] - E_{\omega;t}^{(n)}[\exp(iuS_{1;t}^{(n)})]| \le |u|\{E_{\omega;t}^{(n)}[(\bar{T}_{1;t}^{(n)a} - S_{1;t}^{(n)})^2]\} = |u|\left\{1 + n^{-1}\left[J_1^2\left(\frac{\omega_2}{n+1}\right) + J_2^2\left(\frac{\omega_1}{n+1}\right)\right]\right\}^{1/2} O(n^{-\delta-1/4}),$$

where $O(\cdot)$ again is uniform in ω and t, as explained in (4.17a′). Since $S_{1;t}^{(n)}$ is (conditionally) a sum of K-dependent summands (not all of them are identically distributed) with finite variance, denoting by $F_{1;\omega;t}^{(n)}$ the (conditional) distribution of $S_{1;t}^{(n)}/[\mathrm{Var}(S_{1;t}^{(n)})]^{1/2}$, we obtain from Erickson ((1974), Theorem B) that

$$(4.21) \qquad \left| E_{\omega;t}^{(n)}\left[\exp\left(\frac{iuS_{1;t}^{(n)}}{[\mathrm{Var}(S_{1;t}^{(n)})]^{1/2}}\right) - \exp\left(-\frac{u^2}{2}\right)\right] \right| = \left| \int e^{iux} d(F_{1;\omega;t}^{(n)}(x) - \Phi(x)) \right| \le |u| \int |F_{1;\omega;t}^{(n)}(x) - \Phi(x)| dx = |u| O(n^{-1/2}),$$

uniformly again in ω and t. Moreover, since $\mathrm{Var}(S_{1;t}^{(n)}) = 1 + (n-K)^{-1}[J_1^2(\frac{\omega_2}{n+1}) + J_2^2(\frac{\omega_1}{n+1}) - 3]$, still uniformly in ω and t,

$$(4.22)\qquad \left| E_{\omega;t}^{(n)}\left[\exp\left(\frac{iuS_{1;t}^{(n)}}{[\mathrm{Var}(S_{1;t}^{(n)})]^{1/2}} - E_{\omega;t}^{(n)}(iuS_{1;t}^{(n)})\right)\right]\right|$$
$$\leq |u||(\mathrm{Var}(S_{1;t}^{(n)}))^{1/2} - 1|$$
$$= |u|O\left(n^{-1/2}\left|J_1^2\left(\frac{\omega_2}{n+1}\right) + J_2^2\left(\frac{\omega_1}{n+1}\right) - 3\right|^{1/2}\right).$$

Denote by $E_{\omega;t;\cdot}^{(n)}$ the conditional expectation $E[\cdot \mid \Omega_t^{(n)}; \boldsymbol{U}_{(\cdot)}^{(n)}]$ taken at $\Omega_t^{(n)} = \omega$. Owing to the fact that $T_{1;t}^{(n)a}$ and $\boldsymbol{U}_{(\cdot)}^{(n)}$ are independent, and combining (4.20), (4.21) and (4.22), we obtain

$$(4.23)\qquad E_{\omega;t;\cdot}^{(n)}[\exp(iuT_{1;t}^{(n)a})]$$
$$= \exp(-u^2/2)$$
$$+ |u|\left[1 + n^{-1/2+2\delta}\left(J_1^2\left(\frac{\omega_2}{n+1}\right) + J_2^2\left(\frac{\omega_1}{n+1}\right)\right)\right]^{1/2}$$
$$\cdot O(n^{-(1/4)-\delta}),$$

uniformly in t and ω, from which we deduce that

$$E[Z_{2;t}^e \exp(iuT_{1;t}^{(n)a})]$$
$$= E\{\{E_{\Omega;t;\cdot}^{(n)}[\exp(iuT_{1;t}^{(n)a})]\}Z_{2;t}^e \exp(iuT_{2;t}^{(n)a})\}$$
$$= O\Bigg([e^{-u^2/2}\{E[Z_{2;t}^e] + |u|E[Z_{2;t}^e T_{2;t}^{(n)a}]\}$$
$$+ |u|E\left[Z_{2;t}^e(1 + n^{-1/2+2\delta}\left(J_1^2\left(\frac{R_{t-K}^{(n)}}{n+1}\right) + J_2^2\left(\frac{R_t^{(n)}}{n+1}\right)\right)^{1/2}\right]$$
$$\cdot O(n^{-\delta-1/4})\Bigg).$$

Now, for $2K+1 \leq t \leq n-K$,

$$E[Z_{2;t}^e] = [n(n-1)]^{-1}\sum_{1\leq t_1 \neq}\sum_{t_2 \leq n} E\{[a_1^e(t_1) - J_1(U_{(t_1)}^{(n)})]J_2(U_{(t_2)}^{(n)})\}$$
$$= -(n-1)^{-1} + [n(n-1)]^{-1}\sum_{i=1}^{n} E\{[a_1^e(i) - J_1(U_{(i)}^{(n)})][a_2^e(i) - J_2(U_{(i)}^{(n)})]\};$$

hence, from Cauchy-Schwarz's inequality and (3.2),

$$(4.24)\qquad E[Z_{2;t}^e] = O(n^{-1}),$$

uniformly in t. Next,

$$\begin{aligned}
&(n-K)^{1/2} Z^e_{2;t} T^{(n)a}_{2;t} \\
&= \{a^e_1(R^{(n)}_t) - J_1(U^{(n)}_t)\} a^e_2(R^{(n)}_{t-K}) J_1\left(\frac{R^{(n)}_t}{n+1}\right) J_2\left(\frac{R^{(n)}_{t-K}}{n+1}\right) \\
&\quad - \{a^e_1(R^{(n)}_t) - J_1(U^{(n)}_t)\}\{a^e_2(R^{(n)}_{t-K}) - J_2(U^{(n)}_{t-K})\} \\
&\quad \cdot J_1\left(\frac{R^{(n)}_t}{n+1}\right) J_2\left(\frac{R^{(n)}_{t-K}}{n+1}\right),
\end{aligned}$$

where the expectation of the first term (in the right-hand side), conditionally upon $\boldsymbol{R}^{(n)}$, hence also unconditionally, is zero, so that, using Cauchy-Schwarz's inequality, (4.5), (3.2) and (3.13) again,

$$|E[Z^e_{2;t} T^{(n)a}_{2;t}]| = O(n^{-2\delta-1/2}). \tag{4.25}$$

In a very similar way, using (3.1), (3.2) and (3.4), we obtain

$$E\left[Z^e_{2;t} J^2_1\left(\frac{R^{(n)}_{t-K}}{n+1}\right)\right] = O(n^{-1-2\delta})$$

and

$$E\left[Z^e_{2;t} J^2_2\left(\frac{R^{(n)}_t}{n+1}\right)\right] = O(n^{-1-2\delta}),$$

whence

$$\begin{aligned}
&E\left\{Z^e_{2;t}\left[1 + n^{-(1/2)+2\delta}\left(J^2_1\left(\frac{R^{(n)}_{t-K}}{n+1}\right) + J^2_2\left(\frac{R^{(n)}_t}{n+1}\right)\right)\right]^{1/2}\right\} \\
&= E[Z^e_{2;t}] + O\left(n^{-1/2+2\delta} E\left\{Z^e_{2;t}\left(J^2_1\left(\frac{R^{(n)}_{t-K}}{n+1}\right) + J^2_2\left(\frac{R^{(n)}_t}{n+1}\right)\right)\right\}\right) \\
&= O(n^{-1}).
\end{aligned} \tag{4.26}$$

Combining (4.23), (4.24), (4.25) and (4.26) yields, for $|u| \leq \log n$ and uniformly in t between $2K+1$ and $n-K$,

$$E[Z^e_{2;t} \exp(iuT^{(n)}_a)] = e^{-u^2/2} O(n^{-1}) + |u| O(n^{-(5/4)-\delta}). \tag{4.27}$$

Consequently, after summing over $t = 2K+1, \ldots, n-K$,

$$\left|(n-K)^{-1/2} \sum_{t=2K+1}^{n-K} E[Z^e_{2;t} \exp(iuT^{(n)}_a)]\right| = e^{-u^2/2} O(n^{-1}) + |u| O(n^{-\delta-3/4}),$$

and, integrating with respect to $u \in [0, 1]$,

$$\int_0^1 \left| (n-K)^{-1/2} \sum_{t=2K+1}^{n-K} E[Z_{2;t}^c \exp(iuT_a^{(n)})] \right| du = O(n^{-1/2}),$$

as was to be shown. □

The main theorem, in the exact score case, follows from piecing together Lemmas 4.2, 4.3, 4.4 and 4.5. The approach we have been using for exact scores however apparently does not extend to the approximate score situation. For instance, we could not obtain the required $O(n^{-1})$ order for terms of the form (for exact scores, cf. (4.25))

$$\begin{aligned} &E[Z_{2;t}^a T_{2;t}^{(n)a}] \\ &\quad = (n-K)^{-1/2} E\left\{ \left[J_1\left(\frac{R_{t-K}^{(n)}}{n+1}\right) - J_1(U_t^{(n)}) \right] J_1^2\left(\frac{R_t^{(n)}}{n+1}\right) J_2\left(\frac{R_{t-K}^{(n)}}{n+1}\right) \right\}. \end{aligned}$$

Another method of attack is thus necessary.

Here again, we need some preparatory lemmas (Lemmas 4.6 and 4.7). For all $\beta \in [0, 1]$, let $m = [n^{1-\beta}]$ be the largest integer smaller than or equal to $n^{1-\beta}$, and denote by $\mathcal{I}_m$ the set of integers $\{1, \ldots, m, n-m+1, \ldots, n\}$.

LEMMA 4.6. *Let the (score-generating) function J satisfy assumptions* (A1) *and* (A2). *Then, for all $\beta \in [0, 1]$ and $l = 1, 2, 3, 4$,*

$$\frac{1}{n} \sum_{i \in \mathcal{I}_m} \left| J\left(\frac{i}{n+1}\right) \right|^l = O(n^{-\beta(1-(l/4)+l\delta)}). \tag{4.28}$$

If J moreover satisfies assumption (A_r) *with $r = 1$, then*

$$\sum_{i \in \mathcal{I}_m} \left| J\left(\frac{i}{n+1}\right) \left[J\left(\frac{i}{n+1}\right) - EJ(U_{(i)}^{(n)}) \right] \right| = O(n^{-2\delta+1/2}). \tag{4.29}$$

PROOF. From assumption (A2),

$$\begin{aligned} \frac{1}{n} \sum_{i \in \mathcal{I}_m} \left| J\left(\frac{i}{n+1}\right) \right|^l &= O\left(\frac{1}{n} \sum_{i=1}^{m} \left(\frac{i}{n+1}\right)^{-(l/4)+l\delta} \right) \\ &= O\left(\int_0^{n^{-\beta}} u^{-(l/4)+l\delta} du \right) = O(n^{-\beta(1-(l/4)+l\delta)}); \end{aligned}$$

as for (4.29), it follows along the same lines as Lemma 3.1.

LEMMA 4.7. *For given $\beta \in [0,1]$, define*

$$(4.30a) \quad Z_{11;t}^{(n)} = J_1\left(\frac{R_t^{(n)}}{n+1}\right)\left[J_2\left(\frac{R_{t-K}^{(n)}}{n+1}\right) - a_2^e(R_{t-K}^{(n)})\right] I[R_{t-K}^{(n)} \notin \mathcal{I}_m],$$

$$(4.30b) \quad Z_{12;t}^{(n)} = J_1\left(\frac{R_t^{(n)}}{n+1}\right)\left[J_2\left(\frac{R_{t-K}^{(n)}}{n+1}\right) - a_2^e(R_{t-K}^{(n)})\right] I[R_{t-K}^{(n)} \in \mathcal{I}_m],$$

$$(4.30c) \quad Z_{21;t}^{(n)} = \left[J_1\left(\frac{R_t^{(n)}}{n+1}\right) - a_1^e(R_t^{(n)})\right] a_2^e(R_{t-K}^{(n)}) I[R_t^{(n)} \notin \mathcal{I}_m],$$

$$(4.30d) \quad Z_{22;t}^{(n)} = \left[J_1\left(\frac{R_t^{(n)}}{n+1}\right) - a_1^e(R_t^{(n)})\right] a_2^e(R_{t-K}^{(n)}) I[R_t^{(n)} \in \mathcal{I}_m].$$

Then, under the assumptions of the main theorem,

$$(4.31) \quad \begin{array}{ll} E[Z_{12;t}^{(n)}] = O(n^{-1-(\beta/2)-2\delta}) & E[Z_{22;t}^{(n)}] = O(n^{-2\delta-3/2}) \\ E[(Z_{12;t}^{(n)})^2] = O(n^{-\delta-1/2}) & E[(Z_{22;t}^{(n)})^2] = O(n^{-\delta-1/2}) \end{array}$$

as $n \to \infty$, uniformly in t, $K+1 \le t \le n$.

PROOF. Starting from the definition of $Z_{12;t}^{(n)}$, we obtain, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
E[Z_{12;t}^{(n)}] &= [n(n-1)]^{-1}\left\{\sum_{1\le i\ne j\le n}\sum J_1\left(\frac{i}{n+1}\right)\left[J_2\left(\frac{j}{n+1}\right) - a_2^e(j)\right] I[j \in \mathcal{I}_m]\right\} \\
&= [n(n-1)]^{-1}\left\{\sum_{i=1}^{n} J_1\left(\frac{i}{n+1}\right)\right\}\sum_{j\in\mathcal{I}_m}\left[J_2\left(\frac{j}{n+1}\right) - a_2^e(j)\right] \\
&\quad - [n(n-1)]^{-1}\sum_{j\in\mathcal{I}_m} J_1\left(\frac{i}{n+1}\right)\left[J_2\left(\frac{i}{n+1}\right) - a_2^e(i)\right] \\
&\le [n(n-1)]^{-1}\left|\sum_{i=1}^{m} J_1\left(\frac{i}{n+1}\right)\right|(2m)^{1/2} \\
&\quad \cdot\left\{\sum_{j\in\mathcal{I}_m}\left[J_2\left(\frac{j}{n+1}\right) - a_2^e(j)\right]^2\right\}^{1/2} \\
&\quad + [n(n-1)]^{-1}\left\{\sum_{i\in\mathcal{I}_m} J_1^2\left(\frac{i}{n+1}\right)\right\}^{1/2} \\
&\quad \cdot\left\{\sum_{j\in\mathcal{I}_m}\left[J_2\left(\frac{j}{n+1}\right) - a_2^e(j)\right]^2\right\}^{1/2}.
\end{aligned}$$

It follows from (3.3), (4.5) and (4.28) that the first term in this latter expression is $O(n^{-1-(\beta/2)-2\delta})$, whereas the second one is $O(n^{-(5/4)-(\beta/4)-\delta(1+\beta)})$. Since $0 < \delta < 1/4$, the desired result is established. Also, from (4.29)

$$E[Z_{22;t}^{(n)}] = -[n(n-1)]^{-1} \sum_{i \subset \mathcal{I}_m} a_2^e(i) \left[J_2\left(\frac{i}{n+1}\right) - a_2^e(i) \right] = O(n^{-2\delta-3/2})$$

and, from (3.3) and (4.5),

$$\begin{aligned} E[(Z_{12;t}^{(n)})^2] &\leq [n(n-1)]^{-1} \left\{ \sum_{i=1}^{n} J_1^2\left(\frac{i}{n+1}\right) \right\} \left\{ \sum_{j \in \mathcal{I}_m} \left[J_2\left(\frac{j}{n+1}\right) - a_2^e(j) \right]^2 \right\} \\ &= O(n^{-\delta-1/2}); \end{aligned}$$

$E[(Z_{22;t}^{(n)})^2]$ is treated similarly. □

We now proceed to the final step. The main theorem indeed follows from Lemmas 4.2 and 4.3 if we are able to prove the following analogue, for approximate scores, of (4.14).

LEMMA 4.8. *Under the assumptions of the main theorem,*

$$\text{(4.32)} \qquad \int_{|u| \leq \log n} |E[(S_K^{(n)} - T_a^{(n)}) \exp(iuT_a^{(n)})]| du = O(n^{-1/2}),$$

as $n \to \infty$.

PROOF. First, note that

$$\text{(4.33)} \qquad \begin{aligned} &\int_{|u| \leq \log n} |E[(S_K^{(n)} - T_a^{(n)}) \exp(iuT_a^{(n)})]| du \\ &\quad \leq \int_{|u| \leq \log n} |E[(S_K^{(n)} - T_e^{(n)}) \exp(iuT_a^{(n)})]| du \\ &\quad\quad + \int_{|u| \leq \log n} |E[(T_e^{(n)} - T_a^{(n)}) \exp(iuT_a^{(n)})]| du. \end{aligned}$$

We know from Lemmas 4.4 and 4.5 that the first integral in the right hand side of (4.33) is $O(n^{-1/2})$. It thus remains to show that the second integral is $O(n^{-1/2})$, too. Therefore, decompose $T_a^{(n)} - T_e^{(n)}$ into

$$\text{(4.34)} \qquad \begin{aligned} T_a^{(n)} - T_e^{(n)} = (n-K)^{-1/2} \Bigg\{ &\sum_{t=K+1}^{n} Z_{11;t}^{(n)} + \sum_{t=K+1}^{n} Z_{12;t}^{(n)} \\ &+ \sum_{t=K+1}^{n} Z_{21;t}^{(n)} + \sum_{t=K+1}^{n} Z_{22;t}^{(n)} \Bigg\}, \end{aligned}$$

where the $Z^{(n)}_{ij;t}$'s are given by (4.30), for fixed but arbitrary $\beta \in [0,1]$. Using the same arguments as in the proof of Lemma 4.1, we obtain, from Lemma 4.7,

$$E\left\{\left[(n-K)^{-1/2}\sum_{t=K+1}^{n} Z^{(n)}_{11;t}\right]^2\right\} = O(n^{-2+(3\beta/2)-2\beta\delta}), \tag{4.35}$$

$$E\left\{\left[(n-K)^{-1/2}\sum_{t=K+1}^{n} Z^{(n)}_{21;t}\right]^2\right\} = O(n^{-2+(3\beta/2)-2\beta\delta}), \tag{4.36}$$

hence, for $\zeta^{(n)}_t = Z^{(n)}_{11;t}$ or $Z^{(n)}_{21;t}$,

$$\left|E\left\{(n-K)^{-1/2}\sum_{t=K+1}^{n} \zeta^{(n)}_t \exp(iuT^{(n)}_a)\right\}\right| = O(n^{-1+(3\beta/4)-\beta\delta}), \tag{4.37}$$

and, for $\xi^{(n)}_t = Z^{(n)}_{12;t}$ or $Z^{(n)}_{22;t}$,

$$\left|E\left\{(n-K)^{-1/2}\sum_{t=K+1}^{2K} \xi^{(n)}_t \exp(iuT^{(n)}_a)\right\}\right| = O(n^{-\delta-3/4}), \tag{4.38}$$

$$\left|E\left\{(n-K)^{-1/2}\sum_{t=K+1}^{n} \xi^{(n)}_t \exp(iuT^{(n)}_a)\right\}\right| = O(n^{-\delta-3/4}). \tag{4.39}$$

The remaining terms require the same conditional approach as in the exact score case. In view of (4.23) and assumption (A1), we obtain, for $2K+1 \le t \le n-K$,

$$\begin{aligned}
E[Z^{(n)}_{12;t}] &= E\{\{E^{(n)}_{\Omega;t;\cdot}[\exp(iuT^{(n)a}_{1;t})]\} Z^{(n)}_{12;t}\exp(iuT^{(n)a}_{2;t})\}\\
&= O\Bigg(e^{-u^2/2}\{E[Z^{(n)}_{12;t}] + |u| E[Z^{(n)}_{12;t}T^{(n)a}_{2;t}]\}\\
&\quad + |u| E\Bigg\{Z^{(n)}_{12;t}\Bigg[1+n^{-(1/2)+2\delta}\\
&\quad \cdot\left(J_1^2\left(\frac{R^{(n)}_{t-K}}{n+1}\right) + J_2^2\left(\frac{R^{(n)}_t}{n+1}\right)\right)\Bigg]^{1/2}\Bigg\} O(n^{-\delta-1/2})\Bigg)
\end{aligned}$$

where, from (4.29),

$$\begin{aligned}
&|(n-K)^{1/2} E[Z^{(n)}_{12;t}T^{(n)a}_{2;t}]|\\
&\quad \le \left\{(n-1)^{-1}\left[\sum_{i=1}^{n} J_1^2\left(\frac{i}{n+1}\right)\right]\right\}\\
&\qquad \cdot\left\{n^{-1}\sum_{j\in\mathcal{I}_m}\left|J_2\left(\frac{j}{n+1}\right)\left[J_2\left(\frac{j}{n+1}\right) - a_2^e(j)\right]\right|\right\}\\
&\quad = O(n^{-2\delta-1/2}).
\end{aligned}$$

It thus follows from (4.29) that, uniformly in t,

$$E[Z_{12;t}^{(n)} T_{2;t}^{(n)a}] = O(n^{-1-2\delta}). \tag{4.40}$$

The same reasoning as in the exact score case (cf. (4.27)) yields, for $|u| \leq \log n$, uniformly in $2K+1 < t < n-K$,

$$E[Z_{12;t}^{(n)} \exp(iuT_a^{(n)})] = e^{-u^2/2} O(n^{-1-2\delta}) + |u| O(n^{-1-2\beta\delta}), \tag{4.41}$$

so that

$$\left| E\left\{ (n-K)^{-1/2} \sum_{t=2K+1}^{n-K} Z_{12;t}^{(n)} \exp(iuT_a^{(n)}) \right\} \right| \\ = e^{-u^2/2} O(n^{-1-2\delta}) + |u| O(n^{-1-2\beta\delta}).$$

Combining (4.38), (4.39) and (4.42) and integrating over $|u| \leq \log n$ eventually yields

$$\int_{|u| \leq \log n} \left| E\left\{ (n-K)^{-1/2} \sum_{t=K+1}^{n} Z_{12;t}^{(n)} \exp(iuT_a^{(n)}) \right\} \right| du = O(n^{-1/2}). \tag{4.43}$$

A similar reasoning leads to

$$\int_{|u| \leq \log n} \left| E\left\{ (n-K)^{-1/2} \sum_{t=K+1}^{n} Z_{22;t}^{(n)} \exp(iuT_a^{(n)}) \right\} \right| du = O(n^{-1/2}). \tag{4.44}$$

The end of the proof simply consists in piecing together (4.43), (4.44) and an integrated version of (4.37), then choosing $\beta = 2/3$. □

The proof of the main theorem is thus complete.

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