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ON THE ASYMPTOTIC EXPECTATIONS OF SOME UNIT ROOT TESTS IN A FIRST ORDER AUTOREGRESSIVE PROCESS IN THE PRESENCE OF TREND

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Abstract. Estimation in a first order autoregressive process with trend is considered. Integral expressions for the asymptotic bias of the estimator under a unit root and for the expectation of the limit distribution of the log likelihood ratio test for a unit root are given, and evaluated numerically.

Key words and phrases: Autoregression with trend, unit root test.

1. Introduction

Consider an AR(1) process with drift $\{X_t\}_{t=1}^{\infty}$, given by

$$X_t = \rho X_{t-1} + \eta + \varepsilon_t, \qquad t = 1, 2, \dots, \tag{1.1}$$

where ρ and η are constants, $\{\varepsilon_t\}_{t=1}^{\infty}$ is a sequence of independent normally distributed random variables with mean 0 and variance σ^2, and assume X_0 to be constant. Furthermore, suppose that we observe $\{X_t\}$ up to time T and want to test the hypothesis $H_0 : \rho = 1$ against $\neg H_0$ (a type of unit root test). If H_0 is true and $\eta \neq 0$, $-2 \log Q_T$ is asymptotically χ^2, where Q_T is the likelihood ratio test statistic. However, if H_0 holds and $\eta = 0$, we have the asymptotic result

$$-2 \log Q_T \xrightarrow{\mathrm{d}} \frac{(\int_0^1 W_t dW_t - W_1 \int_0^1 W_t dt)^2}{\int_0^1 W_t^2 dt - (\int_0^1 W_t dt)^2} \stackrel{\mathrm{def}}{=} Z \quad \text{as} \quad T \to \infty, \tag{1.2}$$

where W_t is a standard Wiener process. This is a special case of the multivariate results of Johansen (1991). (See also Phillips and Durlauf (1986) and Phillips and Perron (1988).)

Similarly, if $\rho = 1$ the least squares estimator $\hat{\rho}$ of ρ satisfies (cf. Phillips and Durlauf (1986) and Billingsley (1968))

$$T(\hat{\rho} - 1) \xrightarrow{\mathrm{d}} \frac{\int_0^1 W_t dW_t - W_1 \int_0^1 W_t dt}{\int_0^1 W_t^2 dt - (\int_0^1 W_t dt)^2} \stackrel{\mathrm{def}}{=} \tilde{Z} \quad \text{as} \quad T \to \infty, \tag{1.3}$$

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and so, $E\tilde{Z}/T$ is the asymptotic bias of $\hat{\rho}$. Of course, the quantity $T(\hat{\rho}-1)$ may be viewed upon as an alternative unit root test statistic.

In a model without drift term η, where the corresponding limit laws are similar (but simpler), the asymptotic bias was calculated by Evans and Savin (1981), Mikulski and Monsour (1994), Le Breton and Pham (1989) and Abadir (1993). Corresponding results for related statistics were given by Pham (1990). Using similar methods, Mikulski and Monsour (1994) also obtained the asymptotic expectation (as well as higher moments) of the log likelihood ratio test statistic for test of $H_0 : \rho = 1$ against $\neg H_0$. (The latter statistic was also studied numerically by Jacobson and Larsson (1997).) The object of the present paper is to extend these results to the nonzero drift case. (In this case, the distribution of $T(\hat{\rho}-1)$ was given numerically by Evans and Savin (1984), and by Andrews (1993).)

Calculations and main results are given in Section 2, whereas Section 3 contains a brief concluding discussion.

2. Results

To find the expectation of $\tilde{Z}$, we at first note that (cf. Billingsley (1968))

$$\tilde{Z}_T \stackrel{\text{def}}{=} T\frac{\sum \tilde{S}_{t-1}\varepsilon_t}{\sum \tilde{S}_{t-1}^2} \stackrel{\text{d}}{\to} \tilde{Z} \qquad \text{as} \quad T \to \infty,$$

(if nothing else is said, summation goes from $t = 1$ to T), implying $E(\tilde{Z}_T) \to E(\tilde{Z})$ (see Lemma 2.1 below), where $\tilde{S}_t \stackrel{\text{def}}{=} S_t - \sum S_{t-1}/(T-1)$, $S_t \stackrel{\text{def}}{=} \sum_{i=1}^t \varepsilon_i$, $S_0 = 0$, and $\{\varepsilon_t\}_{t=1}^\infty$ is a sequence of independent normally distributed random variables with mean 0 and variance 1. Hence,

$$E\tilde{Z}_T = TE\left(\frac{\sum S_{t-1}\varepsilon_t}{\sum \tilde{S}_{t-1}^2}\right) - E\left(\frac{\sum S_{t-1}S_T}{\sum \tilde{S}_{t-1}^2}\right) + o(1). \tag{2.1}$$

Now, consider the process $\{X_t^*\}_{t=1}^\infty$, satisfying

$$X_t^* = \rho X_{t-1}^* + \varepsilon_t, \qquad t = 1, 2, \ldots,$$

where $X_0^* = 0$, and the ε_t's are as above (with unit variance) and introduce the Laplace transform

$$\varphi(s, u, v; \rho) \stackrel{\text{def}}{=} E(e^{-s\sum \tilde{X}_{t-1}^2 - u\sum X_{t-1}^* - vX_T^*}),$$

where $\tilde{X}_t \stackrel{\text{def}}{=} X_t^* - \sum X_{t-1}^*/(T-1)$. (The variable ρ enters the function φ through the X_t^*'s.) By successive differentiation of the equality (cf. Mikulski and Monsour (1994))

$$\int_0^\infty \varphi(s, u, v; \rho)ds = E\left(\frac{1}{\sum \tilde{S}_{t-1}^2}\right)$$
$$= \int \cdots \int \frac{1}{\sum \tilde{x}_{t-1}^2}(2\pi)^{-T/2}e^{-1/2\sum(X_t^* - \rho X_{t-1}^*)^2}dx_1 \cdots dx_T,$$

we obtain

$$E\left(\frac{\sum S_{t-1}\varepsilon_t}{\sum \tilde{S}_{t-1}^2}\right) = \int_0^\infty \frac{\partial}{\partial\rho}\varphi ds \tag{2.2}$$

and

$$E\left(\frac{\sum S_{t-1}S_T}{\sum \tilde{S}_{t-1}^2}\right) = \int_0^\infty \frac{\partial^2}{\partial u\partial v}\varphi ds, \tag{2.3}$$

where, here and henceforth, the derivatives are taken at $u = v = 0$, $\rho = 1$. Similarly,

$$Z_T \stackrel{\text{def}}{=} \frac{(\sum \tilde{S}_{t-1}\varepsilon_t)^2}{\sum \tilde{S}_{t-1}^2} \xrightarrow{\text{d}} Z \quad \text{as} \quad T \to \infty$$

(by Lemma 2.1, $E(Z_T) \to E(Z)$), implying

$$EZ_T = E\left(\frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum \tilde{S}_{t-1}^2}\right) - \frac{2}{T}E\left(\frac{\sum S_{t-1}\varepsilon_t \sum S_{t-1}S_T}{\sum \tilde{S}_{t-1}^2}\right) + \frac{1}{T^2}E\left(\frac{(\sum S_{t-1})^2 S_T^2}{\sum \tilde{S}_{t-1}^2}\right) + o(1).$$

But, since

$$\frac{1}{T^2}\sum \tilde{S}_{t-1}^2 = \frac{1}{T^2}\left(\sum S_{t-1}^2 - \frac{1}{T-1}\left(\sum S_{t-1}\right)^2\right) + o_p(1),$$

where $o_p(1)$ is a term which tends to zero in probability as $T \to \infty$, it follows as above that

$$\int_0^\infty \frac{\partial^2}{\partial\rho^2}\varphi ds = E\left(\frac{-\sum S_{t-1}^2 + (\sum S_{t-1}\varepsilon_t)^2}{\sum \tilde{S}_{t-1}^2}\right) = -1 - \frac{1}{T}E\left(\frac{(\sum S_{t-1})^2}{\sum \tilde{S}_{t-1}^2}\right) + E\left(\frac{(\sum S_{t-1}\varepsilon_t)^2}{\sum \tilde{S}_{t-1}^2}\right) + o(1),$$

and by repeating similar arguments and rearranging, we get

$$EZ_T = 1 + \int_0^\infty \frac{\partial^2}{\partial\rho^2}\varphi ds + \frac{1}{T}\int_0^\infty \frac{\partial^2}{\partial u^2}\varphi ds - \frac{2}{T}\int_0^\infty \frac{\partial^3}{\partial u\partial v\partial\rho}\varphi ds + \frac{1}{T^2}\int_0^\infty \frac{\partial^4}{\partial u^2\partial v^2}\varphi ds + o(1). \tag{2.4}$$

The equalities (2.1)–(2.4) provide the basis for proving our main results, collected in

THEOREM 2.1.

$$E(\tilde{Z}) = -\frac{1}{2}\int_0^\infty \frac{x^{3/2}}{\sqrt{\sinh x}}dx \approx -5.3791, \tag{2.5}$$

$$E(Z) = \int_0^\infty \sqrt{\frac{x}{\sinh x}}\left(\frac{x}{4} + \frac{1}{x} - 2\frac{\cosh x - 1}{x^2 \sinh x}\right)dx \approx 3.0560. \tag{2.6}$$

As is seen by Taylor expansion, the integrands of (2.5) and (2.6) tend to zero as $x \to 0$. The figures are obtained by numerical integration, and naturally, the figure of (2.6) is more reliable than the simulation results $E(Z) \approx 3.030$ (Johansen and Juselius (1990)) and $E(Z) \approx 3.025$ (Jacobson and Larsson (1997)).

To prove the theorem, we will need some lemmas:

LEMMA 2.1.

$$\lim_{T\to\infty} E(\tilde{Z}_T) = E(\tilde{Z}), \tag{2.7}$$

$$\lim_{T\to\infty} E(Z_T) = E(Z). \tag{2.8}$$

PROOF. Choose a $\delta > 0$ and let

$$\tilde{Z}_T = X_T + Y_T, \qquad X_T \stackrel{\text{def}}{=} \tilde{Z}_T 1_{\{T^{-2}\sum \tilde{S}_{t-1}^2 \geq \delta\}}, \qquad Y_T \stackrel{\text{def}}{=} \tilde{Z}_T 1_{\{T^{-2}\sum \tilde{S}_{t-1}^2 < \delta\}}.$$

(The notation 1_A means that $1_A = 1$ if condition A holds, and 0 otherwise.) Naturally, $E(\tilde{Z}_T) = E(X_T) + E(Y_T)$, and by the Hölder inequality,

$$|E(Y_T)| \leq E(|\tilde{Z}_T| 1_{\{T^{-2}\sum \tilde{S}_{t-1}^2 < \delta\}}) \leq \left(E(\tilde{Z}_T^2) P\left(T^{-2}\sum \tilde{S}_{t-1}^2 < \delta\right)\right)^{1/2}. \tag{2.9}$$

But, as is proved in the sequel without using this lemma, $E(\tilde{Z}_T)$ converges to the r.h.s. of (2.5) as $T \to \infty$, and similar arguments may be adopted (cf. Larsson (1994) for the model with trend zero) to prove that $E(\tilde{Z}_T^2)$ converges to a finite limit as $T \to \infty$. Hence, $E(\tilde{Z}_T^2) < \infty$ for all T sufficiently large, and since furthermore $T^{-2}\sum \tilde{S}_{t-1}^2$ has a non-degenerate distribution for all T, δ may be chosen such that the r.h.s. of (2.9) becomes arbitrarily small.

Now, if $\{X_T\}$ is a uniformly integrable sequence, we have since (cf. Billingsley (1968))

$$T^{-2}\sum \tilde{S}_{t-1}^2 \xrightarrow{\text{d}} V \stackrel{\text{def}}{=} \int_0^1 W_t^2 dt - \left(\int_0^1 W_t dt\right)^2 \quad \text{as} \quad T \to \infty,$$

$$\lim_{T\to\infty} E(X_T) = E\left(\lim_{T\to\infty} X_T\right) = E(\tilde{Z} 1_{\{V \geq \delta\}}), \tag{2.10}$$

where δ may be chosen such that the r.h.s. of (2.10) becomes arbitrarily close to $E(\tilde{Z})$.

Uniform integrability follows from the fact that, for all T,

$$|E(X_T)| \leq E\left(\left|\frac{T^{-1}\sum \tilde{S}_{t-1}\varepsilon_t}{T^{-2}\sum \tilde{S}_{t-1}^2}\right| 1_{\{T^{-2}\sum \tilde{S}_{t-1}^2 \geq \delta\}}\right) \leq \frac{1}{\delta} E\left(\left|T^{-1}\sum \tilde{S}_{t-1}\varepsilon_t\right|\right),$$

where, for T sufficiently large, the rightmost expectation is finite because (cf. Billingsley (1968))

$$\lim_{T\to\infty} E\left(\left|T^{-1}\sum \tilde{S}_{t-1}\varepsilon_t\right|\right) = E\left(\left|\int_0^1 W_t dW_t - W_1 \int_0^1 W_t dt\right|\right) < \infty,$$

and the proof of (2.7) is completed.

The proof of (2.8), which is entirely similar, is omitted. □

Turning to the actual computation of the expectations, we will need

LEMMA 2.2.
$$\varphi(s,u,v;1-\theta) = \frac{1}{\sqrt{\det Q}} e^{1/2 q' Q^{-1} q},$$

where $Q = Q_0 + \theta h_1 + \theta^2 h_2$, $q = uc + vd$, $Q_0 = P_0 - \frac{2s}{T-1}cc'$, $c' \stackrel{\text{def}}{=} (1,1,\ldots,1,0)$, $d' \stackrel{\text{def}}{=} (0,0,\ldots,0,1)$,

$$P_0 \stackrel{\text{def}}{=} \begin{pmatrix} \alpha & -1 & 0 & 0 & & & & \\ -1 & \alpha & -1 & 0 & & & & \\ 0 & -1 & \alpha & -1 & & & & \\ & & & & \cdots & & & \\ & & & & & -1 & \alpha & -1 \\ & & & & & 0 & -1 & 1 \end{pmatrix}, \quad \alpha \stackrel{\text{def}}{=} 2(1+s),$$

$$h_1 \stackrel{\text{def}}{=} \begin{pmatrix} -2 & 1 & 0 & 0 & & & & \\ 1 & -2 & 1 & 0 & & & & \\ 0 & 1 & -2 & 1 & & & & \\ & & & & \cdots & & & \\ & & & & & 1 & -2 & 1 \\ & & & & & 0 & 1 & 0 \end{pmatrix} \quad \textit{and}$$

$$h_2 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ & & & \cdots & & & \\ & & & & 1 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 \end{pmatrix}.$$

The matrices are all $T \times T$ and the vectors are T dimensional.

PROOF. The lemma follows directly from the identity

$$\varphi(\rho;s,u) = \int \cdots \int (2\pi)^{-T/2} e^{-1/2\underline{x}'Q\underline{x} - q'\underline{x}} dx_1 \cdots dx_T = \frac{1}{\sqrt{\det Q}} e^{1/2 q' Q^{-1} q},$$

with $\underline{x}' \stackrel{\text{def}}{=} (x_1, \ldots, x_T)$ and Q as in the statement of the lemma (cf. Larsson (1994)). □

The following lemma gives expressions for the derivatives appearing in (2.1)–(2.4).

LEMMA 2.3. *At* $u = v = 0$, $\rho = 1$,

$$\frac{\partial\varphi}{\partial\rho} = \frac{1}{2} b_1 \frac{1}{\sqrt{\det Q_0}}, \tag{2.11}$$

$$\frac{\partial^2\varphi}{\partial\rho^2} = \left(-b_2 + \frac{1}{4}b_1^2 + \frac{1}{2}b_{11}\right) \frac{1}{\sqrt{\det Q_0}}, \tag{2.12}$$

$$\frac{\partial^2\varphi}{\partial u \partial v} = \frac{1}{2} A_{12} \frac{1}{\sqrt{\det Q_0}}, \tag{2.13}$$

$$\frac{\partial^2\varphi}{\partial u^2} = A_{11} \frac{1}{\sqrt{\det Q_0}}, \tag{2.14}$$

$$\frac{\partial^3\varphi}{\partial u \partial v \partial\rho} = \left(\frac{1}{4} b_1 A_{12} + \frac{1}{2} B_{12}\right) \frac{1}{\sqrt{\det Q_0}} \quad \textit{and} \tag{2.15}$$

$$\frac{\partial^4\varphi}{\partial u^2 \partial v^2} = \left(A_{11}A_{22} + \frac{1}{2}A_{12}^2\right) \frac{1}{\sqrt{\det Q_0}}, \tag{2.16}$$

where $b_i \stackrel{\text{def}}{=} \operatorname{tr}(Q_0^{-1}h_i)$, $i = 1, 2$, $b_{11} \stackrel{\text{def}}{=} \operatorname{tr}((Q_0^{-1}h_1)^2)$, *and the* A_{ij}*'s and* B_{ij}*'s*, $i, j = 1, 2$, *are defined through*

$$q' Q_0^{-1} q = u^2 A_{11} + uv A_{12} + v^2 A_{22}$$

and

$$q' Q_0^{-1} h_1 Q_0^{-1} q = u^2 B_{11} + uv B_{12} + v^2 B_{22},$$

respectively.

PROOF. As in Larsson (1994), (2.11) and (2.12) follow from the identity

$$\frac{1}{\sqrt{\det Q}} = \frac{1}{\sqrt{\det Q_0}} E(e^{-1/2\underline{X}'h\underline{X}}), \qquad h = \theta h_1 + \theta^2 h_2, \tag{2.17}$$

where $\underline{X}$ is a T-variate normally distributed random vector with covariance matrix Q^{-1}, Taylor expansion of the exponential and the equalities

$$E(\underline{X}'h\underline{X}) = \operatorname{tr}(Q_0^{-1}h), \qquad E((\underline{X}'h\underline{X})^2) = \operatorname{tr}^2(Q_0^{-1}h) + 2\operatorname{tr}((Q_0^{-1}h)^2)$$

(see Magnus (1978)). Moreover, using (2.17) and

$$Q^{-1} = Q_0^{-1} - \theta Q_0^{-1} h_1 Q_0^{-1} + O(\theta^2),$$

the φ expression in Lemma 2.2 is readily Taylor expanded to yield (2.13)–(2.16). □

We will also need the following result, proved in Larsson (1994) (cf. also Le Breton and Pham (1989)):

LEMMA 2.4. *Denoting an arbitrary element of the $T \times T$ matrix P_0^{-1} by p_{ij}, we have*

$$p_{ij} = \begin{cases} \dfrac{1}{\det P_0} D^*_{i-1} D_{T-j}, & i \leq j, \\ p_{ji}, & j < i, \end{cases}$$

where $D_0 = D_0^* = 1$, and for $k \geq 1$,

$$D_k = \left(\frac{\alpha}{2}\right)^{k-1} \left(\frac{(1+\xi)^{k-1} + (1-\xi)^{k-1}}{2} + \left(1 - \frac{2}{\alpha}\right) \frac{(1+\xi)^{k-1} - (1-\xi)^{k-1}}{2\xi} \right),$$

$$D_k^* = \left(\frac{\alpha}{2}\right)^{k-1} \left(\alpha \frac{(1+\xi)^{k-1} + (1-\xi)^{k-1}}{2} + \left(\alpha - \frac{2}{\alpha}\right) \frac{(1+\xi)^{k-1} - (1-\xi)^{k-1}}{2\xi} \right),$$

and

$$\xi = \sqrt{1 - \frac{4}{\alpha^2}}. \tag{2.18}$$

Moreover,

$$\det P_0 = D_T.$$

By (2.18) and the substitution $x = \xi T$,

$$s = \frac{1}{\sqrt{1-\xi^2}} - 1 = \frac{x^2}{2T^2} + O(T^{-4}),$$

implying

$$\det P_0 = D_T = \cosh x + O(T^{-1}), \tag{2.19}$$
$$D_T^* = T\frac{\sinh x}{x} + O(1), \qquad \Delta D_T \overset{\text{def}}{=} D_T - D_{T-1} = \frac{1}{T}\frac{\sinh x}{x} + O(T^{-2}),$$

etc. Furthermore, putting $y = i/T$, we get

$$\frac{1}{T}\sum_{i=1}^{T} D_i = \frac{1}{T}\sum_{i=1}^{T} \frac{1}{2}\left(\left(1+\frac{x}{T}\right)^{i-1} + \left(1-\frac{x}{T}\right)^{i-1} \right) + O(T^{-1})$$
$$= \int_0^1 \cosh(xy)dy + O(T^{-1}),$$

etc., approximations which will turn out to be useful below.

PROOF OF THEOREM 2.1. The derivation of (2.5) and (2.6) rests on the lemmas, the approximation technique outlined above, and the identities

$$\det(A + ww') = \det A(1 + w'A^{-1}w) \tag{2.20}$$

and

$$(A + ww')^{-1} = A^{-1} - \frac{1}{1 + w'A^{-1}w} A^{-1}ww'A^{-1}, \tag{2.21}$$

where A is a $T \times T$ matrix and w is a T dimensional vector. To begin with, because (cf. Lemma 2.1)

$$Q_0 = P_0 - \frac{x^2}{T^3} cc'(1 + O(T^{-1})), \tag{2.22}$$

we have by (2.20) (put $w = \sqrt{-x^2/T^3 c}$)

$$\det Q_0 = \det P_0 \left(1 - \frac{x^2}{T^3} c' P_0^{-1} c(1 + O(T^{-1}))\right),$$

and using Lemma 2.4 together with approximation arguments as above,

$$\begin{aligned}
\det P_0 c' P_0^{-1} c &= \sum_{i=1}^{T-1} \left(D_{T-i} \sum_{k=1}^{i} D_{k-1}^* + D_{i-1}^* \sum_{l=i+1}^{T-1} D_{T-l} \right) \\
&= T^3 \int_0^1 \left(\cosh(x(1-y)) \int_0^y \frac{\sinh(xz)}{x} dz \right. \\
&\qquad \left. + \frac{\sinh(xy)}{x} \int_y^1 \cosh(x(1-z)) dz \right) dy + O(T^2) \\
&= \frac{T^3}{x^2} \left(\cosh x - \frac{\sinh x}{x} \right) + O(T^2).
\end{aligned} \tag{2.23}$$

Hence, via (2.19),

$$\det Q_0 = \frac{\sinh x}{x} + O(T^{-1}). \tag{2.24}$$

Furthermore, (2.21) and (2.22) yield

$$\begin{aligned}
b_1 &= \operatorname{tr}(Q_0^{-1} h_1) \\
&= \operatorname{tr}(P_0^{-1} h_1) + \frac{1}{1 - \frac{x^2}{T^3} c' P_0^{-1} c} \frac{x^2}{T^3} \operatorname{tr}(P_0^{-1} cc' P_0^{-1} h_1)(1 + O(T^{-1})) \\
&= \operatorname{tr}(P_0^{-1} h_1) + \frac{\det P_0}{\det Q_0} \frac{x^2}{T^3} (P_0^{-1} c)' h_1 (P_0^{-1} c)(1 + O(T^{-1})).
\end{aligned} \tag{2.25}$$

But, as a consequence of Lemma 2.4 (cf. Larsson (1994)),

$$\begin{aligned}
&\det P_0 \operatorname{tr}(P_0^{-1} h_1) \\
&\quad = -\sum_{i=2}^{T-2} (D_{i-2}^* \Delta D_{T-i+1} + \Delta D_{i-1}^* D_{T-i+1}) + D_{T-2}^* + O(1) \\
&\quad = -T \int_0^1 \left(\frac{\sinh(xy)}{x} x \sinh(x(1-y)) + \cosh(xy) \cosh(x(1-y)) \right) dy \\
&\qquad + T \frac{\sinh x}{x} + O(1) \\
&\quad = -T \left(\cosh x - \frac{\sinh x}{x} \right) + O(1).
\end{aligned} \tag{2.26}$$

Moreover, Lemma 2.4 implies that the i-th element of the vector $(\det P_0)P_0^{-1}c$ equals

$$D_{T-i}\sum_{k=1}^{i} D_{k-1}^* + D_{i-1}^* \sum_{l=i+1}^{T-1} D_{T-l}, \qquad i=1,\dots,T,$$

and so

$$(\det P_0)^2(P_0^{-1}c)'h_1(P_0^{-1}c) = -2\sum_{i=1}^{T-1} c_i d_i,$$

where

$$\begin{aligned}
c_i &= D_{T-i}\sum_{k=1}^{i} D_{k-1}^* + D_{i-1}^* \sum_{l=i+1}^{T-1} D_{T-l} \\
&= T^2\left(\cosh(x(1-y))\int_0^y \frac{\sinh(xz)}{x}dz + \frac{\sinh(xy)}{x}\int_y^1 \cosh(x(1-z))dz\right) \\
&\quad + O(T) \\
&= \frac{T^2}{x^2}(\cosh x - \cosh(x(1-y))) + O(T)
\end{aligned}$$

and

$$\begin{aligned}
d_i &= D_{T-i}\sum_{k=1}^{i} D_{k-1}^* + D_{i-1}^* \sum_{l=i+1}^{T-1} D_{T-l} - D_{T-i-1}\sum_{k=1}^{i+1} D_{k-1}^* - D_i^* \sum_{l=i+2}^{T-1} D_{T-l} \\
&= \Delta D_{T-i}\sum_{k=1}^{i} D_{k-1}^* - \Delta D_i^* \sum_{l=i+2}^{T-1} D_{T-l} - D_{T-i-1}D_i^* + D_{i-1}^* D_{T-i-1} \\
&= T\left(x\sinh(x(1-y))\int_0^y \frac{\sinh(xz)}{x}dz - \cosh(xy)\int_y^1 \cosh(x(1-z))dz\right) \\
&\quad + O(1) \\
&= -\frac{T}{x}\sinh(x(1-y)) + O(1).
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
(2.27)\quad &(\det P_0)^2(P_0^{-1}c)'h_1(P_0^{-1}c) \\
&= 2\frac{T^4}{x^3}\int_0^1 (\cosh x - \cosh(x(1-y)))\sinh(x(1-y))dy + O(T^3) \\
&= 2\frac{T^4}{x^4}\left(\frac{1}{2}\sinh^2 x + 1 - \cosh x\right) + O(T^3),
\end{aligned}$$

after some simplification. Hence, plugging in (2.19), (2.24), (2.26) and (2.27) into (2.25) and simplifying, we get

$$b_1 = -T\left(1 - 2\frac{\cosh x - 1}{x\sinh x}\right) + O(1), \tag{2.28}$$

and by inserting (2.24) and (2.28) into (2.11),

$$\frac{\partial\varphi}{\partial\rho} = -\frac{T}{2}\sqrt{\frac{x}{\sinh x}}\left(1 - 2\frac{\cosh x - 1}{x\sinh x}\right) + O(1). \tag{2.29}$$

Now, to deduce (2.5), (2.1) and (2.3) tell us that we also need $\partial^2\varphi/(\partial u\partial v)$ i.e., in view of (2.13), A_{12}. But expanding Q_0^{-1} via (2.21), we find

$$\begin{aligned}\frac{1}{2}A_{12} = c'Q_0^{-1}d &= c'P_0^{-1}d + \frac{1}{1 - \frac{x^2}{T^3}c'P_0^{-1}c}\frac{x^2}{T^3}c'P_0^{-1}cc'P_0^{-1}d(1 + O(T^{-1})) \\ &= \frac{\det P_0}{\det Q_0}cP_0^{-1}d(1 + O(T^{-1})),\end{aligned}$$

and it follows from Lemma 2.4 that

$$\begin{aligned}\det P_0 c'P_0^{-1}d = \sum_{i=1}^{T-1} D_{i-1}^* &= T^2\int_0^1 \frac{\sinh(xy)}{x}dy + O(T) \\ &- \frac{T^2}{x^2}(\cosh x - 1) + O(T),\end{aligned} \tag{2.30}$$

i.e., by (2.24),

$$\frac{1}{2}A_{12} = T^2\frac{\cosh x - 1}{x\sinh x} + O(T), \tag{2.31}$$

and so, (2.13) and (2.24) yield

$$\frac{\partial^2\varphi}{\partial u\partial v} = T^2\frac{\cosh x - 1}{\sqrt{x}(\sinh x)^{3/2}} + O(T). \tag{2.32}$$

Finally, since $ds = xdx/T^2 + O(T^{-4})$, (2.29) and (2.32) inserted in (2.1)–(2.3) imply (2.5).

Deriving (2.6), we start with $\partial^2\varphi/\partial\rho^2$ (cf. (2.4)), and to this end, (2.21) implies (cf. (2.25))

$$b_2 = \operatorname{tr}(Q_0^{-1}h_2) = \operatorname{tr}(P_0^{-1}h_2) + \frac{\det P_0}{\det Q_0}\frac{x^2}{T^3}(P_0^{-1}c)'h_2(P_0^{-1}c)(1 + O(T^{-1})),$$

and by Lemma 2.4,

$$\begin{aligned}\det P_0 \operatorname{tr}(P_0^{-1}h_2) = \sum_{i=1}^{T-1} D_{i-1}^* D_{T-i} &= T^2\int_0^1 \frac{\sinh(xy)}{x}\cosh(x(1-y))dy + O(T) \\ &= \frac{T^2}{2}\frac{\sinh x}{x} + O(T).\end{aligned}$$

Moreover,

$$\begin{aligned}
&(\det P_0)^2 (P_0^{-1}c)' h_2 (P_0^{-1}c) \\
&= \sum_{i=1}^{T-1} \left(D_{T-i} \sum_{k=1}^{i} D_{k-1}^* + D_{i-1}^* \sum_{l=i+1}^{T-1} D_{T-l} \right)^2 \\
&= T^5 \int_0^1 \left(\cosh(x(1-y)) \int_0^y \frac{\sinh(xz)}{x} dz \right. \\
&\qquad\qquad \left. + \frac{\sinh(xy)}{x} \int_y^1 \cosh(x(1-z)) dz \right)^2 dy + O(T^4) \\
&= \frac{T^5}{x^4} \left(\cosh^2 x - \frac{3}{2} \cosh x \frac{\sinh x}{x} + \frac{1}{2} \right) + O(T^4),
\end{aligned}$$

and so, by (2.19) and (2.24), we get after simplification

$$b_2 = T^2 \frac{3}{2} \frac{1}{x \sinh x} \left(\cosh x - \frac{\sinh x}{x} \right) + O(T). \tag{2.33}$$

Finally, by (2.21),

$$\begin{aligned}
b_{11} &= \operatorname{tr}((Q_0^{-1}h_1)^2) \\
&= \operatorname{tr}\left(\left(P_0^{-1}h_1 + \frac{\det P_0}{\det Q_0} \frac{x^2}{T^3} (P_0^{-1}c)(P_0^{-1}c)' h_1 (1 + O(T^{-1})) \right)^2 \right) \\
&= \operatorname{tr}((P_0^{-1}h_1)^2) + 2 \frac{\det P_0}{\det Q_0} \frac{x^2}{T^3} (P_0^{-1}c)' h_1 P_0^{-1} h_1 (P_0^{-1}c)(1 + O(T^{-1})) \\
&\quad + \left(\frac{\det P_0}{\det Q_0} \right)^2 \frac{x^4}{T^6} ((P_0^{-1}c)' h_1 (P_0^{-1}c))^2 (1 + O(T^{-1})).
\end{aligned}$$

But, in the same manner as above, it follows that

$$(\det P_0)^2 \operatorname{tr}((P_0^{-1}h_1)^2) = {D_{T-2}^*}^2 + O(T) = T^2 \left(\frac{\sinh x}{x} \right)^2 + O(T).$$

Furthermore,

$$\begin{aligned}
(\det P_0)^3 (P_0^{-1}c)' h_1 P_0^{-1} h_1 (P_0^{-1}c) &= D_{T-2}^* \left(\sum_{i=1}^{T} D_{i-1}^* \right)^2 + O(T^4) \\
&= T^5 \frac{\sinh x}{x} \left(\int_0^1 \frac{\sinh(xy)}{x} dy \right)^2 + O(T^4) \\
&= \frac{T^5}{x^5} \sinh x (\cosh x - 1)^2 + O(T^4),
\end{aligned}$$

and so, via (2.19), (2.24), (2.27) and simplifications

$$b_{11} = 4\frac{T^2}{x^2}\left(1 - 2\frac{\cosh x - 1}{\sinh^2 x}\right) + O(T^3). \tag{2.34}$$

Hence, (2.24), (2.28), (2.33) and (2.34) inserted in (2.12) yield

$$\begin{aligned}\frac{\partial^2\varphi}{\partial\rho^2} = T^2\sqrt{\frac{x}{\sinh x}}\Bigg(& -\frac{3}{2}\frac{1}{x\sinh x}\left(\cosh x - \frac{\sinh x}{x}\right) \\ & + \frac{1}{4}\left(1 - 2\frac{\cosh x - 1}{x\sinh x}\right)^2 \\ & + \frac{2}{x^2}\left(1 - 2\frac{\cosh x - 1}{\sinh^2 x}\right)\Bigg) + O(T).\end{aligned} \tag{2.35}$$

Dealing with $\partial^2\varphi/\partial u^2$, (2.14) tells us that we need A_{11}, and by expanding Q_0^{-1} we get

$$A_{11} = c'Q_0^{-1}c = c'P_0^{-1}c + \frac{1}{1 - \frac{x^2}{T^3}c'P_0^{-1}c}\frac{x^2}{T^3}(c'P_0^{-1}c)^2 = \frac{\det P_0}{\det Q_0}c'P_0^{-1}c,$$

and so, (2.23) and (2.24) imply

$$A_{11} = T^3\frac{1}{x\sinh x}\left(\cosh x - \frac{\sinh x}{x}\right) + O(T^2), \tag{2.36}$$

and by (2.14) and (2.24),

$$\frac{\partial^2\varphi}{\partial u^2} = T^3\frac{1}{\sqrt{x}(\sinh x)^{3/2}}\left(\cosh x - \frac{\sinh x}{x}\right) + O(T^2). \tag{2.37}$$

As for $\partial^3\varphi/\partial u\partial v\partial\rho$, the only new term needed is B_{12} (cf. (2.15)). But, expanding Q_0^{-1} and simplifying,

$$\begin{aligned}\frac{1}{2}B_{12} &= c'Q_0^{-1}h_1Q_0^{-1}d \\ &= \left(\frac{\det P_0}{\det Q_0}\right)^2\Bigg((P_0^{-1}c)'h_1(P_0^{-1}d) + \frac{x^2}{T^3}((c'P_0^{-1}d)(P_0^{-1}c)'h_1(P_0^{-1}c) \\ &\qquad - (c'P_0^{-1}c)(P_0^{-1}c)'h_1(P_0^{-1}d))\Bigg)(1 + O(T^{-1})).\end{aligned}$$

The only "unknown" component of this expression is $(P_0^{-1}c)'h_1(P_0^{-1}d)$, and via Lemma 2.4, approximations and simplifications in the usual manner, we obtain

$$(\det P_0)^2(P_0^{-1}c)'h_1(P_0^{-1}d) = \frac{T^3}{x^3}\sinh x(\cosh x - 1) + O(T^2),$$

which together with (2.19), (2.23), (2.24), (2.27) and (2.30) yields, after simplification,

$$\frac{1}{2}B_{12} = 2\frac{T^3}{x^2}\frac{(\cosh x - 1)^2}{\sinh^2 x} + O(T^2). \tag{2.38}$$

Hence, putting (2.28), (2.31) and (2.38) into (2.15), it follows that

$$\frac{\partial^3 \varphi}{\partial u \partial v \partial \rho} = -\frac{T^3}{2}\frac{\cosh x - 1}{\sqrt{x}(\sinh x)^{3/2}}\left(1 - 6\frac{\cosh x - 1}{x \sinh x}\right) + O(T^2). \tag{2.39}$$

We now have

$$A_{22} = d'Q_0^{-1}d = d'P_0^{-1}d + \frac{\det P_0}{\det Q_0}\frac{x^2}{T^3}(c'P_0^{-1}d)^2,$$

and since by Lemma 2.4,

$$\det P_0 d'P_0^{-1}d = D_{T-1}^* = T\frac{\sinh x}{x} + O(1),$$

(2.19), (2.24) and (2.30) imply

$$A_{22} = 2T\frac{\cosh x - 1}{x \sinh x} + O(1), \tag{2.40}$$

and by inserting (2.24), (2.31), (2.36) and (2.40) into (2.16), we obtain

$$\frac{\partial^4 \varphi}{\partial u^2 \partial v^2} = 2T^4 \frac{1}{x^{3/2}(\sinh x)^{5/2}}(\cosh x - 1)\left(2\cosh x - \frac{\sinh x}{x} - 1\right) + O(T^3). \tag{2.41}$$

Finally, since $ds = x dx/T^2 + O(T^{-4})$ and by partial integration

$$\int_0^\infty \frac{\sqrt{x}}{(\sinh x)^{3/2}}\left(\cosh x - \frac{\sinh x}{x}\right) dx = 2,$$

it follows readily from (2.4), (2.35), (2.37), (2.39) and (2.41) that, via simplifications

$$\begin{aligned} EZ = \int_0^\infty \frac{x^{3/2}}{\sqrt{\sinh x}}\Bigg(&\frac{1}{4}\left(1 - 2\frac{\cosh x - 1}{x\sinh x}\right)^2 + \frac{2}{x^2}\left(1 - 2\frac{\cosh x - 1}{\sinh^2 x}\right) \\ &+ \frac{\cosh x - 1}{x \sinh x}\left(1 - 6\frac{\cosh x - 1}{x\sinh x}\right) \\ &+ 2\frac{\cosh x - 1}{x^2 \sinh^2 x}\left(2\cosh x - \frac{\sinh x}{x} - 1\right)\Bigg) dx \\ = \int_0^\infty \sqrt{\frac{x}{\sinh x}}&\left(\frac{x}{4} + \frac{1}{x} - 2\frac{\cosh x - 1}{x^2 \sinh x}\right) dx, \end{aligned}$$

which was to be proved. □

Remark. An alternative method of proving the theorem could be via stochastic calculus (cf. Le Breton and Pham (1989) and Liptser and Shiryayev ((1978), Vol. 2, pp. 206–209)). However, our method is preferable in the sense that it may (in principle) easily be generalized to obtain correction terms in T^{-1}, etc. (cf. Larsson (1994)). This is not the case with stochastic calculus methods.

3. Concluding remarks

By generalizing the methods of the present paper, it would be possible to calculate higher moments for the test statistics as well as corresponding results for first order autoregression processes with a polynomial in t in place of the constant η in the defining equation. We may also, by some refining of the technique, calculate first (or higher) order correction terms in T^{-1} of the expectations. These terms are interesting in the context of small sample (Bartlett) correction, cf. Larsson (1994, 1995) and Nielsen (1995). Finally, our ideas might be useful for a generalization to vector-valued autoregressive processes, where the unit root test carries over to a test for cointegration (cf. Johansen (1988)).

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