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REPLENISHMENT-DEPLETION URN IN EQUILIBRIUM

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Abstract. An urn has balls of colors C_1 and C_2. It is replenished (R) by balls of both colors and then depleted by (D) the same number; this constitutes a cycle. When $R = D$, the system is closed and equilibrium will be reached after many cycles. The ultimate distribution is found only when the replenishment is the same for each color. Asymptotic normal and asymptotic binomial distributions arise when the parameters reach extreme values. For the multicolor urn an expression is given for the correlation between the number of balls of any two colors.

Key words and phrases: Bernard's urn, beta integral transforms, finite difference calculus, generating functions, hypergeometric distributions, hypergeometric functions, moments, replenishment-depletion urn.

1. Introduction

In a previous paper, Shenton and Bowman (1996) consider a Bernard (1977) urn with balls of s colors, each color being subject to a certain replenishment; a random depletion takes place in the sense that a fixed number of balls are randomly removed from the urn. This constitutes a cycle, the replenishment-depletion (R-D) pattern being repeated in form; identical cycles are possible. Solutions depend on the usage of the finite difference calculus, and factorial moment generating functions (fmgf).

The R D phase may result in a closed system; thus R=D at a cycle, so that the total number of balls in the system at the completion of any cycle is constant. This is the urn in equilibrium ultimately. Under what circumstances do simple solutions exist? How is the problem affected by the number of colors involved, for example, univariate (2 colors) against 3 colors (bivariate).

We shall use the same notation as described in the 1996 paper; in particular formula (2.2) and those in Sections 5.2 and 5.3. Note that all parameters involved are assumed to belong to the class of positive integers—one exception is the cycle parameter j which may be zero corresponding to the initial state of the urn.

2. The 2-color urn in equilibrium

Initially there are 0 balls of color C_1, and k balls of color C_2. At the j-th cycle the replenishments are p balls of C_1, and q balls of C_2, with depletion $p+q$.

Thus the R-D pattern is $(p, q; p+q)$. The system is closed. The fmgf is, at cycle m, in terms of finite difference operators,

(2.1)
$$f_m(\alpha) = \left(1 + \frac{\alpha}{E_{x_m}}\right)^p \left(1 + \frac{\alpha}{E_{x_m}E_{x_{m-1}}}\right)^p \cdots \\ \cdots \left(1 + \frac{\alpha}{E_{x_m}E_{x_{m-1}}\cdots E_{x_1}}\right)^p \prod_{r=1}^{m} \frac{x_r^{(R)}}{(k+R)^{(R)}} \Bigg|_{x_r=k+R}$$
$$(m = 1, 2, \ldots; R = p+q)$$

and this refers to the fmgf of balls of color C_1. An operator E_{x_λ} $(\lambda = 1, 2, \ldots, m)$ operates on the corresponding component in $x_r^{(R)}$.

When $m \to \infty$ we set $f_m = f$, where

$$f(\alpha) = 1 + \alpha y + \alpha^2 y_2 + \cdots$$

where $y_j = \mu'_{(j)}/j!$ and is a **reduced factorial moment**, the factorial moment being $\mu'_{(j)}$.

This notation is merely a convenience device.

Now (2.1) may be factorized. Thus

(2.2)
$$f_m(\alpha) = \left(1 + \frac{\alpha}{E_{x_m}}\right)^p f_{m-1}\left(\frac{\alpha}{E_{x_m}}\right) \frac{x_m^{(R)}}{(k+R)^{(R)}} \Bigg|_{x_m=k+R}$$

and in the limit there is the finite difference equation

$$(k+R)^{(R)} f(\alpha) = \left(1 + \frac{\alpha}{E}\right)^p f\left(\frac{\alpha}{E}\right) x^{(R)} \Bigg|_{x=k+R},$$

or equating coefficients of α^r,

$$y_r = \left[y_r + \binom{p}{1} y_{r-1} + \binom{p}{2} y_{r-2} + \cdots + \binom{p}{p} y_{r-p}\right] \frac{(k+R-r)^{(R)}}{(k+R)^{(R)}}$$
$$(r = 1, 2, \ldots; y_0 = 1; y_r = 0, r < 0 \text{ and } r > k).$$

This equation may be further simplified to

$$y_r = \left(\sum_{s=0}^{p} \binom{p}{s} y_{r-s}\right) \frac{k^{(r)}}{(k+R)^{(r)}},$$

so that

(2.3)
$$y_r = \frac{\left(\sum_{s=1}^{p} \binom{p}{s} y_{r-s}\right) k^{(r)}}{(k+R)^{(r)} - k^{(r)}} = \frac{k^{(r)} \sum_{s=1}^{r} \binom{p}{s} y_{r-s}}{\sum_{s=1}^{r} \binom{r}{s} R^{(s)} k^{(r-s)}}.$$

For example

$$(2.4)\qquad \begin{cases} y_1 = pk/R \\ y_2 = \dfrac{p}{2R^2}\dfrac{k^{(2)}[2pk + R(p-1)]}{(2k + R - 1)}. \end{cases}$$

If $p = q$, the symmetric case, then

$$y_1 = \frac{k}{2}, \qquad y_2 = \frac{\binom{k}{2}(k+p)^{(2)}}{(2k+2p)^{(2)}}, \qquad (\mu_{(2)} = 2y_2)$$

Higher factorial moments become complicated since the numerator and denominator in (2.3) do not have factors in common, unless $p = q$.

3. The symmetrical case, $p = q$

3.1 *Formulas*

Using (2.3) and simplifying the algebra, we find

$$(3.1)\qquad y_3 = \binom{k}{3}\frac{(k+p)^{(3)}}{(2k+2p)^{(3)}},$$

and

$$(3.2)\qquad y_4 = \binom{k}{4}\frac{(k+p)^{(4)}}{(2k+2p)^{(4)}}.$$

There is the obvious conjecture for the r-th reduced factorial moment y_r when equilibrium is reached, that

$$(3.3)\qquad y_{(r)} = \binom{k}{r}\frac{(k+p)^{(r)}}{(2k+2p)^{(r)}}, \qquad (s = 1, 2, \ldots, k).$$

Inserting this in (2.3) linearized, there is the conjecture that

$$\sum_{s=0}^{p}\binom{p}{s}y_{r-s} = \frac{(k+2p)^{(2p)}}{(k+2p-r)^{(2p)}}y_r,$$

or, in polynomial form

$$(3.4)\qquad \sum_{r=0}^{p} k^{(s-r)}(k+p)^{(s-r)}s^{(r)}\binom{p}{r}(2k+2p-s+r)^{(r)}$$
$$= (k+p)^{(s)}(k+2p)^{(s)}. \qquad (s = 1, 2, \ldots, k)$$

The left side will terminate if $r > p$ or $r > s$. Using the Maple mathematical manipulation language, (3.4) has been checked as an identity in k and p for $s =$

$1, 2, \ldots, 10, 15$. An algebraic proof appears by dividing both sides of (3.4) by $k^{(s-p)}(k+p)^{(s-p)}$ to obtain an equivalent identity

$$\binom{k+2p}{2p}\binom{2p}{p} = \sum_r \binom{k+p-s}{r}\binom{k+2p-s}{r}\binom{s}{p-r}\binom{2k+2p-s+r}{r} \Big/ \binom{p}{r}.$$

The equality is a special case of a more elegant polynomial identity,

$$\binom{x+z}{n}\binom{y+x}{n} = \sum_k \binom{x}{k}\binom{y}{k}\binom{z}{n-k}\binom{x+y+z-k}{n-k} \Big/ \binom{n}{k},$$

which is the formula (Gould (1972), p. 70, (16.3)). Gould credits the formula to H. L. Krall, but does not cite his paper in the bibliography.

3.2 *Numerical illustrations*

We can test out the formulas for the moments in equilibrium given in (2.4), (3.1) and (3.2) by considering an urn subject to finite cycles. Formula (2.2) with $p = q$, $R = 2p$ now concerns the factorial moments for cycles of m, $m = 1, 2, \ldots$. Recursive schemes are easily set up and indeed are described in Shenton and Bowman (1996), Section 5.2. For example

$$f_{m,1} = (pf_{m-1,0} + f_{m-1,1})\frac{k}{k+2p} \qquad (m = 1, 2, \ldots;)$$

$$f_{m,2} = \left(\binom{p}{2}f_{m-1,0} + \binom{p}{1}f_{m-1,1} + f_{m-1,2}\right)\frac{k^{(2)}}{(k+2p)^{(2)}}$$

and so on for higher moments.

It does not matter what the R-D comes to in the first set of cycles, $m = 1, 2, \ldots, m^*$, as long as from the (m^*+1)-th cycle onwards replenishment of each color is the same $(p-q)$ and the k-parameter is held constant. We have chosen the scheme in Table 1.

Thus

$$f_1(\alpha) = \left(1 + \frac{\alpha}{E_x}\right)^6 \frac{x^{(11)}}{(k+11)^{(11)}}\Bigg|_{x=k+11}$$

$$f_2(\alpha) = \left(1 + \frac{\alpha}{E_{x_2}}\right)^4 \left(1 + \frac{\alpha}{E_{x_2}E_{x_1}}\right)^6 \frac{x_1^{(11)}}{(k+11)^{(11)}}\frac{x_1^{(14)}}{(k+14)^{(14)}}\Bigg|_{x_1=k+11, x_2=k+14}$$

and so on, there being a break for 4 or more cycles.

Tables 2.1 and 2.2 show that the distributions of C_1 balls are on the whole close to normality as indicated by the skewness $(\sqrt{\beta_1} - \mu_3/\mu_2^{3/2} - \gamma_1)$ and kurtosis $(\beta_2 = \mu_4/\mu_2^2 = \gamma_2 + 3)$, especially as cycle values increase. Moreover the last few cycles in each case show moments with only slight differences. We shall give the asymptotic moments in the next section.

Table 1 Parameters for urns approaching the equilibrium state.

m	C_1	C_2	
0	0	k	Depletion
1	6	5	11
2	4	10	14
3	1	11	12
$m > 3$	p	p	$2p$

3.3 *Asymptotic normality*

Consider the equilibrium urn ($p = q$) as k, the initial input of C_2 balls, becomes large. This can be done by evaluating the skewness and kurtosis; at least this provides information on non-normality as measured by departures of $\sqrt{\beta_1}$ and β_2 from 0 and 3 respectively.

We start with expressions for reduced factorial moments, given in (2.4), (3.1) and (3.2). These are transformed to factorial moments, then to non-central moments, and finally to central moments. Using $\mu'_{(s)}$, μ'_s, and μ_s, for these moments, we have for example

$$\mu'_s = \sum_{r=0}^{s} S_s^{(r)} \mu'_{(r)}$$

where $S_s^{(r)}$ is a Stirling number of the second kind, and with

$$\mu_r = \sum_{s=0}^{r} (-1)^s (\mu'_1)^s \mu'_{r-s}.$$

Now a glance at the expressions for y_r ($r = 1-4$) shows some common factors. In fact the basic denominators are $(2k + 2p - 1)$ and $(2k + 2p - 3)$. So we look for geometric type series terms in the parameters of $(2p - 1)/k$ and $(2p - 3)/k$, since $k \to \infty$.

$$(3.5) \qquad \begin{cases} \mu'_1 = k/2, \\ \mu_2 = \mu_{(2)} + \mu'_1 - \mu'^2_1 = \dfrac{k}{8} + \dfrac{1}{16} \displaystyle\sum_{i=0}^{\infty} (-1)^i (2p+1) \left(\dfrac{(2p-1)}{2k} \right)^i \\ \quad = \dfrac{k}{8} + \dfrac{k(2p+1)}{8(2k+2p-1)}, \\ \mu_3 = 0, \\ \mu_4 = \dfrac{3k^2}{64} + \dfrac{k(3p+1)}{32} - \dfrac{(2p+1)(6p-7)}{256} + \dfrac{(2p+1)}{512} \displaystyle\sum_{r=1}^{\infty} \dfrac{(-1)^s}{k^s} \\ \qquad \times \left[(2p^2+5)\left(p - \dfrac{1}{2}\right)^s - 3(2p-1)(2p+3)\left(p - \dfrac{3}{2}\right)^s \right]. \\ \qquad \left(k > p - \dfrac{1}{2}; p = 1, 2, \ldots \right) \end{cases}$$

Table 2.1. Convergence of moments to limiting form.

		$k = 16$			
Input p	Cycle	Mean	σ	$\sqrt{\beta_1}$	β_2
3	1	13.0370	1.0107	0.1246	2.7236
	2	9.0864	1.4755	−0.0066	2.9386
	3	5.7637	1.5264	0.0544	2.9336
	4	6.3736	1.5186	0.0515	2.9405
	5	6.8171	1.5218	0.0404	2.9437
	10	7.7593	1.5398	0.0087	2.9435
	15	7.9510	1.5420	0.0018	2.9432
	20	7.9900	1.5422	0.0004	2.9432
	25	7.9980	1.5422	0.0001	2.9432
	30	7.9996	1.5422	0.0000	2.9432
	35	7.9999	1.5422	0.0000	2.9432
	40	8.0000	1.5422	0.0000	2.9432
	45	8.0000	1.5422	0.0000	2.9432
	50	8.0000	1.5422	0.0000	2.9432
5	1	13.0370	1.0107	0.1246	2.7236
	2	9.0864	1.4755	−0.0066	2.9386
	3	5.7637	1.5264	0.0544	2.9336
	4	6.6238	1.5534	0.0427	2.9422
	5	7.1531	1.5726	0.0284	2.9431
	10	7.9253	1.5923	0.0027	2.9416
	15	7.9934	1.5927	0.0002	2.9416
	20	7.9994	1.5927	0.0000	2.9416
	25	7.9999	1.5927	0.0000	2.9416
	26	8.0000	1.5927	0.0000	2.9416
25	1	13.0370	1.0107	0.1246	2.7236
	2	9.0864	1.4755	−0.0066	2.9386
	3	5.7637	1.5264	0.0544	2.9336
	4	7.4579	1.7869	0.0234	2.9194
	5	7.8686	1.8041	0.0060	2.9169
	10	7.9999	1.8053	0.0000	2.9167
	11	8.0000	1.8053	0.0000	2.9167
50	1	13.0370	1.0107	0.1246	2.7236
	2	9.0864	1.4755	-0.0066	2.9386
	3	5.7637	1.5264	0.0544	2.9336
	4	7.6915	1.8748	0.0157	2.9034
	5	7.9575	1.8819	0.0022	2.9021
	6	7.9941	1.8820	0.0003	2.9021
	7	7.9992	1.8820	0.0000	2.9021
	8	7.9999	1.8820	0.0000	2.9021
	9	8.0000	1.8820	0.0000	2.9021

$\sqrt{\beta_1} = \gamma_1$, $\beta_2 = \gamma_2 + 3$.

Table 2.2. Convergence of moments to limiting form.

$k = 50$

Input p	Cycle	Mean	σ	$\sqrt{\beta}_1$	β_2
3	1	45.9016	0.8305	0.6654	3.0167
	2	38.9857	1.5256	0.1803	2.9161
	3	32.2465	1.9391	0.0663	2.9528
	4	31.4701	2.0648	0.0400	2.9623
	5	30.7769	2.1638	0.0228	2.9688
	10	28.2779	2.4327	−0.0076	2.9807
	15	26.8600	2.5280	−0.0094	2.9818
	20	26.0554	2.5625	−0.0067	2.9815
	25	25.5989	2.5751	−0.0041	2.9812
	30	25.3398	2.5795	−0.0024	2.9811
	35	25.1928	2.5811	−0.0014	2.9811
	30	25.1094	2.5817	−0.0008	2.9811
	45	25.0621	2.5819	−0.0005	2.9811
	50	25.0352	2.5820	−0.0003	2.9811
5	1	45.9016	0.8305	0.6654	3.0167
	2	38.9857	1.5256	0.1803	2.9161
	3	32.2465	1.9391	0.0663	2.9528
	4	31.0387	2.1442	0.0288	2.9671
	5	30.0323	2.2836	0.0095	2.9747
	10	27.0224	2.5588	−0.0093	2.9821
	15	25.8127	2.6109	−0.0052	2.9817
	20	25.3266	2.6208	−0.0023	2.9814
	25	25.1313	2.6227	−0.0009	2.9814
	30	25.0528	2.6230	−0.0004	2.9814
	35	25.0212	2.6231	−0.0002	2.9814
	40	25.0085	2.6231	−0.0001	2.9814
	45	25.0034	2.6231	0.0000	2.9814
	50	25.0014	2.6231	0.0000	2.9814
25	1	45.9016	0.8305	0.6654	3.0167
	2	38.9857	1.5256	0.1803	2.9161
	3	32.2465	1.9391	0.0663	2.9528
	4	28.6232	2.6667	−0.0078	2.9806
	5	26.8116	2.8355	−0.0093	2.9810
	10	25.0566	2.8963	−0.0004	2.9800
	15	25.0018	2.8964	0.0000	2.9800
	20	25.0001	2.8964	0.0000	2.9800
	21	25.0000	2.8964	0.0000	2.9800
50	1	45.9016	0.8305	0.6654	3.0167
	2	38.9857	1.5256	0.1803	2.9161
	3	32.2465	1.9391	0.0663	2.9528
	4	27.4155	2.9535	−0.0130	2.9786
	5	25.8052	3.0556	−0.0058	2.9769
	10	25.0033	3.0695	0.0000	2.9766
	14	25.0000	3.0695	0.0000	2.9766

$\sqrt{\beta_1} = \gamma_1$, $\beta_2 = \gamma_2 + 3$.

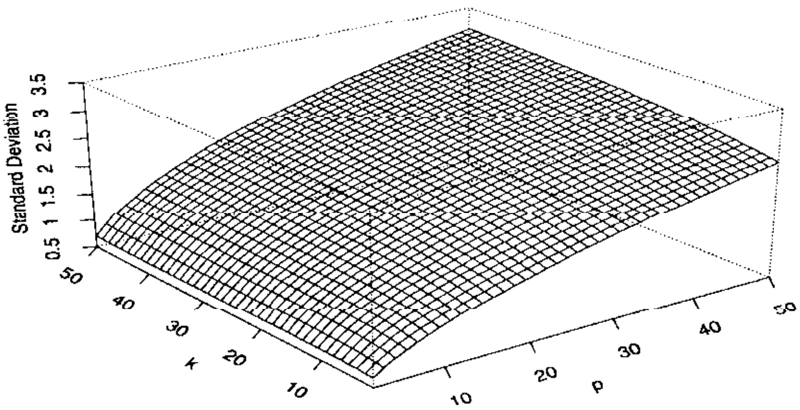

Fig. 1. Equilibrium case—asymptotic standard deviation.

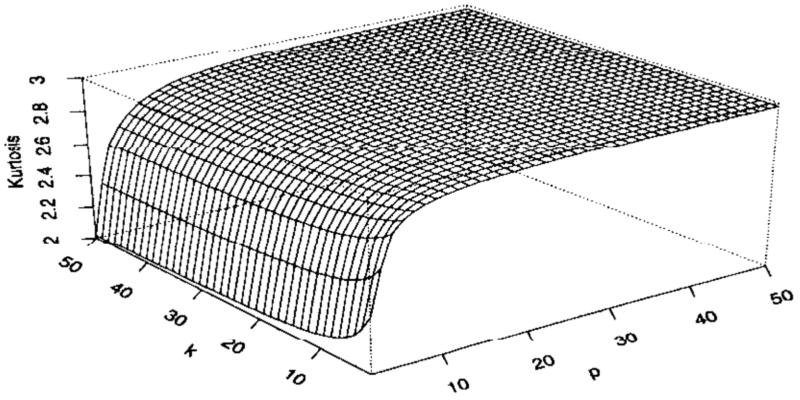

Fig. 2. Equilibrium case—asymptotic kurtosis.

The dominant asymptotic values can now be formed. Thus

$$
(3.6) \qquad \begin{cases} \mu_1' = k/2, \qquad \mu_2 \sim k/8, \\ \mu_3 = 0, \qquad \sqrt{\beta_1} = 0, \\ \beta_2 = \mu_4/\mu_2^2 \sim 3 - \dfrac{1}{k} + \dfrac{(2p+1)}{2k^2} - \dfrac{(2p+1)(10p-3)}{4k^3} \\ \qquad + \dfrac{(2p+1)(52p^2-36p+9)}{8k^4} \\ \qquad - \dfrac{(2p+1)(232p^3-228p^2+126p-27)}{16k^5}. \end{cases}
$$

Alternative forms for μ_2 and μ_4 are:

$$\text{(3.7)}\qquad \begin{cases} \mu_2 = \dfrac{k}{8} + \dfrac{2p+1}{10} - \dfrac{(4p^2-1)}{16(2k+2p-1)}, \\ \mu_4 = \dfrac{3k^2}{64} + \dfrac{(3p+1)k}{32} - \dfrac{(2p+1)(6p-7)}{256} \\ \qquad - \dfrac{(4p^2-1)(12p^2+5)}{512(2k+2p-1)} + \dfrac{3(4p^2-1)(4p^2-9)}{512(2k+2p-3)}. \end{cases}$$

The asymptotic expression for β_2 may turn out to be convergent for $k > p - 1/2$, but it is of doubtful use, since an exact value can be found (see (3.7)). We note that the response of kurtosis relates to the two parameters k and p where k is an initial value and p refers to the R-D phase. As a general descriptive the kurtosis seems to be just less than three, with skewness zero, provided k is 4 or more. If k is large β_2 is only slightly affected by changes in p, and near normality exists.

If p is large compared to k, then expansions (3.5) and (3.6) are useless. Actually if $p \to \infty$, then the r-th factorial moment for C_1 balls is,

$$\mu_{(r)} \sim k^{(r)}/2^r \qquad (p \to \infty)$$

which relates to a symmetric binomial distribution with index k.

Graphic representations for the standard deviation and kurtosis are given in Figs. 1 and 2.

4. Probability generating function in the equilibrium state

From (3.3) we have for the fmgf

$$f(\alpha) = (E_x + \alpha\Delta_x)^{k+p} x^{(k)}/(2k+2p)^{(k)} \,|_{x=k+p}$$

so the p.g.f. is

$$P(t) = (1 + t\Delta_x)^{k+p} x^{(k)}/(2k+2p)^{(k)} \,|_{x=k+p}$$

and

$$\text{(4.1)}\qquad P_r(N_1 = s) = \binom{k}{s}(k+p)^{(s)}(k+p)^{(k-s)}/(2k+2p)^{(k)}. \qquad (s = 0, 1, \ldots, k)$$

The reader will not fail to notice the association of this hypergeometric form to the Vandermond expansion of $(2k+2p)^{(k)}$ written as $[(k+p)+(k+p)]^{(k)}$.

Table 3 gives the exact moments of balls of color C_1 when equilibrium is reached. There are two values of k and 4 values of p. The agreement with those in Tables 2.1 and 2.2 is remarkable, and we note that for the latter the moments were computed from models with different initial inputs (actually the first 3 cycles; see Table 1).

Table 3. Exact moment parameters for equilibrium.

k	p	μ_1'	σ	β_2
16	3	8	1.5422	2.9432
	5	8	1.5927	2.9416
	25	8	1.8053	2.9167
	50	8	1.8820	2.9021
50	3	25	2.5820	2.9811
	5	25	2.6231	2.9814
	25	25	2.8064	2.9800
	50	25	3.0695	2.9766

($\sqrt{\beta_1} = 0$ through out)

5. Generalizations

5.1 *Two colors and asymmetry*

In the case when the reinforcements are p balls of C_1, q balls of C_2, and depletion $R = p + q$, the fmgf when equilibrium is reached is defined by

$$f(\alpha) = \left(1 + \frac{\alpha}{E_x}\right)^p f\left(\frac{\alpha}{E_x}\right) \frac{x^{(R)}}{(k+R)^{(R)}} \Bigg|_{x=k+R} . \tag{5.1}$$

This is a finite difference equation. A more general form is

$$f(\alpha) = g\left(\frac{\alpha}{E}\right) f\left(\frac{\alpha}{E}\right) x^{(R)}/(k+R)^{(R)} \,|_{x=k+R}$$

with restrictions on $g(\cdot)$ such as $g(0) = 1$. Little is known about solutions except in the case of the model described in (3.1)–(3.3) with $p = q$ as given in Section 3. Note that as far as the E operator is concerned the coefficient of α^s is

$$cE^{-s}x^{(R)}/(k+R)^{(R)}$$

so that this is zero for $s > k$. Thus if we consider the corresponding p.g.f., it will be a polynomial of degree k.

An implicit solution to (5.1) is given by

$$x_s = \left(x_s + px_{s-1} + \cdots + \binom{p}{p} x_{s-p}\right) \frac{k^{(s)}}{(k+R)^{(s)}}$$

or

$$x_s = \frac{\left(\sum_{r=0}^{s} x_r \binom{p}{r}\right)}{(k+R)^{(s)} - R^{(s)}}. \qquad (s = 1, 2, \ldots, k)$$

This will simplify provided that the denominator factors into the numerator. We are only aware of one non-trivial case, i.e. the case $p = q$.

5.2 *The bivariate model*

Here the reinforcements are p, q, r with depletion $R = p + q + r$; the colors are C_1, C_2, and C_3. In equilibrium

$$f(\alpha,\beta) = \left(1+\frac{\alpha}{E_x}\right)^p \left(1+\frac{\beta}{E_x}\right)^q f\left(\frac{\alpha}{E_x},\frac{\beta}{E_x}\right) \frac{x^{(R)}}{(k+R)^{(R)}}\Bigg|_{x=k+R}$$

A search for closed solutions proved negative. So we turned to deriving the correlations between balls of colors C_1 and C_2. Means are

$$\mu'_{10} = kp/R, \qquad \mu'_{01} = kq/R.$$

For central moments after simplification, we have

$$\begin{cases} \mu_{20} = \dfrac{kp(R-p)(k+R)}{R^2(2k+R-1)}, & (\text{color } C_1) \\ \mu_{11} = \dfrac{-kpq(k+R)}{R^2(2k+R-1)}, & (C_1 \ \& \ C_2) \\ \mu_{02} = \dfrac{kq(R-q)(k+R)}{R^2(2k+R-1)}, & (C_2) \end{cases}$$

and it can be seen that μ_{02} is derived from μ_{20} by interchanging p for q.

Thus for the correlation ρ for colors C_1 and C_2,

$$\rho(1,2) = \frac{-\sqrt{pq}}{\sqrt{(r+p)(r+q)}} \tag{5.2}$$

with similar expressions for $\rho(1,3)$ and $\rho(2,3)$.

6. Pólya-Eggenberger distribution

The result in (4.1) for the probability function for balls of color C_1 is somewhat similar to the case of the Pólya-Eggenberger distribution as quoted by Berg (1988). Here at the initial cycle there are a red balls and b black balls. One ball is drawn at random and then replaced together with c balls of same color. After n cycles, X referring to red balls, according to Berg

$$\Pr(X=x) = \binom{n}{x} \frac{\prod_{i=0}^{x-1}(a+ic)\prod_{j=0}^{n-x-1}(b+jc)}{\prod_{i=0}^{n-1}(a+b+ic)}, \qquad (x=0,1,\ldots,n)$$

or with a change of notation

$$\Pr(X=x) = \binom{n}{x} \alpha^{[x]}\beta^{[n-x]}/(\alpha+\beta)^{[n]}$$

where $\alpha^{[n]} = \alpha(\alpha+1)\cdots(\alpha+x-1)$, and with r-th factorial moment

$$\mathcal{E}(X^{(n)}) = n^{(r)}\alpha^{[r]}/(\alpha+\beta)^{[r]}.$$

Although the basic structures of this distribution and that for the urn (4.1) are quite different, yet there is a curious similarity of probability forms.

7. Conclusion

A new urn distribution in equilibrium arises when there are two colors, each reinforced by the same amount with $R = D$. When $R \neq D$ no simple solutions have been found. With $R = D$ there is asymptotic normality ($k > p$, $k \to \infty$) and asymptotic binomiality ($p = q \to \infty$, k fixed). In general the symmetric urn is such that the distribution of balls of color C_1 has skewness zero and kurtosis a little less than three.

An awkward polynomial equality awaits an algebraic solution. A new finite difference equation turns up and is a curiosity. Some related references are Bowman *et al.* (1985) and Shenton (1981, 1983).

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