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PROBABILITY DISTRIBUTION FUNCTIONS OF SUCCESSION QUOTAS IN THE CASE OF MARKOV DEPENDENT TRIALS

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Abstract. The exact probability distribution functions (pdf's) of the sooner and later waiting time random variables (rv's) for the succession quota problem are derived presently in the case of Markov dependent trials. This is done by means of combinatorial arguments. The probability generating functions (pgf's) of these rv's are then obtained by means of enumerating generating functions (enumerators). Obvious modifications of the proofs provide analogous results for the occurrence of frequency quotas and such a result is established regarding the pdf of a frequency and succession quotas rv. Longest success and failure runs are also considered and their joint cumulative distribution function (cdf) is obtained.

Key words and phrases: Sooner and later problems, frequency quota, succession quota, Markov dependent trials, probability distribution function, probability generating function, cumulative distribution function, binomial expressions, longest run, enumerator.

1. Introduction

Ebneshahrashoob and Sobel (1990) considered independent Bernoulli trials and studied the waiting times arising by imposing succession quotas (SQ's) and frequency quotas (FQ's) on both successes and failures. The sooner (later) case arises when one (both) of the quotas is (are) observed. New results and generalizations of this type of waiting time problems were studied by Aki (1992), Aki and Hirano (1993), Balasubramanian *et al.* (1993), Chryssaphinou *et al.* (1994), Ling (1992), Ling and Low (1993), Sobel and Ebneshahrashoob (1992) and Uchida and Aki (1995). The main scope of the aforementioned authors was the derivation of the pgf's of the SQ and/or FQ rv's, whereas a few obtained recurrences of their pdf's and Ling (1992) derived the pdf's of the FQ rv's. None of them, however, established exact formulas for the pdf's of the SQ rv's. Recently, Antzoulakos and Philippou (1996), derived exact formulas for the pdf's of the SQ rv's in terms of multinomial coefficients in the case of a binary sequence of order k and hence in the case of independent Bernoulli trials. In the latter case, they also obtained formulas in terms of binomial coefficients.

In the present paper we derive exact formulas for the pdf's of the SQ rv's in the case of Markov dependent trials in terms of binomial coefficients (see Theorems 2.1 and 2.2). We do this by means of combinatorial arguments. We also obtain their pgf's by means of enumerators (see Propositions 2.1 and 2.2). Obvious modifications of our methodology provide analogous results for the occurrence of frequency quotas and such a result is established regarding the pdf of a frequency and succession quotas rv (see Theorem 3.1). Longest success and failure runs are also considered, and their joint cdf is established (see Theorem 3.2). Finally we indicate in Section 4 how our results can be transformed in order to cover other Markov dependent models. In ending this section we note that, apart from Propositions 2.1 and 2.2 which represent new derivations of known pgf's, all main results of the present paper, namely Theorems 2.1, 2.2, 3.1 and 3.2, are new. Even the corollaries of Theorems 2.1 and 3.2 (Corollaries 2.1(a) and 3.1) are also new, and they compare well with related results of Aki and Hirano (1993) and Mohanty (1994).

2. Succession quotas

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of Markov dependent trials, each trial being either a success (1) or a failure (0), with $P(X_1 = 1) = p_0$, $P(X_1 = 0) = q_0$, and for $n \geq 1$ $P(X_{n+1} = 1 \mid X_n = 1) = p_1$, $P(X_{n+1} = 0 \mid X_n = 1) = q_1$, $P(X_{n+1} = 1 \mid X_n = 0) = p_2$ and $P(X_{n+1} = 0 \mid X_n = 0) = q_2$. Let E_1 (E_0) be the event that a success (failure) run of length k (r) occurs. Let W_S (W_L) be a rv denoting the waiting time until E_1 or (and) E_0 occurs, whichever comes sooner (later). For $j = 0, 1$, let $P_S^{(j)}(n)$ ($P_L^{(j)}(n)$) be the probability that at the n-th trial the sooner (later) event between E_1 and E_0 occurs and the sooner (later) event is E_j. Then

$$(2.1) \qquad P(W_\alpha = n) = P_\alpha^{(1)}(n) + P_\alpha^{(0)}(n), \qquad \text{for} \quad \alpha = S \text{ or } L.$$

The pgf $G_\alpha(t)$ of the rv W_α is given by

$$(2.2) \qquad G_\alpha(t) = G_\alpha^{(1)}(t) + G_\alpha^{(0)}(t), \qquad \text{for} \quad \alpha = S \text{ or } L,$$

where $G_\alpha^{(1)}(t)$ and $G_\alpha^{(0)}(t)$ are the generating functions of $P_\alpha^{(1)}(n)$ and $P_\alpha^{(0)}(n)$, respectively.

In the present section we derive the pdf's of the rv's W_S and W_L by means of combinatorial arguments, and we rederive their pgf's by means of enumerators. To this end we shall employ the following three lemmas from combinatorial analysis (see, e.g. p. 105 of Riordan (1958)).

LEMMA 2.1. *Let $Q(n, m, s)$ be the number of ways in which n identical objects can be arranged to form m groups with no group containing more than s*

objects. Then

$$Q(n,m,s)=\sum_{j=0}^{m}(-1)^{j}\binom{m}{j}\binom{n-1-js}{m-1} \quad \textit{and}$$
$$\sum_{n} Q(n,m,s)t^{n}=\left(t\frac{1-t^{s}}{1-t}\right)^{m}.$$

LEMMA 2.2. *Let $R(n,m,s)$ be the number of ways in which n identical objects can be arranged to form m groups with at least one group containing more than s objects. Then*

$$R(n,m,s)=\sum_{j=1}^{m}(-1)^{j+1}\binom{m}{j}\binom{n-1-js}{m-1} \quad \textit{and}$$
$$\sum_{n} R(n,m,s)t^{n}=\left(\frac{t}{1-t}\right)^{m}-\left(t\frac{1-t^{s}}{1-t}\right)^{m}.$$

LEMMA 2.3. *Let $M(n,m)$ be the number of ways in which n identical objects can be arranged to form m groups. Then*

$$M(n,m)=\binom{n-1}{m-1} \quad \textit{and} \quad \sum_{n} M(n,m)t^{n}=\left(\frac{t}{1-t}\right)^{m}.$$

We now treat the sooner waiting time problem.

THEOREM 2.1. *Let $Q(n,m,s)$ be as in Lemma 2.1. Then,*

$$P(W_S=n)=P_S^{(1)}(n)+P_S^{(0)}(n), \qquad n\geq \min\{k,r\},$$

where $P_S^{(1)}(k)=p_0p_1^{k-1}$, $P_S^{(0)}(r)=q_0q_2^{r-1}$, and for $n\geq 1$

$$\begin{aligned} P_S^{(1)}(n+k)=p_2p_1^{k-1}\Bigg\{\sum_{i=1}^{n}p_1^{n-i}q_2^{i}\sum_{m=1}^{i}Q(i,m,r-1) \\ \cdot\Bigg[Q(n-i,m,k-1)\frac{p_0}{p_2}\left(\frac{p_2q_1}{p_1q_2}\right)^{m} \\ +Q(n-i,m-1,k-1)\frac{q_0}{q_2}\left(\frac{p_2q_1}{p_1q_2}\right)^{m-1}\Bigg]\Bigg\} \end{aligned}$$

and

$$P_S^{(0)}(n+r) = q_1 q_2^{r-1}\left\{\sum_{i=1}^{n} q_2^{n-i} p_1^i \sum_{m=1}^{i} Q(i,m,k-1)\right.$$
$$\cdot\left[Q(n-i,m,r-1)\frac{q_0}{q_1}\left(\frac{p_2q_1}{p_1q_2}\right)^m\right.$$
$$\left.\left.+\,Q(n-i,m-1,r-1)\frac{p_0}{p_1}\left(\frac{p_2q_1}{p_1q_2}\right)^{m-1}\right]\right\}.$$

PROOF. We shall first derive $P_S^{(1)}(n+k)$, $n \geq 0$. It is obvious that $P_S^{(1)}(k) = p_0p_1^{k-1}$. Let $L_n^{(1)}$, $L_n^{(0)}$ and F_n denote, respectively, the length of the longest success run, the length of the longest failure run and the number of failures in n (≥ 1) Markov dependent trials. Let $A_{n,i}$ be the event defined by

$$A_{n,i} = \{L_n^{(1)} \leq k-1, L_n^{(0)} \leq r-1, F_n = i, X_n = 0\}, \qquad 1 \leq i \leq n.$$

We observe that

$$\begin{aligned} P_S^{(1)}(n+k) &= p_2p_1^{k-1}P(L_n^{(1)} \leq k-1, L_n^{(0)} \leq r-1, X_n = 0) \\ &= p_2p_1^{k-1}\sum_{i=1}^{n} P(A_{n,i}), \qquad n \geq 1. \end{aligned}$$

Any occurrence of the event $A_{n,i}$ is a series of i failures and $n-i$ successes ($n \geq 1, 1 \leq i \leq n$). Let m ($1 \leq m \leq i$) be the number of runs of failures in $A_{n,i}$. These runs can be formed in $Q(i,m,r-1)$ ways. There are two alternatives for the $n-i$ successes. They can form m runs in $Q(n-i,m,k-1)$ ways or $m-1$ runs in $Q(n-i,m-1,k-1)$ ways. In the first case the series can begin only with a run of successes and its probability is $p_0p_2^{m-1}p_1^{n-i-m}q_1^m q_2^{i-m}$, while in the second case the series can begin only with a run of failures and its probability is $p_2^{m-1}p_1^{n-i-m+1}q_0q_1^{m-1}q_2^{i-m}$. Then, $P_S^{(1)}(n+k)$ follows using the multiplication principle. By reasons of symmetry $P_S^{(0)}(n+r)$, $n \geq r$, can be obtained from $P_S^{(1)}(n+k)$ by replacing p_0, p_1, p_2, k and r by q_0, q_2, q_1, r and k respectively. The proof of the theorem is completed.

Balasubramanian *et al.* (1993) derived the pgf of the rv W_S and introduced a recurrence relation for calculating its pdf. In the following proposition we give a new derivation of the pgf of the rv W_S.

PROPOSITION 2.1. *Let $G_S(t)$ be the pgf of the rv W_S. Then $G_S(t) = (A + \bar{A})/Q$, where*

$$\begin{aligned} A &= (1-p_1t)(p_1t)^{k-1}[p_0t(1-q_2t) + q_0p_2t^2(1-(q_2t)^{r-1})], \\ \bar{A} &= (1-q_2t)(q_2t)^{r-1}[q_0t(1-p_1t) + p_0q_1t^2(1-(p_1t)^{k-1})], \\ Q &= (1-p_1t)(1-q_2t) - q_1p_2t^2(1-(p_1t)^{k-1})(1-(q_2t)^{r-1}). \end{aligned}$$

PROOF. We shall first derive $G_S^{(1)}(t)$. By Theorem 2.1, we have

$$G_S^{(1)}(t) = \sum_{n=k}^{\infty} P_S^{(1)}(n)t^n = \sum_{n=0}^{\infty} P_S^{(1)}(n+k)t^{n+k} = p_0 p_1^{k-1} t^k + t^k \sum_{n=1}^{\infty} P_S^{(1)}(n+k)t^n.$$

Substitute $P_S^{(1)}(n+k)$ with its equivalent formula proposed in Theorem 2.1. Note that the double summation $\sum_{n=1}^{\infty}\sum_{i=1}^{n}$ is equivalent to $\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}$, which becomes $\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}$ upon setting $n-i=j$. Set $a = p_2 q_1 (p_1 q_2)^{-1}$, $b = (p_1 t)[1-(p_1 t)^{k-1}](1-p_1 t)^{-1}$ and sum over j. Then by Lemma 2.1 we get

$$\begin{aligned} G_S^{(1)}(t) &= p_0 p_1^{k-1} t^k \\ &\quad + p_2 p_1^{k-1} t^k \sum_{i=1}^{\infty}\sum_{m=1}^{i} Q(i,m,r-1)(q_2 t)^i \left[\frac{p_0}{p_2}(ab)^m + \frac{q_0}{q_2}(ab)^{m-1}\right]. \end{aligned}$$

The double summation $\sum_{i=1}^{\infty}\sum_{m=1}^{i}$ is equivalent to $\sum_{m=1}^{\infty}\sum_{i=m}^{\infty}$. Set $c = (q_2 t)[1-(q_2 t)^{r-1}](1-q_1 t)^{-1}$ and sum over i. Then by Lemma 2.1 we get

$$G_S^{(1)}(t) = p_0 p_1^{k-1} t^k + p_2 p_1^{k-1} t^k \sum_{m=1}^{\infty}\left[\frac{p_0}{p_2}(abc)^m + c\frac{q_0}{q_2}(abc)^{m-1}\right],$$

which after some algebra gives $G_S^{(1)}(t) = A/Q$. By reasons of symmetry we have that $G_S^{(0)}(t) = \bar{A}/Q$. Then the proof of the proposition follows by (2.2).

Letting $r \to \infty$, and noting that in this case $Q(n,m,r) \to M(n,m)$ and $P_S^{(0)}(n) \to 0$, Theorem 2.1 and Proposition 2.1 reduce to the following corollary.

COROLLARY 2.1. *Let W be a rv denoting the number of trials until the occurrence of the first success run of length k in the Markov dependent trials, and let $G(t)$ be its pgf. Then,*

(a) $P(W=k) = p_0 p_1^{k-1}$, *and for* $n \geq 1$

$$\begin{aligned} P(W=n+k) &= p_2 p_1^{k-1} \sum_{i=1}^{n} p_1^{n-i} q_2^i \sum_{m=1}^{i} M(i,m) \\ &\quad \cdot \left[Q(n-i,m,k-1)\frac{p_0}{p_2}\left(\frac{p_2 q_1}{p_1 q_2}\right)^m \right. \\ &\qquad \left. + Q(n-i,m-1,k-1)\frac{q_0}{q_2}\left(\frac{p_2 q_1}{p_1 q_2}\right)^{m-1} \right]; \end{aligned}$$

(b) $$G(t) = \frac{(1-p_1 t)(p_1 t)^{k-1}[p_0 t(1-q_2 t) + q_0 p_2 t^2]}{(1-p_1 t)(1-q_2 t) - q_1 p_2 t^2 (1-(p_1 t)^{k-1})}.$$

Next, we deal with the later waiting time problem.

THEOREM 2.2. *Let $Q(n,m,s)$ and $R(n,m,s)$ be as in Lemmas* 2.1 *and* 2.2, *respectively. Then*

$$P(W_L = n) = P_L^{(1)}(n) + P_L^{(0)}(n), \qquad n \geq k + r,$$

where

$$P_L^{(1)}(n+k) = p_2 p_1^{k-1} \left\{ \sum_{i=r}^{n} p_1^{n-i} q_2^i \sum_{m=1}^{i-r+1} R(i,m,r-1) \cdot \left[Q(n-i,m,k-1) \frac{p_0}{p_2} \left(\frac{p_2 q_1}{p_1 q_2} \right)^m + Q(n-i,m-1,k-1) \frac{q_0}{q_2} \left(\frac{p_2 q_1}{p_1 q_2} \right)^{m-1} \right] \right\}, \qquad n \geq r,$$

and $P_L^{(0)}(n)$ can be obtained by $P_L^{(1)}(n)$ by replacing p_0, p_1, p_2, k and r by q_0, q_2, q_1, r and k respectively.

PROOF. It follows along the same lines as those in the proof of Theorem 2.1 by noting that

$$P_L^{(1)}(n+k) = p_2 p_1^{k-1} \sum_{i=r}^{n} P(L_n^{(1)} \leq k-1, L_n^{(0)} \geq r, F_n = i, X_n = 0), \qquad n \geq r,$$

which implies the replacement of $Q(i,m,r-1)$ by $R(i,m,r-1)$.

Following the steps of the proof of Proposition 2.1 and using Lemmas 2.1 and 2.2 we can obtain the following proposition.

PROPOSITION 2.2. *Let $G_L(t)$ be the pgf of the rv W_L. Then*

$$G_L(t) = (B/R) + (\bar{B}/\bar{R}) - (A + \bar{A})/Q,$$

where

$$\begin{aligned}
B &= (1 - p_1 t)(p_1 t)^{k-1}[p_0 t(1 - q_2 t) + q_0 p_2 t^2], \\
\bar{B} &= (1 - q_2 t)(q_2 t)^{r-1}[q_0 t(1 - p_1 t) + p_0 q_1 t^2], \\
R &= (1 - p_1 t)(1 - q_2 t) - q_1 p_2 t^2 (1 - (p_1 t)^{k-1}), \\
\bar{R} &= (1 - p_1 t)(1 - q_2 t) - q_1 p_2 t^2 (1 - (q_2 t)^{r-1}),
\end{aligned}$$

and A, $\bar{A}$ and Q are as in Proposition 2.1.

3. Frequency quotas and longest runs

In this section we discuss the sooner and later waiting time problem when frequency quotas are imposed on successes or (and) failures (see, Balasubramanian *et al.* (1993), Ebneshahrashoob and Sobel (1990), Ling (1992) and Sobel and Ebneshahrashoob (1992)). The methodology employed in the previous section can be easily modified in order to obtain expressions in terms of binomial coefficients for the pdf's of the sooner and later waiting time rv's, by noting that frequency quotas are closely related to the number $M(n,m)$. For instance, consider Markov dependent trials which are performed sequentially until either k successes in total (event $\tilde{E}_1$) or r consecutive failures (event $\tilde{E}_0$) are observed, whichever comes sooner, and denote the waiting time by $\tilde{W}_S$. In the following theorem we derive the pdf of the rv $\tilde{W}_S$.

THEOREM 3.1. *Let $Q(n,m,s)$ and $M(n,m)$ be as in Lemmas 2.1 and 2.3, respectively. Then*

$$P(\tilde{W}_S = n) = \tilde{P}_S^{(1)}(n) + \tilde{P}_S^{(0)}(n), \qquad n \geq \min\{k, r\},$$

where $\tilde{P}_S^{(1)}(k) = p_0 p_1^{k-1}$, $\tilde{P}_S^{(0)}(r) = q_0 q_2^{r-1}$, and for $n \geq 1$

$$\tilde{P}_S^{(1)}(n+k) = \sum_{m=1}^{k} M(k,m)\left[Q(n,m,r-1)\frac{q_0}{q_1}p_1^k q_2^n \left(\frac{p_2 q_1}{p_1 q_2}\right)^m + Q(n,m-1,r-1)p_0 p_1^{k-1} q_2^n \left(\frac{p_2 q_1}{p_1 q_2}\right)^{m-1}\right]$$

and

$$\tilde{P}_S^{(0)}(n+r) = \sum_{i=1}^{\min(n,k-1)} \sum_{m=1}^{i} M(i,m) \cdot \left[Q(n-i,m,r-1)q_0 p_1^i q_2^{n-i+r-1}\left(\frac{p_2 q_1}{p_1 q_2}\right)^m + Q(n-i,m-1,r-1)p_0 q_1 p_1^{i-1} q_2^{n-i+r-1}\left(\frac{p_2 q_1}{p_1 q_2}\right)^{m-1}\right].$$

PROOF. It can be shown by noting that for $n \geq 1$

$$\tilde{P}_S^{(1)}(n+k) = P(L_{n+k}^{(0)} \leq r-1, F_{n+k} = n, X_{n+k} = 1)$$

and

$$\tilde{P}_S^{(0)}(n+r) = q_1 q_2^{r-1} \sum_{i=1}^{\min(n,k-1)} P(L_n^{(0)} \leq r-1, F_n = n-i, X_n = 1).$$

Next we derive the joint cdf of the lengths of the longest success and longest failure runs in Markov dependent trials.

THEOREM 3.2. *Let $L_n^{(1)}$ and $L_n^{(0)}$ denote, respectively, the lengths of the longest success and longest failure runs in n (≥ 1) Markov dependent trials. Then*

$$\begin{aligned} P(L_n^{(1)} \leq k, L_n^{(0)} \leq r) = \zeta(n,k)p_0p_1^{n-1} + \sum_{i=1}^{n} p_1^{n-i}q_2^i \sum_{m=1}^{i} Q(i,m,r) \\ \cdot \left[Q(n-i, m-1, k)\frac{q_0}{q_2}\left(\frac{p_2q_1}{p_1q_2}\right)^{m-1} \right. \\ + Q(n-i,m,k)\left(\frac{p_0}{p_2}+\frac{q_0}{q_1}\right)\left(\frac{p_2q_1}{p_1q_2}\right)^{m} \\ \left. + Q(n-i,m+1,k)\frac{p_0}{p_1}\left(\frac{p_2q_1}{p_1q_2}\right)^{m} \right], \qquad n \geq 1, \end{aligned}$$

where $\zeta(\cdot,\cdot)$ is the function defined by $\zeta(u,v) = 1$ if $v \geq u$ and 0 otherwise.

PROOF. It can be shown by noting that whenever there are m runs of failures there are three alternatives for the successes: (a) there are $m-1$ runs of successes and the series can begin only with a run of failures, (b) there are m runs of failures and the series can begin with a run of successes or a run of failures and (c) there are $m+1$ runs of successes and the series can begin only with a run of successes.

The above theorem can be easily modified in order to cover all possible inequalities. For instance, in case we want to derive $P(L_n^{(1)} \leq k, L_n^{(0)} \geq r)$ we replace $Q(i,m,r)$ by $R(i,m,r)$.

Letting $r \to \infty$, Theorem 3.2 reduces to the following corollary.

COROLLARY 3.1. *Let L_n denote the length of the longest success run in n (≥ 1) Markov dependent trials. Then $P(L_n \leq k)$ is obtained by replacing $Q(i,m,r)$ by $M(i,m)$ in the corresponding expression in Theorem 3.2.*

4. Other Markov structures

Let $\{X_n, n = 0,1,2,\ldots\}$ be a $\{0,1\}$-valued Markov chain with $P(X_0 = 1) = \pi_1$, $P(X_0 = 0) = \pi_0$, and for $n \geq 0$ $P(X_{n+1} = 1 \mid X_n = 1) = p_{11}$, $P(X_{n+1} = 0 \mid X_n = 1) = p_{10}$, $P(X_{n+1} = 1 \mid X_n = 0) = p_{01}$ and $P(X_{n+1} = 0 \mid X_n = 0) = p_{00}$. Under this structure, Aki and Hirano (1993) derived the pgf of the rv W_α ($\alpha = S$ or L) for the SQ problem, using the generalized pgf method proposed by Ebneshahrashoob and Sobel (1990). We can obtain the pdf of the rv W_α by noting that

$$\begin{aligned} P(W_\alpha = n) &= \pi_0 P(W_\alpha = n \mid X_0 = 0) + \pi_1 P(W_\alpha = n \mid X_0 = 1) \\ &= \pi_0[P(W_\alpha^{(1)} = n \mid X_0 = 0) + P(W_\alpha^{(0)} = n \mid X_0 = 0)] \\ &\quad + \pi_1[P(W_\alpha^{(1)} = n \mid X_0 = 1) + P(W_\alpha^{(0)} = n \mid X_0 = 1)], \end{aligned}$$

where the conditional probabilities may be deduced from Theorems 2.1 and 2.2 by replacing p_1, q_1, p_2 and q_2 by p_{11}, p_{10}, p_{01} and p_{00}, respectively, and by replacing

p_0 and q_0 by p_{01} and p_{00}, respectively, if $X_0 = 0$, and by p_{11} and p_{10}, respectively, if $X_0 = 1$. Following this analysis we can modify the results of the present paper to cover cases of the above Markov structure. We note that the modifications referring to the pdf's and cdf's, including the modification of Corollary 2.1, are all new results.

Mohanty (1994) formulated Markov dependent trials as a two-coin tossing game. His transition probabilities coincide with ours of Section 2, and when his game starts with coin 1(2) this situation corresponds to $P(X_1 = 1) = p_1$ and $P(X_1 = 0) = q_1$ ($P(X_1 = 1) = p_2$ and $P(X_1 = 0) = q_2$). This analysis solves the sooner and later waiting time problems under the influence of this Markov structure. Among other results, and given that the game starts with coin i ($i = 1, 2$), Mohanty (1994) derived the pdf of the number of trials X_i needed to get k consecutive successes for the first time and the probability of the event $\{L_i(n) \leq k\}$ where $L_i(n)$ represents the length of the longest success run in n trials, all in terms of multinomial coefficients (see Mohanty (1994)'s Propositions 2.1 and 2.3 and Theorem 3.4). Our Corollaries 2.1 and 3.1, modified as indicated above, provide binomial expressions for these pdf's and probabilities.

Finally, we note that when $p_0 = p_1 = p_2 = p$ our results reduce to respective ones regarding independent Bernoulli trials. We note that in this case the binomial expressions for the pdfs of the rv's W_S and W_L are different from the ones derived by Antzoulakos and Philippou (1996).

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