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COMPARISON OF NORMAL LINEAR EXPERIMENTS BY QUADRATIC FORMS*

CZESŁAW STĘPNIAK

Department of Mathematics, Agricultural University of Lublin, P.O. Box 158, PL 20 950 Lublin 1, Poland

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Abstract. Let X and Y be observation vectors in normal linear experiments $\mathcal{E} = \mathcal{N}(A\beta, \sigma V)$ and $\mathcal{F} = \mathcal{N}(B\beta, \sigma W)$. We write $\mathcal{E} \succ \mathcal{F}$ if for any quadratic form $Y'GY$ there exists a quadratic form $X'HX$ such that $E(X'HX) = E(Y'GY)$ and $\mathrm{var}(X'HX) \leq \mathrm{var}(Y'GY)$. The relation $\succ$ is characterized by the matrices A, B, V and W. Moreover some connections with known orderings of linear experiments are given.

Key words and phrases: Normal linear experiment, comparison of experiments, sufficiency, linear sufficiency, quadratic sufficiency.

1. Introduction

Any statistical experiment can be perceived as an information channel transforming a deterministic quantity (parameter) into a random quantity (observation) according to a **design** indicated by the experimentator. This rises some problems of design choice. Further specifications may lead to different concepts of comparison of statistical experiments.

We restrict ourselves to normal linear experiments $\mathcal{N}(A\beta, \sigma V)$ and $\mathcal{N}(B\beta, \sigma W)$. The classical concepts in this subject adopted the well known notions of **sufficiency** and **linear sufficiency** (cf., among others, Boll (1955), Ehrenfeld (1955), Kiefer (1959), Hansen and Torgersen (1974), Torgersen (1984, 1991), Stępniak and Torgersen (1981), and Stępniak *et al.* (1984)). On the other hand, it follows from the frame of the minimal sufficient statistic, that all reasonable decision rules in such experiments are based on linear and quadratic forms. Thus a study of **quadratic sufficiency** may be an interesting complement to the known results. We take an effort to lay the foundations of such a work.

As we shall see, quadratic sufficiency is stronger than linear one but weaker than the usual notion of sufficiency.

In Section 3 the problem of quadratic sufficiency is reduced to some simpler experiments and in Section 4—to the problem of linear sufficiency for corresponding

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linear models with unknown variance components. A comprehensive characterization of the quadratic sufficiency in term of the matrices A, B, V and W is presented in Sections 5 and 6.

2. Preliminaries

Throughout this paper the following matrix notation is used. If M is a matrix then M', $R(M)$, $N(M)$, $r(M)$, M^- and M^+ denote, respectively, transposition, the range (column space), the null space, the rank, a generalized inverse and the Moore-Penrose generalized inverse of M. If M is square then $\text{tr}(M)$ denotes its trace. The symbols R^n and $\mathcal{S}_n$ stand, respectively, for the space of real $n \times 1$ vectors and for the space of real $n \times n$ symmetric matrices. Moreover, the symbol $M \geq 0$, where $M \in \mathcal{S}_n$, means that M is nonnegative definite (n.n.d. for short).

Let X and Y be random columns, possibly of different length, with distributions depending on a parameter $\theta \in \Theta$.

DEFINITION 2.1. (Lehmann (1988)) We say that X is **sufficient** for Y if there exists a random quantity Z, independent on X, with a known distribution and a function h of X and Z such that

$$h(X, Z) \text{ has the same distribution as } Y. \tag{2.1}$$

DEFINITION 2.2. Assume that $E(X'X) < \infty$ and $E(Y'Y) < \infty$. We say that X is **linearly sufficient** for Y if for any linear form $b'Y$ there exists a linear form $a'X$ such that

$$E_\theta(a'X) = E_\theta(b'Y) \quad \text{and} \quad \text{var}_\theta(a'X) \leq \text{var}_\theta(b'Y) \quad \text{for all} \quad \theta \in \Theta. \tag{2.2}$$

DEFINITION 2.3. Assume that $E(X'X)^2 < \infty$ and $E(Y'Y)^2 < \infty$. We say that X is **quadratically sufficient** for Y if for any quadratic form $Y'GY$ there exists a quadratic form $X'HX$ such that

$$E_\theta(X'HX) = E_\theta(Y'GY) \quad \text{and} \quad \text{var}_\theta(X'HX) \leq \text{var}_\theta(Y'GY) \quad \text{for all} \quad \theta \in \Theta. \tag{2.3}$$

Remark 2.1. Definition 2.2 extends a notion introduced by Drygas (1983).

Remark 2.2. Our notion of quadratic sufficiency does not coincide with the same term used by Mueller (1987).

In this paper we assume that X and Y are, respectively, observation vectors in the normal linear experiments $\mathcal{E} = \mathcal{N}(A\beta, \sigma V)$ and $\mathcal{F} = \mathcal{N}(B\beta, \sigma W)$, where A and B are known $n \times p$ and $m \times p$ matrices, V and W are known n.n.d. matrices belonging to $\mathcal{S}_n$ and $\mathcal{S}_m$, while $\beta \in R^p$ and $\sigma > 0$ are unknown parameters. So, instead of X and Y, we shall often refer the relations of sufficiency, linear sufficiency and quadratic sufficiency for the experiments $\mathcal{E}$ and $\mathcal{F}$.

Definition 2.2 uses only first two moments of the observation vectors X and Y and one can apply it to the **usual** linear models $\mathcal{L}_1 = \mathcal{L}(A\beta, \sigma V)$ and $\mathcal{L}_2 = \mathcal{L}(B\beta, \sigma W)$. For example, instead to say that X is linearly sufficient for Y, we shall often write $\mathcal{L}_1 \rhd \mathcal{L}_2$. It is well known that $\mathcal{L}_1 \rhd \mathcal{L}_2$ if and only if $A'(V + AA')^- A - B'(W + BB')^- B$ is n.n.d. Similarly, assuming X and Y are observation vectors in the **normal** experiments $\mathcal{E}$ and $\mathcal{F}$, the symbol $\mathcal{E} \succ \mathcal{F}$ will mean that X is quadratically sufficient for Y.

Our goal is to characterize the relation $\succ$ by the matrices A, B, V and W. In fact, by Definition 2.3, the characterization will be valid not only for normal linear experiments $\mathcal{E}$ and $\mathcal{F}$ but for arbitrary random vectors X and Y such that

$$E(X) = A\beta, \qquad E(Y) = B\beta, \tag{2.4}$$

$$\begin{aligned} E(X'HX) &= \sigma \operatorname{tr}(VH) + \beta' A' H A \beta, \\ E(Y'GY) &= \sigma \operatorname{tr}(WG) + \beta' B' G B \beta, \end{aligned} \tag{2.5}$$

and,

$$\begin{aligned} \operatorname{cov}(X'H_1X, X'H_2X) &= 2\sigma^2 \operatorname{tr}(H_1 V H_2 V) + 4\sigma\beta' A' H_1 V H_2 A \beta, \\ \operatorname{cov}(Y'G_1Y, Y'G_2Y) &= 2\sigma^2 \operatorname{tr}(G_1 W G_2 W) + 4\sigma\beta' B' G_1 W G_2 B \beta \end{aligned} \tag{2.6}$$

for all $H, H_1, H_2 \in \mathcal{S}_n$, $G, G_1, G_2 \in \mathcal{S}_m$, $\beta \in R^p$ and $\sigma > 0$ (cf. e.g., Searle (1971), p. 57 and p. 451).

Considering the relation $\succ$ for the normal linear experiments one can ask about a practical sense of Definition 2.3 in this context. By formulae (2.5) the expectation $(X'HX)$ is a parametric function of the form $c\sigma + \beta' C \beta$, where $c \in R$ and $C \in \mathcal{S}_p$. Such functions appear in estimating the mean squared error of linear estimators (cf. Stępniak (1995)).

Some properties of the relations of sufficiency, linear sufficiency and quadratic sufficiency are collected in the following lemmas.

LEMMA 2.1. (Transformation invariance) *For any nonsingular linear transformations $X \to T_1X$ and $Y \to T_2Y$:*

(i) *X is sufficient for Y if and only if T_1X is sufficient for T_2Y,*

(ii) *X is linearly sufficient for Y if and only if T_1X is linearly sufficient for T_2Y,*

(iii) *X is quadratically sufficient for Y if and only if T_1X is quadratically sufficient for T_2Y.*

LEMMA 2.2. (Parameterization invariance) *For arbitrary matrix K such that $R(A') \subset R(K)$ and $R(B') \subset R(K)$:*

(i) *$\mathcal{N}(A\beta, \sigma V)$ is sufficient for $\mathcal{N}(B\beta, \sigma W)$ if and only if $\mathcal{N}(AK\xi, \sigma V)$ is sufficient for $\mathcal{N}(BK\xi, \sigma W)$,*

(ii) *$\mathcal{L}(A\beta, \sigma V)$ is linearly sufficient for $\mathcal{L}(B\beta, \sigma W)$ if and only if $\mathcal{L}(AK\xi, \sigma V)$ is linearly sufficient for $\mathcal{L}(BK\xi, \sigma W)$,*

(iii) *$\mathcal{N}(A\beta, \sigma V)$ is quadratically sufficient for $\mathcal{N}(B\beta, \sigma W)$ if and only if $\mathcal{N}(AK\xi, \sigma V)$ is quadratically sufficient for $\mathcal{N}(BK\xi, \sigma W)$.*

LEMMA 2.3. (Smoothing)

(i) *$\mathcal{N}(A\beta, \sigma V)$ is sufficient for $\mathcal{N}(B\beta, \sigma W)$ if and only if $\mathcal{N}(A\beta, \sigma(V+AA'))$ is sufficient for $\mathcal{N}(B\beta, \sigma(W+BB'))$.*

(ii) *$\mathcal{L}(A\beta, \sigma V)$ is linearly sufficient for $\mathcal{L}(B\beta, \sigma W)$ if and only if $\mathcal{L}(A\beta, \sigma(V+AA'))$ is linearly sufficient for $\mathcal{L}(B\beta, \sigma(W+BB'))$.*

(iii) *$\mathcal{N}(A\beta, \sigma V)$ is quadratically sufficient for $\mathcal{N}(B\beta, \sigma W)$ if and only if $\mathcal{N}(A\beta, \sigma(V+AA'))$ is quadratically sufficient for $\mathcal{N}(B\beta, \sigma(W+BB'))$.*

For proof of the statements (i) and (ii) in Lemma 2.3 we refer to Stępniak ((1987), Theorem 1) and Torgersen ((1991), pp. 465–467). In order to show the statement (iii) we only need to verify that under the assumptions $\mathrm{tr}(VH) = \mathrm{tr}(WG)$ and $A'HA = B'GB$ the conditions

$$\text{(2.7)} \qquad \mathrm{tr}(HVHV) \le \mathrm{tr}(GWGW), \quad B'GWGB - A'HVHA \quad \text{is n.n.d.}$$

and

$$\text{(2.8)} \qquad \begin{aligned} &\mathrm{tr}[H(V+AA')H(V+AA')] \le \mathrm{tr}[G(W+BB')G(W+BB')], \\ &B'G(W+BB')GB - A'H(V+AA')HA \quad \text{is n.n.d.} \end{aligned}$$

are equivalent.

The following theorem provides a connection between the orderings $\succ$ and $\vartriangleright$.

THEOREM 2.1. *For arbitrary normal linear experiments $\mathcal{E} = \mathcal{N}(A\beta, \sigma V)$ and $\mathcal{F} = \mathcal{N}(B\beta, \sigma W)$ let us consider the linear models $\mathcal{L}_1 = \mathcal{L}(A\beta, \sigma V)$ and $\mathcal{L}_2 = \mathcal{L}(B\beta, \sigma W)$. If $\mathcal{E} \succ \mathcal{F}$ then*

$$\text{(a)} \quad r(V+AA') - r(A) \ge r(W+BB') - r(B)$$

and

$$\text{(b)} \quad \mathcal{L}_1 \vartriangleright \mathcal{L}_2.$$

PROOF. Assume that $\mathcal{E} \succ \mathcal{F}$.

(a) It follows directly by the well known fact that σ is unbiasedly estimable in $\mathcal{E}$ if and only if $v = r(V+AA') - r(A) > 0$ and if so, then the variance of its Best Quadratic Unbiased Estimator is equal to $2\sigma^2/\nu$ (cf., e.g., Rao (1973), pp. 294–300).

Let X and Y be observation vectors in the experiments $\mathcal{E}$ and $\mathcal{F}$, respectively. It is known (cf. Seely (1978), Theorem 2.1 and Corollary 2.2) that the BLUE $\hat{\mu} = \hat{\mu}(X)$ of $A\beta$ and the BQUE $\hat{\sigma} = \hat{\sigma}(X)$ of σ are stochastically independent and jointly constitute a complete sufficient statistic in $\mathcal{E}$, provided $r(V+AA') > r(A)$. Otherwise, if $r(V+AA') = r(A)$, the initial vector X is complete.

(b) For a given linear form $b'Y$ let us consider the parametric functions $\phi = E(b'Y)$ and $\psi = E[(b'Y)^2]$. By assumption $\mathcal{E} \succ \mathcal{F}$, via (2.4) and (2.5), the both functions are unbiasedly estimable in $\mathcal{E}$. Moreover, by completeness, the BQUE of ψ can be written in the form $\hat{\psi} = c\hat{\sigma} + (\hat{\phi})^2$, where $\hat{\phi} = \hat{\phi}(X)$ is the BLUE of ϕ and $c \in R$. Therefore $\mathrm{var}[(\hat{\phi})^2] \le \mathrm{var}(\hat{\psi}) \le \mathrm{var}[(b'Y)^2]$. We need to show that $\mathrm{var}(\hat{\phi}) \leq \mathrm{var}(b'Y)$. The desired result follows directly by the fact that for arbitrary normal variables Z_1 and Z_2 with a common mean, $\mathrm{var}(Z_1^2) \leq \mathrm{var}(Z_2^2)$ if and only if $\mathrm{var}(Z_1) \leq \mathrm{var}(Z_2)$. This completes the proof of the theorem. □

Now let us consider a particular case, when $\mathcal{F} = \mathcal{N}(0, \sigma W)$. Then $E(Y'GY) = c\sigma$ for arbitrary quadratic form $Y'GY$. In consequence, by the argument used at the beginning of the proof we get the following result.

LEMMA 2.4. *For arbitrary normal linear experiments $\mathcal{E} = \mathcal{N}(A\beta, \sigma V)$ and $\mathcal{F} = \mathcal{N}(0, \sigma W)$, $\mathcal{E} \succ \mathcal{F}$ if and only if $r(V + AA') - r(A) \geq r(W)$.*

Henceforth we shall assume that the matrix B, appearing in the experiment $\mathcal{F} = \mathcal{N}(B\beta, \sigma W)$, is different from zero.

3. Reduction by invariance

In this section we focus on the normal experiments $\mathcal{E} = \mathcal{N}(A\beta, \sigma I_n)$ and $\mathcal{F} = \mathcal{N}(B\beta, \sigma I_m)$. It is well known (cf. Hansen and Torgersen (1974)) that $\mathcal{E}$ is sufficient for $\mathcal{F}$ if and only if $A'A - B'B$ is nonnegative definite and $r(A'A - B'B) \leq n - m$. Thus, under condition $A'A - B'B \geq 0$, the relation of sufficiency may be expressed by the integers n, m and $r(A'A - B'B)$. It will appear that a corresponding expression for the quadratic sufficiency needs some additional information about A and B or, more precisely, about the eigenvalues of the matrix $(A'A)^+B'B$.

By Theorem 2.1(b) the condition

$$R(B') \subseteq R(A') \tag{3.1}$$

is necessary for the relation $\mathcal{E} \succ \mathcal{F}$. Let A and B be arbitrary nonzero $n \times p$ and $m \times p$ matrices satisfying (3.1) and let r and q be ranks of A and B. By Lemmas A.1 and A.2 in the Appendix all eigenvalues of $(A'A)^+B'B$ are nonnegative. Moreover, $A'A - B'B \geq 0$ if and only if the eigenvalues are not greater than 1.

Now for the initial experiments $\mathcal{E}$ and $\mathcal{F}$ let us consider some reduced experiments

$$\mathcal{E}_0 = \begin{cases} \mathcal{N}(\alpha, \sigma I_q) & \text{if} \quad r = n \\ \mathcal{N}\left(\begin{bmatrix} \alpha \\ 0_{n-r} \end{bmatrix}, \sigma I_{n-r+q}\right) & \text{otherwise}, \end{cases} \tag{3.2}$$

and

$$\mathcal{F}_0 = \begin{cases} \mathcal{N}(D\alpha, \sigma I_q) & \text{if} \quad q = m \\ \mathcal{N}\left(\begin{bmatrix} D\alpha \\ 0_{m-q} \end{bmatrix}, \sigma I_m\right) & \text{otherwise}, \end{cases} \tag{3.3}$$

where $\alpha \in R^q$ and

$$D = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_q}), \tag{3.4}$$

while $\lambda_1 \geq \cdots \geq \lambda_q$ are nonzero eigenvalues of the matrix $(A'A)^+B'D$ taken with their multiplicities.

THEOREM 3.1. (Reduction to canonical experiments) *For arbitrary experiments $\mathcal{E} = \mathcal{N}(A\beta, \sigma I_n)$ and $\mathcal{F} = \mathcal{N}(B\beta, \sigma I_m)$ such that $B \neq 0$ and $R(B') \subseteq R(A')$ let us consider the reduced experiments $\mathcal{E}_0$ and $\mathcal{F}_0$ defined by (3.2) and (3.3). Then the following are equivalent:*

(a) $\mathcal{E} \succ \mathcal{F}$,

(b) $\mathcal{E}_0 \succ \mathcal{F}_0$.

PROOF. By Lemma A.3 (see Appendix) there exists a $p \times r$ matrix M such that

$$R(M) = R(A'), \qquad M'A'AM = I_r \qquad \text{and}$$

$$M'B'BM = \begin{cases} D^2 & \text{if } q = r \\ \begin{bmatrix} D^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \text{otherwise.} \end{cases} \tag{3.5}$$

Rewrite the matrix M as $M = [M_1, M_2]$, where M_1 has q columns. Let F_1 and F_2 be arbitrary $(n-r) \times n$ and $(m-q) \times m$ matrices satisfying the conditions $F_1 A = \mathbf{0}$, $F_2 B = \mathbf{0}$, $F_1 F_1' = I_{n-r}$ and $F_2 F_2' = I_{m-q}$. We define an auxiliary experiment

$$\mathcal{E}_1 = \begin{cases} \mathcal{N}\left(\begin{bmatrix} \alpha \\ \vartheta \end{bmatrix}, \sigma I_r\right) & \text{if } r = n \\ \mathcal{N}\left(\begin{bmatrix} \alpha \\ \vartheta \\ \mathbf{0}_{n-r} \end{bmatrix}, \sigma I_n\right) & \text{otherwise,} \end{cases} \tag{3.6}$$

where $\alpha \in R^q$ and $\vartheta \in R^{r-q}$. It follows form Lemmas 2.1 and 2.2 by setting $K = M$,

$$T_1 = \begin{bmatrix} M'A' \\ \cdots\cdots \\ F_1 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} D^{-1}M'B' \\ \cdots\cdots\cdots \\ F_2 \end{bmatrix},$$

that $\mathcal{E} \succ \mathcal{F}$ if and only if $\mathcal{E}_1 \succ \mathcal{F}_0$. Now we only need to show that the relation $\mathcal{E}_1 \succ \mathcal{F}_0$ is equivalent to $\mathcal{E}_0 \succ \mathcal{F}_0$.

Let us consider an arbitrary quadratic form $X'QX$, where X is observation vector in the experiment $\mathcal{E}_1$ and let

$$Q = \begin{cases} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} & \text{if } r = n \\ \begin{bmatrix} Q_{11} & Q_{22} & Q_{13} \\ Q_{12}' & Q_{22} & Q_{23} \\ Q_{13}' & Q_{23}' & Q_{33} \end{bmatrix} & \text{otherwise,} \end{cases}$$

be the block representation of the matrix Q, corresponding to (3.6). By formula (2.5) the expectation $E(X'QX)$ does not depend on ϑ if and only if Q_{12}, Q_{22} and Q_{23} vanish. This implies the desired equivalence and completes the proof. □

4. Reduction by sufficiency

Theorem 3.1 provides an initial reduction of the problem of quadratic sufficiency to some canonical experiments. In this section the second problem will be further reduced to the problem of linear sufficiency for corresponding linear models with unknown variance components. To this aim we need to collect some results on sufficient statistics for a canonical normal experiment.

PROPOSITION 4.1. *Let Y be observation vector in a normal linear experiment $\mathcal{N}(D\alpha, \sigma I_q)$ such that $D = \mathrm{diag}(d_1, \ldots, d_q)$ is nonsingular. Then*

(a) *Y is a complete statistic in the experiment.*

(b) *A parametric function $\psi = c\sigma + \alpha' C\alpha$, where $c \in R$ and $C \in \mathcal{S}_q$, is estimable by quadratic forms of Y if and only if*

$$c = \mathrm{tr}(D^{-1}CD^{-1}). \tag{4.1}$$

(c) *If the condition (4.1) is satisfied then the Best Quadratic Unbiased Estimator of ψ may be presented in the form $\hat{\psi} = Y'D^{-1}CD^{-1}Y$ and its variance is* $\mathrm{var}(\hat{\psi}) = 2\sigma^2\,\mathrm{tr}(CD^{-2}CD^{-2}) + 4\sigma\alpha' CD^{-2}C\alpha$.

PROPOSITION 4.2. *Let Y be observation vector in a normal linear experiment $\mathcal{N}(\left[\begin{smallmatrix} D \\ 0 \end{smallmatrix}\right]\alpha, \sigma I_{q+k})$ such that $D = \mathrm{diag}(d_1, \ldots, d_q)$ is nonsingular. Then*

(a) *The joint statistic $s(Y) = \left[\begin{smallmatrix} Y_* \\ s(Y) \end{smallmatrix}\right]$, where $Y_* = [Y_1, \ldots, Y_q]'$ and $s(Y) = \frac{1}{k}\sum_{i=q+1}^{q+k} Y_i^2$ is complete and sufficient in the experiment.*

(b) *Any parametric function $\psi = c\sigma + \alpha' C\alpha$, where $c \in R$ and $C \in \mathcal{S}_q$, is estimable by quadratic forms of Y.*

(c) *The BQUE of ψ may be presented in the form $\hat{\psi} = Y_*'D^{-1}CD^{-1}Y_* + [c - \mathrm{tr}(D^{-1}CD^{-1})]s(Y)$ and its variance is*

$$\mathrm{var}(\hat{\psi}) = \frac{2}{k}[c - \mathrm{tr}(D^{-1}CD^{-1})]^2\sigma^2 + 2\sigma^2\,\mathrm{tr}(CD^{-2}CD^{-2}) + 4\sigma\alpha' CD^{-2}C\alpha. \tag{4.2}$$

PROOF OF THE PROPOSITIONS. The sufficiency follows by factorization theorem for exponential families of distributions while the completeness follows from a theorem by Lehmann ((1986), p. 142). Now we only need to verify the conditions (b) and (c) in the Propositions 4.1 and 4.2. To this aim it suffices to use the formulae (2.5) and (2.6). □

THEOREM 4.1. *Let X and Y be observation vectors in arbitrary normal linear experiments $\mathcal{E}_0$ and $\mathcal{F}_0$ of the form (3.2) and (3.3) for some diagonal matrix D with positive diagonal elements. Then $\mathcal{E}_0 \succ \mathcal{F}_0$ if and only if for any parametric function $\psi = c_0\sigma + \sum_{i=1}^{q} c_i\alpha_i^2$ and for any unbiased estimator $\hat{\psi} = Y'GY$ there exists an unbiased estimator $\tilde{\psi} = X'HX$ such that $\mathrm{var}(\tilde{\psi}) \le \mathrm{var}(\hat{\psi})$.*

PROOF. The necessity is evident.

Sufficiency. If $q = m$ then the sufficiency follows by formula (4.1). Otherwise, by Theorem 2.1 (b) we get

$$I_q - D \geq 0 \tag{4.3}$$

and, in consequence, $CD^{-1}C - C^2 \geq 0$ for any $C \in \mathcal{S}_q$. Moreover $\operatorname{tr}(D^{-1}CD^{-1})$ depends on the matrix C through its diagonal elements only. Thus, by formula (4.2) we only need to show that the condition (4.3) implies $\operatorname{tr}(CD^{-2}CD^{-2}) \geq \operatorname{tr}(C^2)$ for any symmetric matrix $C = (c_{ij})$ of order q. Really, $\operatorname{tr}(CD^{-2}CD^{-2}) = \operatorname{tr}[(D^{-1}CD^{-1})^2] = \sum_{i,j} \frac{c_{ij}^2}{d_i^2 d_j^2} \geq \sum_{i,j} c_{ij}^2 = \operatorname{tr}(C^2)$, completing the proof. □

Now for arbitrary normal experiments $\mathcal{E} = \mathcal{N}(A\beta, \sigma I_n)$ and $\mathcal{F} = \mathcal{N}(B\beta, \sigma I_m)$ such that $R(B') \subseteq R(A')$ let us consider the linear models $\mathcal{L}_1 = \mathcal{L}(T_1\tau, V_1(\gamma_1, \ldots, \gamma_q))$ and $\mathcal{L}_2 = \mathcal{L}(T_2\tau, V_2(\gamma_1, \ldots, \gamma_q))$, where T_i and V_i, $i = 1, 2$, are defined by

$$T_1 = \begin{cases} [I_q \ \vdots\ \mathbf{1}_q] & \text{if} \quad r = n \\ \begin{bmatrix} I_q & \vdots & \mathbf{1}_q \\ \mathbf{0}'_q & \vdots & 1 \end{bmatrix} & \text{otherwise,} \end{cases} \tag{4.4}$$

$$T_2 = \begin{cases} [\Lambda \ \vdots\ \mathbf{1}_q] & \text{if} \quad q = m \\ \begin{bmatrix} \Lambda & \vdots & \mathbf{1}_q \\ \mathbf{0}'_q & \vdots & 1 \end{bmatrix} & \text{otherwise,} \end{cases} \tag{4.5}$$

$$V_1 = \begin{cases} I_q + \Gamma & \text{if} \quad r = n \\ \begin{bmatrix} I_q + \Gamma & \vdots & \mathbf{0} \\ \mathbf{0}' & \vdots & \dfrac{1}{n-q} \end{bmatrix} & \text{otherwise,} \end{cases} \tag{4.6}$$

$$V_2 = \begin{cases} I_q + \Lambda\Gamma & \text{if} \quad q = m \\ \begin{bmatrix} I_q + \Lambda\Gamma & \vdots & \mathbf{0} \\ \mathbf{0}' & \vdots & \dfrac{1}{m-q} \end{bmatrix} & \text{otherwise,} \end{cases} \tag{4.7}$$

where $r = r(A)$, $q = r(B)$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_q)$ is the known matrix furnished by the nonzero eigenvalues of the matrix $(A'A)^+B'B$ taken with their multiplicities, while $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_q)$ is a diagonal matrix furnished by unknown nonnegative elements $\gamma_1, \ldots, \gamma_q$.

THEOREM 4.2. *For arbitrary $n \times p$ and $m \times p$ matrices A and B such that $B \neq 0$ and $R(B') \subseteq R(A')$ the following are equivalent:*

(a) *The experiment $\mathcal{E} = \mathcal{N}(A\beta, \sigma I_n)$ is quadratically sufficient for the experiment $\mathcal{F} = \mathcal{N}(B\beta, \sigma I_m)$.*

(b) *The linear model* $\mathcal{L}_1 = \mathcal{L}(T_1\tau, V_1(\gamma_1,\ldots,\gamma_q))$ *is linearly sufficient for the model* $\mathcal{L}_2 = \mathcal{L}(T_2\tau, V_2(\gamma_1,\ldots,\gamma_q))$, *where* T_i *and* V_i, $i = 1, 2$, *are defined by* (4.4)–(4.7).

PROOF. For the initial experiments $\mathcal{E}$ and $\mathcal{F}$ let us consider the canonical experiments $\mathcal{E}_0$ and $\mathcal{F}_0$ defined by (3.2) and (3.3) with observation vectors $X = (X_1,\ldots,X_{n-r+q})'$ and $Y = (Y_1,\ldots,Y_m)'$. Let

$$U_1(X) = \begin{cases} X^{(2)} & \text{if} \quad r = n \\ \begin{bmatrix} X^{(2)} \\ s(X) \end{bmatrix} & \text{otherwise} \end{cases}$$

and

$$U_2(Y) = \begin{cases} Y^{(2)} & \text{if} \quad q = m \\ \begin{bmatrix} Y^{(2)} \\ s(Y) \end{bmatrix} & \text{otherwise} \end{cases}$$

be statistics in $\mathcal{E}_0$ and $\mathcal{F}_0$ defined by $X^{(2)} = (X_1^2,\ldots,X_q^2)'$, $Y^{(2)} = (Y_1^2,\ldots,Y_q^2)'$, $s(X) = \frac{1}{n-r}\sum_{i=q+1}^{n-r+q} X_i^2$ and $s(Y) = \frac{1}{m-q}\sum_{i=q+1}^{m} Y_i^2$. Moreover, let $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ be the linear models induced by the random vectors $U_1 = U_1(X)$ and $U_2 = U_2(Y)$, respectively.

By Propositions 4.1 and 4.2 and Theorem 4.1 the relation $\mathcal{E} \succ \mathcal{F}$ is equivalent to $\mathcal{L}_1^* \rhd \mathcal{L}_2^*$. Moreover, by using formulae (2.5) and (2.6) and by setting $\tau = (\alpha_1^2,\ldots,\alpha_q^2,\sigma)'$ and $\gamma_i = 2\alpha_i^2/\sigma$, $i = 1,\ldots,q$, the models $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ may be presented in the form $\mathcal{L}_1^* = \mathcal{L}(T_1\tau, 2\sigma^2 V_1)$ and $\mathcal{L}_2^* = \mathcal{L}(T_2\tau, 2\sigma^2 V_2)$, where T_i and V_i, $i = 1, 2$, are defined by (4.4)–(4.7). On the other hand, by Stępniak and Torgersen (1981) the relations $\mathcal{L}_1^* \rhd \mathcal{L}_2^*$ and $\mathcal{L}_1 \rhd \mathcal{L}_2$ are equivalent, completing the proof. □

5. Main result

We are ready to prove the main result in this paper.

THEOREM 5.1. *An experiment* $\mathcal{E} = \mathcal{N}(A\beta, \sigma I_n)$ *is quadratically sufficient for an experiment* $\mathcal{F} = \mathcal{N}(B\beta, \sigma I_m)$ *such that* $B \neq 0$, *if and only if*,

$$\text{(i)} \quad A'A - B'B \quad \textit{is n.n.d.},$$

and

$$\text{(ii)} \quad \sum_{i=1}^{q} \frac{1-\lambda_i}{1+\lambda_i} \leq n - m - r(A) + r(B),$$

where λ_i, $i = 1,\ldots,q$, *are the positive eigenvalues of the matrix* $(A'A)^+B'B$ *taken with their multiplicities.*

Remark 5.1. Other necessary and sufficient conditions for the relation $\mathcal{E} \succ \mathcal{F}$ may be obtained by combining respective items in Lemmas A.1 and A.2 (see Appendix).

Proof of Theorem 5.1. In view of Theorem 4.2 and by Stępniak and Torgersen (1981) we only need to show that (ii) is a necessary and sufficient condition for the matrix inequality

$$T_1'V_1^{-1}(\gamma_1,\ldots,\gamma_q)T_1 - T_2'V_2^{-1}(\gamma_1,\ldots,\gamma_q)T_2 \geq \mathbf{0} \tag{5.1}$$

for all nonnegative $\gamma_1,\ldots,\gamma_q$ providing (i) holds. Defining $M_i = T_i'V_i^{-1}T_i$, $i=1,2$, we can write

$$M_1 = \begin{bmatrix} \frac{1}{1+\gamma_1} & 0 & \cdots & 0 & \frac{1}{1+\gamma_1} \\ 0 & \frac{1}{1+\gamma_2} & \cdots & 0 & \frac{1}{1+\gamma_2} \\ \hdotsfor{5} \\ 0 & 0 & \cdots & \frac{1}{1+\gamma_q} & \frac{1}{1+\gamma_q} \\ \frac{1}{1+\gamma_1} & \frac{1}{1+\gamma_2} & \cdots & \frac{1}{1+\gamma_q} & n-r+\sum_{i=1}^{q}\frac{1}{1+\gamma_i} \end{bmatrix}$$

and

$$M_2 = \begin{bmatrix} \frac{\lambda_1^2}{1+\lambda_1\gamma_1} & 0 & \cdots & 0 & \frac{\lambda_1}{1+\lambda_1\gamma_1} \\ 0 & \frac{\lambda_2^2}{1+\lambda_2\gamma_2} & \cdots & 0 & \frac{\lambda_2}{1+\lambda_2\gamma_2} \\ \hdotsfor{5} \\ 0 & 0 & \cdots & \frac{\lambda_q^2}{1+\lambda_q\gamma_q} & \frac{\lambda_q}{1+\lambda_q\gamma_q} \\ \frac{\lambda_1}{1+\lambda_1\gamma_1} & \frac{\lambda_2}{1+\lambda_2\gamma_2} & \cdots & \frac{\lambda_q}{1+\lambda_q\gamma_q} & m-q+\sum_{i=1}^{q}\frac{1}{1+\lambda_i\gamma_i} \end{bmatrix}.$$

By the identities

$$\frac{1}{1+\gamma_i} - \frac{\lambda_i}{1+\lambda_i\gamma_i} = \frac{1-\lambda_i}{(1+\gamma_i)(1+\lambda_i\gamma_i)},$$
$$\frac{1}{1+\gamma_i} - \frac{\lambda_i^2}{1+\lambda_i\gamma_i} = \frac{(1-\lambda_i)(1+\lambda_i+\lambda_i\gamma_i)}{(1+\gamma_i)(1+\lambda_i\gamma_i)},$$

and

$$\frac{1}{1+\gamma_i} - \frac{1}{1+\lambda_i\gamma_i} = -\frac{\gamma_i(1-\lambda_i)}{(1+\gamma_i)(1+\lambda_i\gamma_i)}$$

we get

$$M_1 - M_2 = \begin{bmatrix} \frac{(1-\lambda_1)(1+\lambda_1+\lambda_1\gamma_1)}{(1+\gamma_1)(1+\lambda_1\gamma_1)} & \cdots & 0 & \frac{1-\lambda_1}{(1+\gamma_1)(1+\lambda_1\gamma_1)} \\ \hdotsfor{4} \\ 0 & \cdots & \frac{(1-\lambda_q)(1+\lambda_q+\lambda_q\gamma_q)}{(1+\gamma_q)(1+\lambda_q\gamma_q)} & \frac{1-\lambda_q}{(1+\gamma_q)(1+\lambda_q\gamma_q)} \\ \frac{1-\lambda_1}{(1+\gamma_q)(1+\lambda_q\gamma_q)} & \cdots & \frac{1-\lambda_q}{(1+\gamma_q)(1+\lambda_q\gamma_q)} & \begin{matrix} n-m-r+q \\ -\sum_{i=1}^{q}\frac{\gamma_i(1-\lambda_i)}{(1+\gamma_i)(1+\lambda_i\gamma_i)} \end{matrix} \end{bmatrix}.$$

Now we shall use the fact that a symmetric block matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

is n.n.d., if and only if, $R(M_{12}) \subseteq R(M_{11})$ and the matrices M_{11} and $M_{22} - M_{21}M_{11}^{-}M_{12}$ are nonnegative definite (cf. Pukelsheim (1993), p. 75). To this aim let us write $M_1 - M_2$ in the form

$$M_1 - M_2 = \begin{bmatrix} S & Sv \\ v'S & g \end{bmatrix},$$

where

$$S = \operatorname{diag}\left[\frac{(1-\lambda_1)(1+\lambda_1+\lambda_1\gamma_1)}{(1+\gamma_1)(1+\lambda_1\gamma_1)}, \ldots, \frac{(1-\lambda_q)(1+\lambda_q+\lambda_q\gamma_q)}{(1+\gamma_q)(1+\lambda_q\gamma_q)}\right],$$

$$v' = \left[\frac{1}{1+\lambda_1+\lambda_1\gamma_1}, \ldots, \frac{1}{1+\lambda_q+\lambda_q\gamma_q}\right]$$

and

$$g = n - m - r + q - \sum_{i=1}^{q} \frac{\gamma_i(1-\lambda_i)}{(1+\gamma_i)(1+\lambda_i\gamma_i)}.$$

We observe that condition (5.1) is equivalent to $g - (v'S)S^{-}(Sv) = g - v'Sv \geq 0$ for all nonnegative $\gamma_1, \ldots, \gamma_q$. By simple algebra we get

$$v'Sv = \sum_{i=1}^{q} \frac{1}{(1+\gamma_i)(1+\lambda_i\gamma_i)(1+\lambda_i+\lambda_i\gamma_i)}.$$

Thus

$$g - v'Sv = n - m - r + q - \sum_{i=1}^{q} \frac{1-\lambda_i}{1+\lambda_i},$$

which implies the desired result. □

In a consequence of Theorem 5.1, via Hansen and Torgersen ((1974), Theorem 3.1) and Corollary A.1 (see Appendix) we get the following corollaries.

COROLLARY 5.1. *For arbitrary standard experiments* $\mathcal{E} = \mathcal{N}(A\beta, \sigma I_n)$ *and* $\mathcal{F} = \mathcal{N}(B\beta, \sigma I_m)$ *the condition*

(a) $\mathcal{E}$ *is sufficient for* $\mathcal{F}$ *(in the sense of Definition* 2.1)

implies

(b) $\mathcal{E}$ *is quadratically sufficient for* $\mathcal{F}$.

COROLLARY 5.2. (Experiments with the same degrees of freedom for error) *For arbitrary* $n \times p$ *matrix* A *and* $m \times p$ *matrix* B *such that* $n - r(A) = m - r(B)$ *the following are equivalent:*

(a) $\mathcal{N}(A\beta, \sigma I_n)$ *is sufficient for* $\mathcal{N}(B\beta, \sigma I_m)$,
(b) $\mathcal{N}(A\beta, \sigma I_n)$ *is quadratically sufficient for* $\mathcal{N}(B\beta, \sigma I_m)$,
(c) $A'A = B'B$.

COROLLARY 5.3. (Experiments with a common sample space) *If* $n = m$ *then the above conditions* (a)–(c) *are equivalent.*

Moreover, by Lemma 2.4 we get

COROLLARY 5.4. *For arbitrary matrix* A *the experiment* $\mathcal{N}(A\beta, \sigma I_n)$ *is sufficient for* $\mathcal{N}(0, \sigma I_m)$ *if and only if* $\mathcal{N}(A\beta, \sigma I_n)$ *is quadratically sufficient for* $\mathcal{N}(0, \sigma I_m)$.

Remark 5.2. In general, the relation $\mathcal{E} \succ \mathcal{F}$ does not imply that $\mathcal{E}$ is sufficient for $\mathcal{F}$. To see it let us suppose that the matrix $(A'A)^+B'B$ has only two distinct eigenvalues: 0 with multiplicity $r - q$, and λ with multiplicity q, where $0 < \lambda < 1$. Then the experiment $\mathcal{E} = \mathcal{N}(A\beta, \sigma I_n)$ is quadratically sufficient for $\mathcal{F} = \mathcal{N}(B\beta, \sigma I_m)$ if and only if $k = n - m - r + q > 0$ and $\frac{1-\lambda}{1+\lambda} \geq \frac{k}{q}$. In particular, if $k = 1$, $q = 3$ and $\lambda = .5$ then $\mathcal{E}$ is quadratically sufficient but not sufficient for $\mathcal{F}$.

6. Experiments with arbitrary covariance matrices

In this section we shall consider arbitrary normal linear experiments $\mathcal{E} = \mathcal{N}(A\beta, \sigma V)$ and $\mathcal{F} = \mathcal{N}(B\beta, \sigma W)$, where V and/or W may be singular.

THEOREM 6.1. *For arbitrary normal linear experiments* $\mathcal{E} = \mathcal{N}(A\beta, \sigma V)$ *and* $\mathcal{F} = \mathcal{N}(B\beta, \sigma W)$ *such that* $B \neq 0$ *the relation* $\mathcal{E} \succ \mathcal{F}$ *does hold if and only if*

$$\text{(i)} \quad A'(V + AA')^- A - B'(W + BB')^- B \geq 0 \quad \textit{is n.n.d.}$$

and

$$\text{(ii)} \quad \sum_{i=1}^{q} \frac{1 - \lambda_i}{1 + \lambda_i} \leq r(V + AA') - r(W + BB') - r(A) + r(B),$$

where λ_i, $i = 1, \ldots, q$, *are the positive eigenvalues of the matrix* $(A'A)^+B'B$, *taken with their multiplicities.*

PROOF. Let X and Y be observation vectors in the experiments $\mathcal{E}$ and $\mathcal{F}$, respectively.

At first assume that the both experiments have trivial deterministic parts, i.e. $R(A) \subseteq R(V)$ and $R(B) \subseteq R(W)$. Let $V = \sum_{i=1}^{r_1} \rho_i v_i v_i'$ and $W = \sum_{i=1}^{r_2} \kappa_i w_i w_i'$ be spectral decompositions of V and W, respectively, with positive eigenvalues ρ_i and κ_j, $i = 1, \ldots, r_1$, $j = 1, \ldots, r_2$. Define matrices

$$F_1 = \begin{bmatrix} \sqrt{\rho_1} v_1' \\ \vdots \\ \sqrt{\rho_{r_1}} v_{r_1}' \end{bmatrix} \tag{6.1}$$

and

$$F_2 = \begin{bmatrix} \sqrt{\kappa_1} w_1' \\ \vdots \\ \sqrt{\kappa_{r_2}} w_{r_2}' \end{bmatrix}. \tag{6.2}$$

By transformations F_1X and F_2Y we reduce the comparison of $\mathcal{E}$ and $\mathcal{F}$ to the same problem for the standard normal experiments $\mathcal{E}_1 = \mathcal{N}(A_1\beta, \sigma I_{r_1})$ and $\mathcal{F}_1 = \mathcal{N}(B_1\beta, \sigma I_{r_2})$, where $A_1 = F_1A$ and $B_1 = F_2B$. We note that $r(A_1) = r(A)$, $r(B_1) = r(B)$, $F_1'F_1 = A'V^-A$ and $F_2'F_2 = B'W^-B$, where $^-$ means a generalized inverse. Now, for the considered case the desired result follows by Theorem 5.1. Finally, to complete the proof we only need to use the smoothing property of the relation $\succ$ (cf. Lemma 2.3). □

Remark 6.1. If $R(A) \subseteq R(V)$ and $R(B) \subseteq R(W)$ then the matrices $V + AA'$ and $W + BB'$ appearing in the conditions (i) and (ii) may be replaced by V and W, respectively.

Remark 6.2. This theorem is an analogue of Torgersen ((1991), Theorem 8.6.6, p. 497).

By Theorem 6.1, the Corollaries 5.1, 5.2, 5.3 and 5.4 in Section 5 may be extended to the following.

COROLLARY 6.1. *For arbitrary normal linear experiments $\mathcal{E} = \mathcal{N}(A\beta, \sigma V)$ and $\mathcal{F} = \mathcal{N}(B\beta, \sigma W)$ let us consider the following relations:*
(a) *$\mathcal{E}$ is sufficient for $\mathcal{F}$,*
(b) *$\mathcal{E}$ is quadratically sufficient for $\mathcal{F}$.*
Then (a) *always implies* (b). *Moreover, the relations* (a) *and* (b) *are equivalent providing that at least one of the following conditions*
(c) $r(V + AA') - r(A) = r(W + BB') - r(B)$,
(d) $r(V) = r(W)$,
(e) $B = 0$
does hold.

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Appendix

Let A and B be arbitrary $n \times p$ and $m \times p$ matrices such that $B \neq 0$ and $R(B') \subseteq R(A')$ and let r and q be ranks of A and B. Moreover let L be an arbitrary $p \times r$ matrix such that $r(AL) = r$. Consider the characteristic polynomial

$$\mathcal{P}(\lambda) = |L'(B'B - \lambda A'A)L|. \tag{A.1}$$

LEMMA A.1. (a) *The roots of the polynomial do not depend on choice of the matrix* L.

(b) *All roots of the polynomial* (A.1) *are nonnegative.*

(c) *For arbitrary set* $S = \{\lambda_1, \ldots, \lambda_q\}$ *of positive numbers the following are equivalent*:

(i) S *is the set of all positive roots of the polynomial* $\mathcal{P}$ *taken with their multiplicities.*

(ii) S *is the set of all positive eigenvalues of the matrix* $(A'A)^+B'B$ *taken with their multiplicities.*

(iii) S *is the set of all positive eigenvalues of the matrix* $B(A'A)^-B'$ *taken with their multiplicities, where* $^-$ *means a generalized inverse.*

(iv) S *is the set of all positive eigenvalues of the matrix* $[(A'A)^+]^{1/2}B'B[(A'A)^+]^{1/2}$ *taken with their multiplicities.*

PROOF. Let $P_{A'}$ be the orthogonal projector onto $R(A')$. Since $L'A'AL = L'P_{A'}A'AP_{A'}L$ and $L'B'BL = L'P_{A'}B'BP_{A'}L$ one can assume, without loss of generality, that $R(L) \subseteq R(A')$ and, in consequence, the matrix L may be presented in the form $L = GF$, where F is a nonsingular $r \times r$ matrix while

$$G = [\rho_1^{-1/2}v_1, \ldots, \rho_r^{-1/2}v_r] \tag{A.2}$$

where ρ_i and v_i, $i = 1, \ldots, r$, are defined by a canonical decomposition $A'A = \sum_{i=1}^r \rho_i v_i v_i'$ of $A'A$. Now the polynomial $\mathcal{P}$ can be presented in the form $|L'(B'B - \lambda A'A)L| = |F|^2|G'(B'B - \lambda A'A)G|$ and hence (a) is proved. On the other hand, $|G'(B'B - \lambda A'A)G| = |G'B'BG - \lambda I_r|$. Thus the roots of $\mathcal{P}$ coincide with the eigenvalues of the matrix $G'B'BG$.

In order to show the equivalence (i)$\Leftrightarrow$(iii) we note that the nonzero eigenvalues of $G'B'BG$ coincide with ones of the matrix $BGG'B' = B(A'A)^+B'$. Moreover $B(A'A)^-B' = B(A'A)^+B'$ because the expression $B(A'A)^-B'$, under condition $R(B') \subseteq R(A')$, does not depend on choice of generalized inverse $(A'A)^-$.

The equivalence (iii)$\Leftrightarrow$(iv) is evident. To show that (ii)$\Leftrightarrow$(iv) we observe that the matrices $M = (A'A)^+B'B$ and $T^{-1}MT$ have the same eigenvalues for every nonsingular T. Now we only need to set $T = [(A'A)^+]^{1/2} + (I - P)$, where P is the orthogonal projector onto $R(A')$ (cf. also Stępniak (1985)).

In this way the lemma is proved. □

Now combining Lemma A.1 and Stępniak ((1985), Theorem 1) we get

LEMMA A.2. *Under assumption* $R(B') \subseteq R(A')$ *the following are equivalent*:

(a) $A'A - B'B$ *is n.n.d.*

(b) *All roots of the polynomial* $\mathcal{P}$ *are not greater than* 1.

(c) *All eigenvalues of the matrix* $(A'A)^+B'B$ *are not greater than* 1.

(d) *All eigenvalues of the matrix* $B(A'A)^-B'$ *are not greater than* 1.

(e) *All eigenvalues of the matrix* $[(A'A)^+]^{1/2}B'B[(A'A)^+]^{1/2}$ *are not greater than* 1.

The most important fact is contained in the following lemma.

LEMMA A.3. *Let A and B be arbitrary $n \times p$ and $m \times p$ matrices such that $B \neq 0$ and $R(B') \subseteq R(A')$ and r and q be ranks of A and B. Then there exists a $p \times r$ matrix M such that*

$$R(M) = R(A'), \qquad M'A'AM = I_r \qquad \text{and}$$

$$M'B'BM = \begin{cases} \Lambda & \text{if } q = r \\ \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} & \text{otherwise,} \end{cases} \tag{A.3}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_q)$, while $\lambda_1 \geq \cdots \geq \lambda_q$ are the positive eigenvalues of the matrix $(A'A)^+B'B$, taken with their multiciplities.

PROOF. We only need to set $M = GU$, where G is defined by (A.2) and $U = [u_1, \ldots, u_r]$ is formed by orthonormal eigenvectors of the matrix $G'B'BG$ corresponding to the eigenvalues λ_i, $i = 1, \ldots, r$. □

In a consequence of Lemmas A.1 and A.3 we get

COROLLARY A.1. *Let $\lambda_1 \geq \cdots \geq \lambda_p$ be the eigenvalues of the matrix $(A'A)^+B'B$ taken with their multiplicities. Then $r(A'A - B'B) \geq \sum_{i=1}^{q} \frac{1-\lambda_i}{1+\lambda_i} + r - q$, with the strict inequality unless $\lambda_i = 1$ for all $i = 1, \ldots, q$.*

PROOF OF THE COROLLARY. By (A.3), via Lemmas A.1 and A.3,

$$r(A'A - B'B) = r[M'(A'A - B'B)M] \geq r - q + \sum_{i=1}^{q}(1 - \lambda_i) \geq r - q + \sum_{i=1}^{q} \frac{1 - \lambda_i}{1 + \lambda_i}$$

with the strict inequality if at least one λ_i, $i = 1, \ldots, q$, is different form 1. □

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