Ann. Inst. Statist. Math.
Vol. 49, No. 3, 541–554 (1997)

DISTRIBUTIONS MINIMIZING FISHER INFORMATION FOR LOCATION IN KOLMOGOROV NEIGHBOURHOODS

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(Received July 11, 1995; revised February 24, 1997)

Abstract. This paper is concerned with the problem of finding those distributions minimizing Fisher information for location over Kolmogorov neighbourhoods of distribution functions G satisfying certain mild conditions. The case when G is symmetric has been considered by quite a few authors. The general form of the solution is discussed. Furthermore we provide the solution of two asymmetric distributions, namely, the extreme value distribution and the Pearson's type IV distribution.

Key words and phrases: Robust estimation of location, minimum Fisher information for location, extreme-value distribution, Pearson's type IV distribution.

1. Introduction and summary

Consider Huber's theory (1964, 1981) of robust M-estimation of a location parameter. An important problem which is of interest is to find those distributions minimizing Fisher information of location over a certain convex set $\mathcal{P}$ of distributions. In the literature, two common types of $\mathcal{P}$ are the Gross errors model

$$\mathcal{G}_\epsilon(G) = \{F \mid F = (1-\epsilon)G + \epsilon H; \epsilon \text{ fixed}, H \text{ arbitrary}\}$$

and the Kolmogorov neighbourhood model

$$\mathcal{K}_\epsilon(G) = \left\{F \mid \sup_{-\infty<x<\infty} |F(x) - G(x)| \le \epsilon; \epsilon \text{ and } G \text{ are fixed}\right\}.$$

Huber (1964) obtained the least informative distribution in both the Gross errors neighbourhood model $\mathcal{G}_\epsilon(G)$ where G has a strongly unimodal density g and the Kolmogorov normal neighbourhood model $\mathcal{K}_\epsilon(\Phi)$, with Φ denotes the standard normal distribution function and $\epsilon \le .0303$. Collins and Wiens (1985) worked out the solution for more general Gross errors neighbourhood models for which the known density g is not necessarily strongly unimodal. Sacks and Ylvisaker (1972) obtained further result for the Kolmogorov normal neighbourhood model for which the solution was found in the extended range of $.0303 \leq \epsilon \leq .5$. Wiens (1986) extended the result to more general Kolmogorov neighbourhood model $\mathcal{K}_\epsilon(G)$ where G is not necessarily normal. Collins and Wiens (1989) further extended these results to Lévy neighbourhood model in which G satisfies conditions similar to those imposed in Wiens (1986). In both of the Kolmogorov and Lévy neighbourhood model cases, all of the work done require an assumption that G is symmetric. In this paper the solution for $\mathcal{K}_\epsilon(G)$ where G is asymmetric is discussed. Symmetric G is viewed as special cases. In particular, we obtain the solution when G is the extreme value distribution or the Pearson's type IV distribution in which both are asymmetric. The extreme value distribution is a commonly used model in survival analysis and reliability (see e.g. Bain (1978), Johnson *et al.* (1994) and Lawless (1982)). The Pearson's type IV distribution is chosen since it is sufficiently general to illustrate the result obtained in this paper.

2. The theory

DEFINITION 1. An M-estimator of location is defined as $T(F_n)$, where F_n is the empirical distribution function based on a sample $X_1, X_2, \ldots, X_n \sim F$, and the functional $T(F)$ is defined implicitly by

$$\int \psi(x - T(F))dF(x) = 0 \tag{2.1}$$

where ψ is an arbitrary function chosen by statistician.

Huber (1964) shows that under certain mild regularity conditions,

$$\sqrt{n}(T(F_n) - T(F)) \xrightarrow{w} N(0, V(\psi, F)),$$

where

$$V(\psi, F) = \frac{\int \psi^2(x - T(F))dF(x)}{[\int \psi'(x - T(F))dF(x)]^2}.$$

DEFINITION 2. The Fisher information for location of a distribution F on the real line is

$$I(F) = \sup_{\psi} \frac{(\int \psi' dF)^2}{\int \psi^2 dF}$$

where the supremum is taken over the set of all continuously differentiable functions with compact support, satisfying $\int \psi^2 dF > 0$.

Huber (1964, 1981) showed that

1. $I(F) < \infty$ is equivalent to that F has a continuous density f and $\int (f'/f)^2 f dx < \infty$. In either case, $I(F) = \int (f'/f)^2 f dx < \infty$.
2. There is an $F_0 \in \mathcal{P}$ minimizing $I(F)$.
3. If the set where $f_0 - F_0'$ is strictly positive is convex, then F_0 is unique.

Now suppose F_0 minimizes the Fisher information of location over $\mathcal{P}$. Let $\psi_0 = -\frac{f_0'}{f_0}$ (the score function), and define $T_0(F)$ by (2.1) with $\psi = \psi_0$. Then the following saddlepoint property

$$V(\psi_0, F) \le V(\psi_0, F_0) = \frac{1}{I(F_0)} \le V(\psi, F_0) \tag{2.2}$$

holds for all $F \in \mathcal{P}_0 = \{F \in \mathcal{P} \mid T_0(F) = T_0(F_0)\}$ and all ψ such that (2.1) holds. The first inequality in (2.2) is established by variational arguments, as in Huber (1964) and the second inequality in (2.2) is essentially the Cramér-Rao inequality. This in turn implies that

$$\sup_{\mathcal{P}_0} V(\psi_0, F) \le \sup_{\mathcal{P}_0} V(\psi, F) \tag{2.3}$$

for all ψ such that (2.1) holds for $F \in \mathcal{P}_0$. Thus ψ_0 is optimal in a minimax sense as it minimizes the maximum asymptotic variance over $\mathcal{P}_0$.

If we define $F_t = (1-t)F_0 + tF_1$, $t \in (0,1)$, then $F_0, F_1 \in \mathcal{P}$ implies $F_t \in \mathcal{P}$. As $I(F_t)$ is a convex function in t, we conclude that $F_0 \in \mathcal{P}$ attains the minimum Fisher information for location if and only if

$$\frac{d}{dt} I(F_t) \mid_{t=0} \ge 0$$

for all $F_1 \in \mathcal{P}$ satisfying $I(F_1) < \infty$.

After some calculation, we find that $F_0 \in \mathcal{P}$ attains the minimum Fisher information for location if and only if

$$\frac{d}{dt} I(F_t) \mid_{t=0} = \int J(\psi_0)(x) d(F_1 - F_0)(x) \ge 0 \tag{2.4}$$

for all $F_1 \in \mathcal{P}$ with $I(F_1) < \infty$ where

$$\psi_0(x) = -\frac{f_0'}{f_0}(x)$$

and

$$J(\psi)(x) = 2\psi'(x) - \psi^2(x)$$

provided that ψ_0 is absolutely continuous and bounded. Extend J by left continuity when ψ' is discontinuous.

Now suppose that $F \in \mathcal{K}_\epsilon(G)$. To obtain $F_0 \in \mathcal{K}_\epsilon(G)$, we observe that the support B of f_0 can be partitioned into three parts, say B_0, B_L, B_U where

$$\begin{aligned} B_0 &= \{x \mid \max(0, G(x) - \epsilon) < F_0(x) < \min(1, G(x) + \epsilon)\} \\ B_U &= \{x \mid F_0(x) = G(x) - \epsilon\} \\ B_L &= \{x \mid F_0(x) = G(x) + \epsilon\} \end{aligned}$$

and

$$B = B_0 \cup B_U \cup B_L.$$

Since B_0 is an open set, there exist finite or countably many disjoint open intervals $B_i = (a_i, b_i)$ such that $B_0 = \bigcup_i B_i$. It turns out that F_0 satisfies

$$J(\psi_0)(x) = \text{constant} \tag{2.5}$$

at each B_i.

Remark 1. The solution of $J(\psi_0)(x) = \lambda^2$, $\lambda > 0$ is of the form $\psi_0(x) = \lambda \tan[\frac{\lambda}{2}(|x| - \omega)]$ and the corresponding f_0 is proportional to $\cos^2[\frac{\lambda}{2}(|x| - \omega)]$.

Remark 2. The solution of $J(\psi_0)(x) = -\lambda^2$, $\lambda > 0$ is of one of the forms $\psi_0(x) = \lambda, -\lambda, \lambda \tanh[-\frac{\lambda}{2}(x-\omega)]$ and $\lambda \coth[-\frac{\lambda}{2}(x-\omega)]$ and the corresponding f_0 is proportional to $e^{-\lambda x}$, $e^{\lambda x}$, $\cosh^2[-\frac{\lambda}{2}(x-\omega)]$ and $\sinh^2[-\frac{\lambda}{2}(x-\omega)]$ respectively.

THEOREM 2.1. *If F_0 possesses the following properties, then it is the unique member of $\mathcal{K}_\epsilon(G)$ minimizing $I(F)$ over $\mathcal{K}_\epsilon(G)$:*

(S1) $F_0 \in \mathcal{K}_\epsilon(G)$, $F_0(\infty) = 1$.

(S2) *F_0 has an absolutely continuous density f_0 and ψ_0 is absolutely continuous on $(-\infty, \infty)$.*

(S3) *There exists a sequence $-\infty \le b_1 \le a_2 < b_2 \le \cdots \le a_{n-1} < b_{n-1} \le a_n \le \infty$ and constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that*

1. $$B_U \cup B_L = \bigcup_{i=1}^{n-1} [b_i, a_{i+1}].$$

2. $$J(\psi_0)(x) = \begin{cases} \lambda_1 < 0, & -\infty \le x \le b_1 \\ \lambda_i, & a_i < x \le b_i, \quad i = 2, \ldots, n-1 \\ \lambda_n < 0, & a_n \le x \le \infty \\ J(\xi)(x), & b_i < x \le a_{i+1}, \quad i = 1, \ldots, n-1 \end{cases}$$

where

$$J(\xi)(x) = 2\xi'(x) - \xi^2(x); \qquad \xi(x) = -\frac{g'}{g}(x).$$

3. (a) *If $a_i \in B_L$, then $J(\psi_0)(a_i^+) \le J(\psi_0)(a_i^-)$.*
   (b) *If $a_i \in B_U$, then $J(\psi_0)(a_i^+) \ge J(\psi_0)(a_i^-)$.*
   (c) *If $b_i \in B_L$, then $J(\psi_0)(b_i^-) \ge J(\psi_0)(b_i^+)$.*
   (d) *If $b_i \in B_U$, then $J(\psi_0)(b_i^-) \le J(\psi_0)(b_i^+)$.*
4. *If (b_i, a_{i+1}) is nonempty and contained in $B_L[B_U]$, then $J(\xi)(x)$ is weakly decreasing [increasing] there.*

The above theorem is a straightforward extension of Wiens ((1986), Theorem 1) and hence the proof is omitted.

The conditions of Theorem 2.1 in previous section are not enough to determine F_0 completely because of lack of information about the behaviour of $J(\xi)(x)$. Wiens (1986) explicitly provides some classes of solutions in which G is assumed to be symmetric. Moreover, as in Wiens (1986), a general principle at works appears to be that for sufficiently small ϵ, ψ_0 and ξ should differ only near the local extrema of $J(\xi)$; and there should have $J(\psi_0) =$ constant, with this constant being less extreme than that attained by $J(\xi)$. According to (2.4), we should have $f_0 > g$ near the local minima of $J(\xi)$ and $f_0 < g$ near the local maxima. As ϵ increases, the regions of constancy of $J(\psi_0)$ coalesce. Now in order to obtain (ψ_0, F_0) explicitly, a general procedure is proposed here. For sufficently small $\epsilon > 0$, we first identify what B_0 looks like and hence the form of $B_0 = \bigcup_i B_i$. And for each B_i, the right form of ψ_0 is then chosen. It is worthy to note that the form of B_i in which $J(\psi_0)(x) = \lambda^2$, $\lambda > 0$ is (a, b) and the form of B_i in which $J(\psi_0)(x) = -\lambda^2$, $\lambda > 0$ is one of the forms $(-\infty, a)$, (a, ∞) and (a, b). Moreover explict form of (ψ_0, F_0) in each B_i can be obtained by one of the following three Lemmas.

LEMMA 2.1. *If $J(\psi_0)(x) = \lambda^2$, $\lambda > 0$ on (a, b), then*

$$\psi_0(x) = \lambda \tan\left[\frac{\lambda}{2}(x-\omega)\right], \qquad x \in (a, b)$$

and

$$f_0(x) = \frac{g(b)}{\cos^2\left[\frac{\lambda}{2}(b-\omega)\right]}\cos^2\left[\frac{\lambda}{2}(x-\omega)\right], \qquad x \in (a, b)$$

$$\text{or} \quad \left(\frac{g(a)}{\cos^2\left[\frac{\lambda}{2}(a-\omega)\right]}\cos^2\left[\frac{\lambda}{2}(x-\omega)\right]\right)$$

where a, b, ω are constants determined by $\psi_0(a) = \xi(a)$, $\psi_0(b) = \xi(b)$ and $\int_b^c f_0 = G(b) - G(a) - 2\epsilon$.

PROOF. The result follows immediately by Remark 1. □

LEMMA 2.2. *If $J(\psi_0)(x) = -\lambda^2$, $\lambda > 0$ on $(-\infty, a)$ for some a, then*

$$\psi_0(x) = \xi(a) > 0, \qquad x \in (-\infty, a)$$

and

$$f_0(x) = g(a)\exp[-\xi(a)(x-a)], \qquad x \in (-\infty, a)$$

where the constant a is determined by $\int_{-\infty}^a f_0 - G(a) + \epsilon$.

Similarly if $J(\psi_0)(x) = -\lambda^2$, $\lambda > 0$ on (a, ∞) for some a, then

$$\psi_0(x) = \xi(a) > 0, \qquad x \in (a, \infty)$$

and

$$f_0(x) = g(a)\exp[-\xi(a)(x-a)], \qquad x \in (a,\infty)$$

where the constant a is determined by $\int_a^\infty f_0 = 1 - G(a) + \epsilon$.

PROOF. By noting the fact that the f_0 corresponding to the choices of $\psi_0(x) = -\lambda$, $\lambda\tanh[-\frac{\lambda}{2}(x-\omega_2)]$ or $\lambda\coth[-\frac{\lambda}{2}(x-\omega_3)]$ are not integrable on half-infinite intervals, the result follows. □

LEMMA 2.3. *Suppose on* (a,b), $\psi_0(a) = \xi(a)$, $\psi_0(b) = \xi(b)$ *and* $J(\psi_0)(x) = -\lambda^2$, $\lambda > 0$. *If* $\xi(x)$ *is increasing on* (a,b), *then*

$$\psi_0(x) = \lambda\coth\left[-\frac{\lambda}{2}(x-\omega)\right], \qquad x \in (a,b)$$

and

$$f_0(x) = \frac{g(a)}{\sinh^2\left[-\frac{\lambda}{2}(a-\omega)\right]}\sinh^2\left[-\frac{\lambda}{2}(x-\omega)\right], \qquad x \in (a,b)$$

$$\left(\text{or} \quad \frac{g(b)}{\sinh^2\left[-\frac{\lambda}{2}(b-\omega)\right]}\sinh^2\left[-\frac{\lambda}{2}(x-\omega)\right] \right)$$

where a, b and ω *are determined by* (i) $\psi_0(a) = \xi(a)$, (ii) $\psi_0(b) = \xi(b)$ *and*

(iii) $$\int_a^b f_0 = G(b) - G(a) - 2\epsilon$$

if (a,b) *is not the leftmost or rightmost interval of* B_i*'s, otherwise by*

(iii′) $$\int_a^b f_0 = G(b) - G(a) - \epsilon.$$

Similarly if $\xi(x)$ *is decreasing on* (a,b), *then*

$$\psi_0(x) = \lambda\tanh\left[-\frac{\lambda}{2}(x-\omega)\right], \qquad x \in (a,b)$$

and

$$f_0(x) = \frac{g(a)}{\cosh^2\left[-\frac{\lambda}{2}(a-\omega)\right]}\cosh^2\left[-\frac{\lambda}{2}(x-\omega)\right], \qquad x \in (a,b)$$

$$\left(\text{or} \quad \frac{g(b)}{\cosh^2\left[-\frac{\lambda}{2}(b-\omega)\right]}\cosh^2\left[-\frac{\lambda}{2}(x-\omega)\right] \right)$$

where a, b *and* ω *are determined by the conditions* (i), (ii) *and* (iii) *or* (iii′) *as above.*

PROOF. By noting the fact that tanh (coth) is an increasing (decreasing) function, the result follows.

3. Examples

Example 1. Consider the extreme value distribution whose density is

$$g(x) = \exp(x - e^x), \qquad -\infty < x < \infty. \tag{3.1}$$

The extreme value distribution is extensively used in a number of areas and has been discussed in Johnson *et al.* (1994) in detail. It is also closely related to the Weibull distribution with density

$$f(t) = \lambda\beta(\lambda t)^{\beta-1}\exp[-(\lambda t)^\beta], \qquad t > 0. \tag{3.2}$$

It can be shown easily that if T is Weibull distributed with density (3.2) then $X = \log T$ has an extreme value distirbuiton

$$h(y;\mu,\sigma) = \frac{1}{\sigma}\exp\left[\frac{y-\mu}{\sigma} - \exp\left(\frac{y-\mu}{\sigma}\right)\right], \qquad -\infty < y < \infty \tag{3.3}$$

where $\mu = -\log\lambda$ and $\sigma = 1/\beta$. The main convenience in working with the extreme value distribution stems from the fact that μ and σ are location and scale parameters and it is much more easily handled. Estimation of the parameters of the extreme value distribution has been studied by many authors and an extensively reference has been given by Johnson *et al.* (1994).

Now it is easy to see that

$$\begin{aligned}
&\xi(x) = -\frac{g'}{g}(x) = e^x - 1\\
&J(\xi)(x) = 2\xi'(x) - \xi^2(x) = -e^{2x} + 4e^x - 1.\\
&J(\xi)(-\infty) = -1, \qquad J(\xi)(\infty) = -\infty
\end{aligned}$$

and

$$\max_x J(\xi)(x) = J(\xi)(\ln 2) = 3$$

There are three forms of (ψ_0, f_0) which are given as follows:

Case 1 (small ϵ). There exists ϵ_0 such that for $0 < \epsilon \le \epsilon_0$, the Fisher information for location is minimized by that F_0 with

$$\psi_0(x) = \begin{cases} \xi(a), & x \in (-\infty, a);\\ \xi(x), & x \in [a, b);\\ \delta\tan\left[\frac{\delta}{2}(x-\omega)\right], & x \in [b, c);\\ \xi(x), & x \in [c, d);\\ \xi(d), & x \in [d, \infty),\end{cases}$$

$$f_0(x) = \begin{cases} g(a)\exp[-\xi(a)(x-a)], & x \in (-\infty, a); \\ g(x), & x \in [a, b); \\ \dfrac{g(b)}{\cos^2\left[\dfrac{\delta}{2}(b-\omega)\right]} \cos^2\left[\dfrac{\delta}{2}(x-\omega)\right], & x \in [b, c); \\ g(x), & x \in [c, d); \\ g(d)\exp[-\xi(d)(x-d)], & x \in [d, \infty), \end{cases}$$

where a, b, c, d, δ, ω are constants determined by the side conditions:

(C1). $\xi(b) = \delta \tan[\frac{\delta}{2}(b-\omega)]$,

(C2). $\xi(c) = \delta \tan[\frac{\delta}{2}(c-\omega)]$,

(C3). $\frac{g(b)}{\cos^2[\frac{\delta}{2}(b-\omega)]} = \frac{g(c)}{\cos^2[\frac{\delta}{2}(c-\omega)]}$,

(C4). $\int_{-\infty}^{a} f_0 = G(a) + \epsilon$,

(C5). $\int_b^c f_0 = G(c) \quad G(b) \quad 2\epsilon$ and

(C6). $\int_d^\infty f_0 = 1 - G(d) + \epsilon$.

Case 2 (medium ϵ). There exists ϵ_1 such that for $\epsilon_0 < \epsilon < \epsilon_1$, the Fisher information for location is minimized by that F_0 with

$$\psi_0(x) = \begin{cases} \xi(a), & x \in (-\infty, a); \\ \xi(x), & x \in [a, b); \\ \delta \tan\left[\dfrac{\delta}{2}(x-\omega)\right], & x \in [b, d); \\ \delta \tan\left[\dfrac{\delta}{2}(d-\omega)\right], & x \in [d, \infty), \end{cases}$$

$$f_0(x) = \begin{cases} g(a)\exp[-\xi(a)(x-a)], & x \in (-\infty, a); \\ g(x), & x \in [a, b); \\ \dfrac{g(b)}{\cos^2\left[\dfrac{\delta}{2}(b-\omega)\right]} \cos^2\left[\dfrac{\delta}{2}(x-\omega)\right], & x \in [b, d); \\ g(d)\exp\left[-\delta \tan\left(\dfrac{\delta}{2}(d-\omega)\right)(x-d)\right], & x \in [d, \infty). \end{cases}$$

The constants a, b, d, δ, ω are determined by the side conditions (C1), (C3), (C4), (C5) and (C6) with $d = c$ in (C6).

Case 3 (large ϵ). For a range $\epsilon_1 < \epsilon < 0.5$, the minimum information $F_0 \in$

$\mathcal{K}_\epsilon(G)$ is given by:

$$\psi_0(x) = \begin{cases} \delta \tan\left[\frac{\delta}{2}(a-\omega)\right], & x \in (-\infty, a); \\ \delta \tan\left[\frac{\delta}{2}(x-\omega)\right], & x \in [a, d); \\ \delta \tan\left[\frac{\delta}{2}(d-\omega)\right], & x \in [d, \infty), \end{cases}$$

$$f_0(x) = \begin{cases} g(a)\exp\left[-\delta\tan\left(\frac{\delta}{2}(a-\omega)\right)(x-d)\right], & x \in (-\infty, a); \\ \dfrac{g(a)}{\cos^2\left[\frac{\delta}{2}(a-\omega)\right]}\cos^2\left[\frac{\delta}{2}(x-\omega)\right], & x \in [a, d); \\ g(d)\exp\left[-\delta\tan\left(\frac{\delta}{2}(d-\omega)\right)(x-d)\right], & x \in [d, \infty). \end{cases}$$

The constants b, c, δ, ω are determined by the side conditions (C3), (C4) and (C6) with $a = b$ and $d = c$ in (C4) and (C6) respectively.

We found that $\epsilon_0 = 0.00415$ and $\epsilon_1 = 0.05466$. See Table 1 for other numerical values.

Example 2. In the previous example we can see that the minimum Fisher information distribution does not cover all possible forms of the general solution. Hence for the purpose of further illustration, let us consider the Pearson's type IV distribution whose density is

$$g(x) = \frac{\sqrt{3}\,\mathrm{csch}(\sqrt{3}\pi)}{(1+x^2)} e^{-2\sqrt{3}\tan^{-1}(x)}, \qquad -\infty < x < \infty. \tag{3.4}$$

This distribution has been discussed briefly in Johnson *et al.* (1994) and its minimum Fisher information distribution for location over the Kolmogorov neighbourhoods covers all possible forms of the result we studied in this paper.

Here

$$\xi(x) = -\frac{g'}{g}(x) = \frac{2(x+\sqrt{3})}{1+x^2}$$
$$\xi'(x) = \frac{2}{(1+x^2)^2}(1 - 2\sqrt{3}x - x^2)$$

and

$$J(\xi)(x) = 2\xi'(x) - \xi^2(x) = -\frac{8}{(1+x^2)^2}(1 + 2\sqrt{3}x + x^2)$$
$$J(\xi)(\infty) = J(\xi)(-\infty) = 0$$
$$\max_x J(\xi)(x) = J(\xi)(-.7463) = 3.395$$

Table 1. Numerical values for (ψ_0, F_0) for the Kolmogorov nieghbourhood with G = Extreme value distribution.

ϵ	a	b	c	d	δ	ω
0.001	−3.123	−1.798	1.438	1.690	1.147	−0.702
0.002	−2.782	−1.815	1.445	1.582	1.143	−0.710
0.003	−2.585	−1.831	1.450	1.514	1.140	−0.717
0.004	−2.445	−1.848	1.457	1.464	1.136	−0.724
4.15E-03	−2.427	−1.850	1.457	1.457	1.136	−0.727
0.005	−2.337	−0.202		1.397	1.542	−0.0490
0.006	−2.250	−0.240		1.379	1.531	−0.0590
0.008	−2.112	−0.310		1.349	1.509	−0.0783
0.010	−2.006	−0.376		1.325	1.480	−0.0962
0.020	−1.682	−0.639		1.254	1.389	−0.167
0.030	−1.497	−0.843		1.216	1.312	−0.218
0.040	−1.368	−1.014		1.194	1.247	−0.256
0.050	−1.270	−1.166		1.179	1.191	−0.286
5.466E-02	−1.232	−1.232		1.175	1.167	−0.297
0.060	−1.267			1.170	1.142	−0.308
0.070	−1.334			1.166	1.096	−0.331
0.080	−1.396			1.165	1.052	−0.350
0.090	−1.450			1.165	1.014	−0.367
0.100	−1.508			1.169	0.974	−0.384
0.150	−1.765			1.205	0.806	−0.456
0.300	−2.598			1.419	0.429	−0.695

and the local minimum of $J(\xi)(x)$ occurs at $x = -4.922$ or 0.4716 in which $J(\xi)(-4.922) = -0.1028$ and $J(\xi)(0.4716) = -15.29$.

The forms of (ψ_0, f_0) which are given as follows:

Case 1 (small ϵ). There exists ϵ_0 such that for $0 < \epsilon \le \epsilon_0$, the Fisher information for location is minimized by that F_0 with

$$\psi_0(x) = \begin{cases} \xi(x), & x \in (-\infty, a); \\ \lambda_1 \tanh\left[-\frac{\lambda_1}{2}(x-\omega_1)\right], & x \in [a,b); \\ \xi(x), & x \in [b,c); \\ \lambda_2 \tan\left[\frac{\lambda_2}{2}(x-\omega_2)\right], & x \in [c,d); \\ \xi(x), & x \in [d,e); \\ \lambda_3 \tanh\left[-\frac{\lambda_3}{2}(x-\omega_3)\right], & x \in [e,f); \\ \xi(x), & x \in [f,\infty), \end{cases}$$

$$f_0(x) = \begin{cases} g(x), & x \in (-\infty, a); \\ \dfrac{g(a)}{\cosh^2\left[-\dfrac{\lambda_1}{2}(a-\omega_1)\right]} \cosh^2\left[-\dfrac{\lambda_1}{2}(x-\omega_1)\right], & x \in [a,b); \\ g(x), & x \in [b,c); \\ \dfrac{g(c)}{\cos^2\left[\dfrac{\lambda_2}{2}(c-\omega)\right]} \cos^2\left[\dfrac{\lambda_2}{2}(x-\omega_2)\right], & x \in [c,d); \\ g(x), & x \in [d,e); \\ \dfrac{g(f)}{\cosh^2\left[-\dfrac{\lambda_1}{2}(f-\omega_3)\right]} \cosh^2\left[-\dfrac{\lambda_3}{2}(x-\omega_3)\right], & x \in [e,f); \\ g(x), & x \in [d,\infty), \end{cases}$$

where a, b, c, d, e, f, λ_1, λ_2, λ_3, ω_1, ω_2 and ω_3 are constants determined by the side conditions that f_0, ψ_0 are continuous and the fact that

(D1). $\int_a^b f_0 = G(b) - G(a) + \epsilon$,

(D2). $\int_b^d f_0 = G(d) - G(b) - 2\epsilon$, and

(D3). $\int_d^f f_0 = G(f) - G(d) + \epsilon$.

Case 2 (medium ϵ). There exists ϵ_1 such that for $\epsilon_0 < \epsilon < \epsilon_1$, the Fisher information for location is minimized by that F_0 with

$$\psi_0(x) = \begin{cases} \xi(x), & x \in (-\infty, a); \\ \lambda_1 \tanh\left[-\dfrac{\lambda_1}{2}(x-\omega_1)\right], & x \in [a,b); \\ \xi(x), & x \in [b,c); \\ \lambda_2 \tan\left[\dfrac{\lambda_2}{2}(x-\omega_2)\right], & x \in [c,d); \\ \lambda_3 \tanh\left[-\dfrac{\lambda_3}{2}(x-\omega_3)\right], & x \in [d,f); \\ \xi(x), & x \in [f,\infty), \end{cases}$$

$$f_0(x) = \begin{cases} g(x), & x \in (-\infty, a); \\ \dfrac{g(a)}{\cosh^2\left[-\frac{\lambda_1}{2}(a-\omega_1)\right]} \cosh^2\left[-\frac{\lambda_1}{2}(x-\omega_1)\right], & x \in [a, b); \\ g(x), & x \in [b, c); \\ \dfrac{g(c)}{\cos^2\left[\frac{\lambda_2}{2}(c-\omega)\right]} \cos^2\left[\frac{\lambda_2}{2}(x-\omega_2)\right], & x \in [c, d); \\ \dfrac{g(f)}{\cosh^2\left[-\frac{\lambda_1}{2}(f-\omega_3)\right]} \cosh^2\left[-\frac{\lambda_3}{2}(x-\omega_3)\right], & x \in [d, f); \\ g(x), & x \in [f, \infty), \end{cases}$$

where a, b, c, d, f, λ_1, λ_2, λ_3, ω_1, ω_2 and ω_3 are constants determined by the side conditions that f_0, ψ_0 are continuous and (D1)–(D3) with $d = e$.

Case 3 (large ϵ). For a range $\epsilon_1 < \epsilon < 0.5$, the minimum information $F_0 \in \mathcal{K}_\epsilon(G)$ is given by:

$$\psi_0(x) = \begin{cases} \xi(x), & x \in (-\infty, a); \\ \lambda_1 \tanh\left[-\frac{\lambda_1}{2}(x-\omega_1)\right], & x \in [a, c); \\ \lambda_2 \tan\left[\frac{\lambda_2}{2}(x-\omega_2)\right], & x \in [c, d); \\ \lambda_3 \tanh\left[-\frac{\lambda_3}{2}(x-\omega_3)\right], & x \in [d, f); \\ \xi(x), & x \in [f, \infty), \end{cases}$$

$$f_0(x) = \begin{cases} g(x), & x \in (-\infty, a); \\ \dfrac{g(a)}{\cosh^2\left[-\frac{\lambda_1}{2}(a-\omega_1)\right]} \cosh^2\left[-\frac{\lambda_1}{2}(x-\omega_1)\right], & x \in [a, c); \\ \dfrac{g(c)}{\cos^2\left[\frac{\lambda_2}{2}(c-\omega)\right]} \cos^2\left[\frac{\lambda_2}{2}(x-\omega_2)\right], & x \in [c, d); \\ \dfrac{g(f)}{\cosh^2\left[-\frac{\lambda_1}{2}(f-\omega_3)\right]} \cosh^2\left[-\frac{\lambda_3}{2}(x-\omega_3)\right], & x \in [d, f); \\ g(x), & x \in [f, \infty), \end{cases}$$

where a, c, d, f, λ_1, λ_2, λ_3, ω_1, ω_2 and ω_3 are constants determined by the side conditions that f_0, ψ_0 are continuous and (D1)–(D3).

Table 2. Numerical values for (ψ_0, F_0) for the Kolmogorov neighbourhood with G = Pearson type IV distribution.

ϵ	a	b	λ_1	ω_1	c	d	λ_2	ω_2	e	f	λ_3	ω_3
1×10^{-6}	−5.475	−4.454	.3191	−11.6798	−0.890	−0.617	1.8243	−1.4115	0.290	0.694	3.8615	1.3411
1×10^{-5}	−5.826	−4.226	.3170	−11.8116	−0.982	−0.546	1.7976	−1.4290	0.195	0.850	3.7857	1.4193
1×10^{-4}	−6.474	−3.897	.3116	−12.1588	−1.143	−0.439	1.7325	−1.4740	0.050	1.181	3.5790	1.6436
5×10^{-4}	−7.274	−3.456	.2984	−12.6971	−1.332	−0.336	1.6416	−1.5439	−0.096	1.663	3.2483	2.0508
1×10^{-3}	−7.771	−3.456	.2984	−13.0769	−1.445	−0.282	1.5840	−1.5925	−0.175	2.019	3.0243	2.3715
1.65×10^{-3}	−8.209	−3.347	.2936	−13.4335	−1.546	−0.238	1.5381	−1.6386	−0.238	2.354	2.8293	2.7014
2×10^{-3}	−8.413	−3.302	.2914	−13.6031	−1.588	−0.247	1.5117	−1.6595		2.510	2.7479	2.8384
5×10^{-3}	−9.576	−3.088	.2823	−14.6231	−1.799	−0.286	1.4019	−1.7668		3.325	2.3776	3.6360
7×10^{-3}	−10.125	−3.007	.2732	−15.1260	−1.961	−0.310	1.3184	−1.8525		4.021	2.1309	4.3265
0.01	−10.824	−2.919	.2662	−15.7798	−2.224	−0.341	1.1927	−1.9928		5.270	1.7171	5.6094
0.016	−11.954	−2.802	.2555	−16.8628	−2.563	−0.368	1.0512	−2.1710		7.130	1.4627	7.4557
0.019	−12.441	−2.759	.2511	−17.3384	−2.711	−0.376	0.9967	−2.2470		8.032	1.3432	8.3680
0.0198	−12.550	−2.749	.2500	−17.4514	−2.749	−0.378	0.9830	−2.2663		8.285	1.3133	8.6241
0.0210	−12.873		.2471	−17.7754	−2.797	−0.369	0.9638	−2.3004		8.400	1.2989	8.7412

We found that $\epsilon_0 = 1.65 \times 10^{-3}$ and $\epsilon_1 = 0.0198$. See Table 2 for other numerical values.

Acknowledgements

We gratefully thank Professor Douglas Wiens for his valuable comments. This work was supported by the Physical Science Panel Direct Grant for Research of the Chinese University of Hong Kong.

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