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# MOMENTS OF LIMITS OF LIGHTLY TRIMMED SUMS OF RANDOM VECTORS IN THE GENERALIZED DOMAIN OF NORMAL ATTRACTION OF NON-GAUSSIAN OPERATOR-STABLE LAWS

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**Abstract.** Existence and nonexistence for moments of limiting random vectors of normalized, lightly trimmed sums of random vectors in the generalized domain of normal attraction of non-Gaussian operator-stable laws are studied. The idea of representing the limiting random vectors by infinite series is essentially used in the proofs.

*Key words and phrases*: Lightly trimmed sum, non-Gaussian operator-stable law, generalized domain of normal attraction, moment, tail behavior.

## 1. Introduction and results

A probability measure $\mu$ on $\boldsymbol{R}^d$ is said to be operator-stable if it is infinitely divisible and there exist a linear operator $B$ on $\boldsymbol{R}^d$ and a function $b : (0, \infty) \to \boldsymbol{R}^d$ such that for every $t > 0$,

$$\hat{\mu}(\theta)^t = \hat{\mu}(t^{B^*}\theta)e^{i\langle b(t), \theta\rangle} \qquad \text{for all} \quad \theta \in \boldsymbol{R}^d,$$

where $\hat{\mu}$ is the characteristic function of $\mu$, $B^*$ is the adjoint operator of $B$, $t^A = \exp\{(\ln t)A\} = \sum_{k=0}^{\infty}(k!)^{-1}(\ln t)^k A^k$ and $\langle , \rangle$ stands for an inner product in $\boldsymbol{R}^d$. $B$ is called an exponent for $\mu$. It is not necessarily unique. It is known that every eigenvalue of $B$ has a real part not less than $\frac{1}{2}$, and that if $\mu$ is *purely non Gaussian* in the sense that $\mu$ does not have a Gaussian component, no eigenvalue of $B$ has a real part equal to $\frac{1}{2}$. Throughout this paper, we assume that $\mu$ is full, namely, $\mu$ is not concentrated on a proper hyperplane in $\boldsymbol{R}^d$ and is purely non-Gaussian operator-stable.

Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. $\boldsymbol{R}^d$-valued random vectors belonging to the generalized domain of normal attraction of a full, purely non-Gaussian operator-stable law with an exponent $B$, namely for some vectors $\{c_n\}$ in $\boldsymbol{R}^d$, as $n \to \infty$

$$(1.1) \qquad n^{-B}\left(\sum_{j=1}^{n} X_j - c_n\right) \xrightarrow{w} Z_B,$$

where $Z_B$ is a purely non-Gaussian operator-stable random vector with the exponent $B$, and $\xrightarrow{w}$ stands for the convergence in law.

Throughout this paper, the norm $\|\cdot\|$ in $\boldsymbol{R}^d$ means the special norm defined below. Let $\|\cdot\|_0$ be the usual Euclidean norm in $\boldsymbol{R}^d$, and let $\mathfrak{S}(\mu)$ be the symmetry group associated with $\mu$, that is, the group of all invertible linear operators $A$ on $\boldsymbol{R}^d$, such that for some $a \in \boldsymbol{R}^d$, $\hat{\mu}(\theta) = \hat{\mu}(A^*\theta)e^{i\langle a,\theta\rangle}$. Since $\mu$ is full, $\mathfrak{S}(\mu)$ is compact, and there exists a Haar probability measure $H$ on the Borel subsets of $\mathfrak{S}(\mu)$. Then we define a norm $\|\cdot\|$ which depends on the operator-stable law $\mu$, but not on the choice of exponent:

$$\|x\| = \int_{\mathfrak{S}(\mu)} \int_0^1 \|gt^B x\|_0 \frac{dt}{t} dH(g),$$

(see Hudson *et al.* (1986) and Hahn *et al.* (1989)). Let $S$ be the unit sphere with respect to the norm $\|\cdot\|$. The norm has the following properties:

(i) $\|\cdot\|$ does not depend on the choice of the exponent $B$.

(ii) The map $t \mapsto \|t^B x\|$ is strictly increasing on $(0,\infty)$ for $x \neq 0$.

Define the norm of the linear operator $A$ on $\boldsymbol{R}^d$ as $\|A\| = \sup_{\|x\|=1} \|Ax\|$. Then by property (ii) above,

(iii) the map $t \mapsto \|t^B\|$ is strictly increasing on $(0,\infty)$, (equivalently $t \mapsto \|t^{-B}\| = \|(t^{-1})^B\|$ is strictly decreasing on $(0,\infty)$).

(iv) The map $\Phi : (0,\infty) \times S \to \boldsymbol{R}^d \setminus \{0\}$, defined by $\Phi(t,x) = t^B x$, is a homeomorphism, where $(0,\infty) \times S$ has the product topology.

It follows from (iv) that one can express every vector $x \in \boldsymbol{R}^d \setminus \{0\}$ as $x = \tau(x)^B \ell(x)$, where $\tau(x) \in (0,\infty)$ is a "radial" coordinate, and $\ell(x) \in S$ is a "direction". For each $n \geq 1$, let $X_k^{(n)}$ be the $k$-th largest among $(X_1,\dots,X_n)$ in terms of the radial coodinate $\tau(\cdot)$, namely

$$\tau(X_1^{(n)}) \geq \tau(X_2^{(n)}) \geq \cdots \geq \tau(X_n^{(n)}),$$

where ties are broken according to priority of index. We then consider the lightly trimmed sum

$$(1.2) \qquad \sum_{j=1}^{n} X_j - \left(\sum_{j=1}^{r} X_j^{(n)}\right) = \sum_{j=r+1}^{n} X_j^{(n)},$$

where $r \geq 1$ is a fixed number of trimmed terms.

Hahn *et al.* (1989) gave a necessary and sufficient condition for the validity of (1.1) by using the idea of series representation of operator-stable random vectors by LePage (1981). Without any change of the proof of Hahn *et al.* (1989), we can conclude that the trimmed sum (1.2) also converges weakly to a random vector $Z_r$ (which will be defined in (2.1) in the next section), under the same normalization as in Hahn *et al.* (1989). This idea was also used for the same convergence problem for Banach space valued random variables by the author (Maejima (1993)).

In this paper, we shall show the following result on moments. Let $\lambda_B$ and $\Lambda_B$ be the minimum and maximum of the real parts of the eigenvalues of $B$, respectively. Our assumption of pure non-Gaussianity is equal to assuming that $\lambda_B > \frac{1}{2}$.

THEOREM 1.1.
(i) *For* $p < \frac{1}{\Lambda_B}(r+1)$, $E[\|Z_r\|^p] < \infty$.
(ii) *For* $p > \frac{1}{\Lambda_B}(r+1)$, $E[\|Z_r\|^p] = \infty$.

*Remark* 1. A general non-Gaussian operator-stable random vector is the convolution of a purely non-Gaussian random vector and a Gaussian random vector. Therefore, the conclusions of Theorem 1.1 are also valid for general non-Gaussian random vectors.

*Remark* 2. Note that $Z_B = Z_0$. For a full non-Gaussian operator-stable random vector $Z_B$, it is known that $E[\|Z_B\|^p] < \infty$ or $= \infty$, depending on whether $p < \frac{1}{\Lambda_B}$ or $p \geq \frac{1}{\Lambda_B}$. (See Luczak (1981), Corollary 3.1.) We expect that for general $r \geq 1$, $E[\|Z_r\|^{(r+1)/\Lambda_B}] = \infty$. This cannot be proved in this paper, but a partial result will be discussed at the end of Section 4.

If an exponent $B$, of an operator-stable law $\mu$, is of the form $B = \frac{1}{\alpha}I$ (necessarily $0 < \alpha \leq 2$), $\mu$ is said to be $\alpha$-stable, and if $\alpha \neq 2$, then $\mu$ is non-Gaussian $\alpha$-stable. In this case, we can get the exact tail behavior of the distribution of $Z_r$ as follows:

COROLLARY 1.1. *If* $\mu$ *is* $\alpha$*-stable*, $0 < \alpha < 2$, *then as* $x \to \infty$

$$P(\|Z_r\| > x) \sim \frac{1}{(r+1)!}x^{-\alpha(r+1)}.$$

When $d = 1$, this corollary can easily be obtained from the exact form of the characteristic function of $Z_r$ given by Hall (1978). In the final section, we will give a proof for Corollary 1.1 by applying (i) of Theorem 1.1.

## 2. Series representation of $Z_r$

Our proof in this paper is based on the LePage type series representation of the limiting random vector $Z_r$. We explain it in this section by following the argument in Hahn *et al.* (1989).

Let $\mu$ be a purely non-Gaussian full operator-stable measure with exponent $B$. The Lévy measure $M$ of $\hat{\mu}$ can be represented as

$$M(A) = \int_S \int_0^\infty I[t^B\xi \in A]t^{-2}dt\nu(d\xi), \qquad A \in \mathfrak{B}(\boldsymbol{R}^d \setminus \{0\}),$$

where $\nu$ is a finite Borel measure on $S$, and $I[\cdot]$ is the indicator function. The measure $\nu$ does not depend on the choice of exponent. The measure $\mu$ is characterized through its characteristic function by a triple $(\gamma, B, \nu)$, where $\gamma$ is the centering constant, $B$ is an exponent of $\mu$, and $\nu$ is the mixing measure of the Lévy measure $M$. Define $\{U_j\}$ as a sequence of i.i.d. random vectors having the common distribution $\nu(\cdot)/\nu(S)$, and let $\{\Gamma_j\}$ be Poisson arrival times with unit rate, namely $\Gamma_j = e_1 + \cdots + e_j$, where $\{e_j\}_{j=1}^{\infty}$ are i.i.d. random variables having an exponential distribution with mean 1. We assume $\{U_j\}$ and $\{\Gamma_j\}$ are independent of each other. Then under assumption (1.1), we have as $n \to \infty$

$$(2.1)\qquad n^{-B} \sum_{j=r+1}^{n} \{X_j^{(n)} - E[I[\|X_j^{(n)}\| \leq 1]X_j^{(n)}]\}$$
$$\xrightarrow{w} Z_r := \sum_{j=r+1}^{\infty} \{\Gamma_j^{-B} U_j - E[I[\Gamma_j \geq 1]\Gamma_j^{-B}]E[U_j]\}.$$

(See Hahn *et al.* (1989).)

## 3. Proof of (i) of Theorem 1.1

LEMMA 3.1. *Let $t > 0$. If $j > t\Lambda_B$, then $E[\|\Gamma_j^{-B}\|^t] < \infty$.*

PROOF. Note that for any $\varepsilon > 0$, there exist $C_1 > 0$ and $C_2 > 0$, such that

$$\|x^{-B}\| < C_1 x^{-(\Lambda_B + \varepsilon)} \quad \text{for} \quad 0 < x \leq 1$$

and

$$\|x^{-B}\| < C_2 x^{-(\lambda_B - \varepsilon)} \quad \text{for} \quad x > 1.$$

Then we have

$$\begin{aligned} E[\|\Gamma_j^{-B}\|^t] &= \frac{1}{(j-1)!}\left(\int_0^1 + \int_1^\infty\right) \|x^{-B}\|^t x^{j-1} e^{-x} dx \\ &\leq \frac{C_1}{(j-1)!} \int_0^1 x^{-t(\Lambda_B - \varepsilon)} x^{j-1} e^{-x} dx \\ &\quad + \frac{C_2}{(j-1)!} \int_1^\infty x^{-t(\lambda_B - \varepsilon)} x^{j-1} e^{-x} dx < \infty, \end{aligned}$$

if $j > t\Lambda_B$. □

LEMMA 3.2. *Let $t > 0$. Then there exists $C_t > 0$ such that for all $j$*

$$E\left[\left|\frac{\Gamma_j - j}{j^{1/2}}\right|^t\right] \leq C_t.$$

PROOF. This lemma is probably already known. However, for the sake of completeness, we give the proof below. Note that $E[\Gamma_j] = j$ and $\mathrm{Var}[\Gamma_j] = E[|\Gamma_j - j|^2] = j$. Let $0 < t \le 2$. Then

$$E[|\Gamma_j - j|^t] \le (E[|\Gamma_j - j|^2])^{t/2} = j^{t/2}.$$

Next, suppose $t > 2$, and put

$$F_j(x) = P\left\{\frac{\Gamma_j - j}{j^{1/2}} \le x\right\}.$$

Since $E[\Gamma_j^{t+1}] < \infty$, by the nonuniform central limit theorem (see, e.g. Petrov (1975), Section 5.4),

$$\lim_{j\to\infty}\sup_{x\in\boldsymbol{R}}(1+|x|)^{t+1}|F_j(x) - \Phi(x)| = 0,$$

where $\Phi(x)$ is the standard normal distribution. Thus, there exists $j_0$ such that for every $j > j_0$

$$\sup_{x\in\boldsymbol{R}}(1+|x|)^{t+1}|F_j(x) - \Phi(x)| \le 1.$$

Therefore, for each $j$

$$\begin{aligned}E\left[\left|\frac{\Gamma_j - j}{j^{1/2}}\right|^t\right] &= t\int_{-\infty}^0 |x|^{t-1}F_j(x)dx + t\int_0^\infty |x|^{t-1}(1-F_j(x))dx \\ &\le t\int_{-\infty}^0 |x|^{t-1}\left(\Phi(x) + \frac{1}{(1+|x|)^{t+1}}\right)dx \\ &\quad + t\int_0^\infty |x|^{t-1}\left(1 - \Phi(x) + \frac{1}{(1+|x|)^{t+1}}\right)dx \\ &= \int_{-\infty}^\infty |x|^t d\Phi(x) + t\int_{-\infty}^\infty \frac{|x|^{t-1}}{(1+|x|)^{t+1}}dx < \infty.\end{aligned}$$

This concludes the lemma. □

We proceed to the proof of (i) of Theorem 1.1. We use the idea of decomposition of $Z_0$ which is used in Hahn *et al.* (1989). Then we have

$$Z_r = \sum_{i=1}^4 I_i,$$

where

$$\begin{aligned}I_1 &= \sum_{j=r+1}^\infty (\Gamma_j^{-B} - j^{-B})U_j, \\ I_2 &= \sum_{j=r+1}^\infty j^{-B}(U_j - E[U_j]), \\ I_3 &= \sum_{j=r+1}^\infty P(\Gamma_j < 1)j^{-B}E[U_j], \\ I_4 &= \sum_{j=r+1}^\infty E[(j^{-B} - \Gamma_j^{-B})I[\Gamma_j \ge 1]]E[U_j].\end{aligned}$$

Almost sure convergence of all $I_i$'s are shown in Hahn *et al.* (1989). Also note that $I_3$ and $I_4$ are non-random. Hence, it is enough to consider $E[\|I_i\|^p]$ for $i = 1$ and 2. However, $I_2$ has all finite moments because it is a convergent series of independent random vectors that are uniformly bounded. (See e.g., Kwapień and Woyczyński (1992), Corollary 2.2.1.) Therefore, it is enough to show that $E[\|I_1\|^p] < \infty$ for $p < \frac{1}{\Lambda_B}(r+1)$.

First consider the case when $\frac{1}{\Lambda_B}(r+1) \le 1$. Since $p < \frac{1}{\Lambda_B}(r+1)$, if $j \ge r+1$, then $j > p\Lambda_B$. Thus by Lemma 3.1, we have

$$E[\|\Gamma_j^{-B}\|^p] < \infty \quad \text{if} \quad j \ge r+1 \tag{3.1}$$

and

$$E[\|\Gamma_j^{-B}\|] < \infty \quad \text{if} \quad j > \Lambda_B. \tag{3.2}$$

Fix $\delta \in (\frac{1}{2}, \lambda_B)$. By Lemma 1 (ii) in Hahn *et al.* (1989), we have

$$E[\|\Gamma_j^{-B} - j^{-B}\|] = O(j^{-(1/2+\delta)}).$$

This, together with (3.2), yields that for some $K_1 > 0$,

$$E[\|\Gamma_j^{-B} - j^{-B}\|] \le K_1 j^{-(1/2+\delta)} \quad \text{for} \quad j > \Lambda_B. \tag{3.3}$$

Since $p(< \frac{1}{\Lambda_B}(r+1))$ is strictly less than 1, we have

$$\begin{aligned} J_n &:= E\left[\left\|\sum_{j=r+1}^{n} (\Gamma_j^{-B} - j^{-B})U_j\right\|^p\right] \\ &\le E\left[\left\|\sum_{j=r+1}^{[\Lambda_B]} (\Gamma_j^{-B} - j^{-B})U_j\right\|^p\right] \\ &\quad + E\left[\left\|\sum_{j=[\Lambda_B]+1}^{n} (\Gamma_j^{-B} - j^{-B})U_j\right\|^p\right] \\ &=: (J_{n1} + J_{n2}). \end{aligned}$$

By (3.1), $J_{n1} < \infty$. As to $J_{n2}$, since $p < 1$,

$$\begin{aligned} J_{n2} &\le \left(E\left[\left\|\sum_{[\Lambda_B]+1}^{n} (\Gamma_j^{-B} - j^{-B})U_j\right\|\right]\right)^p \\ &\le \left(\sum_{j=[\Lambda_B]+1}^{n} E[\|\Gamma_j^{-B} - j^{-B}\|]\right)^p, \end{aligned}$$

which is dominated by some constant $M_p$ independent of $n$, because of (3.3). (Recall $\delta > \frac{1}{2}$.)

Next, consider the case when $\frac{1}{\Lambda_B}(r+1) > 1$. In this case, it is enough to prove that $J_n < \infty$ for $p \in (1, \frac{1}{\Lambda_B}(r+1))$. Fix $p \in (1, \frac{1}{\Lambda_B}(r+1))$. Then by Minskowski's inequality

$$J_n \leq \left\{ \sum_{j=r+1}^{n} (E[\|\Gamma_j^{-B} - j^{-B}\|^p])^{1/p} \right\}^p . \tag{3.4}$$

Following the argument in the proof of the Lemma 1 (ii) in Hahn *et al.* (1989), we also get

$$E[\|\Gamma_j^{-B} - j^{-B}\|^p] = O(j^{-p(1/2+\delta)}). \tag{3.5}$$

For some $C_3 > 0$,

$$\begin{aligned} E[\|\Gamma_j^{-B} - j^{-B}\|^p] &\leq C_3 E[|\Gamma_j - j|^p |\min(\Gamma_j, j)|^{-p(1+\delta)}] \\ &\leq C_3 (E[|\Gamma_j - j|^{2p}])^{1/2} (E[|\min(\Gamma_j, j)|^{-2p(1+\delta)}])^{1/2}. \end{aligned} \tag{3.6}$$

Here, as in Hahn *et al.* (1989), we have

$$E[|\min(\Gamma_j, j)|^{-2p(1+\delta)}] = O(j^{-2p(1+\delta)}). \tag{3.7}$$

On the other hand, by Lemma 3.2

$$E[|\Gamma_j - j|^{2p}] = O(j^p). \tag{3.8}$$

Combining (3.6)–(3.8), we get (3.5). Then, by (3.1) and (3.5), for some $K_2 > 0$,

$$E[\|\Gamma_j^{-B} - j^{-B}\|^p] \leq K_2 j^{-p(1/2+\delta)} \quad \text{for} \quad j \geq r+1. \tag{3.9}$$

It follows from (3.4) and (3.9) that

$$J_n \leq \left\{ \sum_{j=r+1}^{n} (K_2 j^{-p(1/2+\delta)})^{1/p} \right\}^p ,$$

which is dominated by some constant $M_p$, independent of $n$ again. Altogether, we conclude $E[\|I_1\|^p] < \infty$, and hence the proof of (i) of Theorem 1.1 is completed.

## 4. Proof of (ii) of Theorem 1.1 and a remark

If we let $\frac{1}{\Lambda_B}(r+1) < p < \frac{1}{\Lambda_B}(r+2)$, we shall prove $E[\|Z_r\|^p] = \infty$. Note that for $\boldsymbol{R}^d$-valued random vectors $X$ and $Y$, if $E[\|X\|^p] = \infty$, and $E[\|Y\|^p] < \infty$, then $E[\|X+Y\|^p] = \infty$.

Let

$$X = \Gamma_{r+1}^{-B} U_{r+1}$$

and

$$Y = Z_{r+1} - E[\Gamma_{r+1}^{-B} I[\Gamma_{r+1} \geq 1]]E[U_{r+1}].$$

Since $p < \frac{1}{\Lambda_B}(r+2)$, by (i) of Theorem 1.1,

$$E[\|Y\|^p] < \infty. \tag{4.1}$$

Thus, it is enough to show that

$$E[\|\Gamma_{r+1}^{-B} U_{r+1}\|^p] = \infty. \tag{4.2}$$

Recall the series representation for $Z_0 (= Z_B)$:

$$Z_0 = \Gamma_1^{-B} U_1 - E[I[\Gamma_1 \geq 1]\Gamma_1^{-B}]E[U_1] + Z_1. \tag{4.3}$$

It is noted in Remark 1 that

$$E[\|Z_0\|^{1/\Lambda_B}] = \infty. \tag{4.4}$$

By (i) of Theorem 1.1 with $r = 1$,

$$E[\|Z_1\|^{1/\Lambda_B}] < \infty. \tag{4.5}$$

It follows from (4.3)–(4.5) that

$$E[\|\Gamma_1^{-B} U_1 - E[I[\Gamma_1 \geq 1]\Gamma_1^{-B}]E[U_1]\|^{1/\Lambda_B}] = \infty,$$

namely

$$E[\|\Gamma_1^{-B} U_1\|^{1/\Lambda_B}] = \infty,$$

which is

$$\frac{1}{\nu(S)} \int_S \nu(d\xi) \int_0^\infty \|x^{-B}\xi\|^{1/\Lambda_B} e^{-x} dx = \infty. \tag{4.6}$$

*Remark* 3. Recall that our $B$ and $\nu$ are ingredients of a purely non-Gaussian full operator-stable measure $\mu$. (4.6) is not necessarily true if $\mu$ is not full. For instance, if $\nu$ is concentrated on $\pm\xi^* \in S$, where $\xi^*$ is the eigenvector of $\lambda_B$, and if $\lambda_B < \Lambda_B$, then the left-hand side of (4.6) is finite. In this case, the corresponding operator stable meaure must not be full.

We now go back to the proof. Since

$$\int_S \nu(d\xi) \int_1^\infty \|x^{-B}\xi\|^{1/\Lambda_B} e^{-x} dx < \infty,$$

we have

$$\int_S \nu(d\xi) \int_0^1 \|x^{-B}\xi\|^{1/\Lambda_B} e^{-x} dx = \infty.$$

Then by Hölder's inequality,

$$\begin{aligned}\infty &= \int_S \int_0^1 \|x^{-B}\xi\|^{1/\Lambda_B} e^{-x} \nu(d\xi)dx \\ &= \int_S \int_0^1 \|x^{-B}\xi\|^{1/\Lambda_B} e^{-x} x^{r/(p\Lambda_B)} x^{-r/(p\Lambda_B)} \nu(d\xi)dx \\ &\le \left(\int_S \int_0^1 \|x^{-B}\xi\|^{p} e^{-p\Lambda_B x} x^{r} \nu(d\xi)dx\right)^{1/(p\Lambda_B)} \\ &\quad \times \left(\int_S \int_0^1 x^{-r/(p\Lambda_B-1)} \nu(d\xi)dx\right)^{(p\Lambda_B-1)/(p\Lambda_B)}.\end{aligned}$$

Since the second integral in the right-hand side is finite, the first integral there must be infinite. Hence, we have

$$\int_S \nu(d\xi) \int_0^1 \|x^{-B}\xi\|^{p} e^{-x} x^{r} dx = \infty,$$

concluding (4.2), and completing the proof of (ii) of Theorem 1.1.

*Remark* 4. As mentioned at the end of Section 1, it is expected to hold that $E[\|Z_r\|^{(r+1)/\Lambda_B}] = \infty$. Here, we give a partial result for that.

THEOREM 4.1. *Suppose $\lambda_B > 1$, and let $r$ be such that $1 \le r < 2\lambda_B - 1$. Then $E[\|Z_r\|^{(r+1)/\Lambda_B}] = \infty$.*

PROOF. As in (4.2), it is enough to show that

$$E[\|\Gamma_{r+1}^{-B} U_{r+1}\|^{(r+1)/\Lambda_B}] = \infty. \tag{4.7}$$

Let $B^* = \frac{1}{r+1}B$. By the assumption that $1 \le r < 2\lambda_B - 1$, we have $\lambda_{B^*} > \frac{1}{2}$. Thus, it is possible to construct a purely non-Gaussian operator-stable random vector $Z_{B^*}$ with the triple $(\gamma, B^*, \nu)$, where $\gamma$ and $\nu$ are the same for $Z_B$. Since $Z_B$ is full, so is $Z_{B^*}$. Then, by the same argument used for (4.6), we have

$$\frac{1}{\nu(S)} \int_S \nu(d\xi) \int_0^\infty \|x^{-B^*}\xi\|^{1/\Lambda_{B^*}} e^{-x} dx = \infty,$$

which implies

$$\frac{1}{\nu(S)} \int_S \nu(d\xi) \int_0^\infty \|x^{-B/(r+1)}\xi\|^{(r+1)/\Lambda_B} e^{-x} dx = \infty.$$

By the change of variables $x^{1/(r+1)} = y$, we have

$$\frac{1}{\nu(S)} \int_S \nu(d\xi) \int_0^\infty \|y^{-B}\xi\|^{(r+1)/\Lambda_B} e^{-y^{r+1}} y^r dy = \infty,$$

which is equivalent to

$$\frac{1}{\nu(S)} \int_S \nu(d\xi) \int_0^\infty \|y^{-B}\xi\|^{(r+1)/\Lambda_B} e^{-y} y^r dy = \infty.$$

We thus conclude (4.7) and complete the proof of Theorem 4.1. □

## 5. Proof of Corollary 1.1

Note that in this case

$$Z_r = \sum_{j=r+1}^{\infty} \{\Gamma_j^{-1/\alpha} U_j - E[\Gamma_j^{-1/\alpha} I[\Gamma_j \geq 1]] E[U_j]\}.$$

LEMMA 5.1. *$P(\Gamma_j^{-1/\alpha} > x) \sim \frac{1}{j!} x^{-\alpha j}$ as $x \to \infty$.*

PROOF. Easy. □

LEMMA 5.2. *Let $X$ and $Y$ be two random vectors. Suppose that the distribution of $\|X\|$ has a regularly varying tail, namely there exists $\theta$, such that for any $c > 1$*

$$\lim_{x \to \infty} \frac{P\{\|X\| > cx\}}{P\{\|X\| > x\}} = c^{-\theta} \tag{5.1}$$

*and that the tail of the distribution of $\|X\|$ dominates that of $\|Y\|$, namely*

$$\lim_{x \to \infty} \frac{P\{\|Y\| > x\}}{P\{\|X\| > x\}} = 0. \tag{5.2}$$

*Then*

$$\lim_{x \to \infty} \frac{P\{\|X + Y\| > x\}}{P\{\|X\| > x\}} = 1.$$

PROOF. This can be proved by exactly the same way for Lemma 1.1 of Samorodnitsky and Szulga (1989). □

We now have

$$P\{\|Z_r\| > x\} = P\{\|\Gamma_{r+1}^{-1/\alpha} U_{r+1} - E[\Gamma_{r+1}^{-1/\alpha} I[\Gamma_{r+1} \geq 1]] E[U_{r+1}] + Z_{r+1}\| > x\}.$$

We apply Lemma 5.2 to

$$X = \Gamma_{r+1}^{-1/\alpha} U_{r+1}$$

and

$$Y = Z_{r+1} - E[\Gamma_{r+1}^{-1/\alpha} I[\Gamma_{r+1} \geq 1]] E[U_{r+1}].$$

By Lemma 5.1, (5.1) holds with $\theta = \alpha(r+1)$. Furthermore, if $x > 2\|E[U_{r+1}]\|$,

$$P\{\|Y\| > x\} \leq P\left\{\|Z_{r+1}\| > \frac{x}{2}\right\}$$

and by (i) of Theorem 1.1 (with the replacement of $r$ by $r+1$), the above is

$$\begin{aligned} &= O(x^{-p}) \qquad (\text{for any } p < \alpha(r+2)) \\ &= o(x^{-\alpha(r+1)}). \end{aligned} \tag{5.3}$$

Thus, by Lemma 5.1 with $j = r+1$ and (5.3), (5.2) is given. Therefore, by Lemma 5.2, we can conclude that the distribution of $\|Z_r\|$ and $\Gamma_{r+1}^{-1/\alpha}$ have the same asymptotic tails, and the conclusion of Corollary 1.1 now follows from Lemma 5.1 with $j = r+1$.

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