Ann. Inst. Statist. Math.
Vol. 49, No. 3, 519–529 (1997)

JOINT DISTRIBUTIONS OF NUMBERS OF SUCCESS-RUNS AND FAILURES UNTIL THE FIRST CONSECUTIVE k SUCCESSES IN A BINARY SEQUENCE

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(Received February 1, 1996; revised July 1, 1996)

Abstract. Joint distributions of the numbers of failures, successes and success-runs of length less than k until the first consecutive k successes in a binary sequence were derived recently by Aki and Hirano (1995, *Ann. Inst. Statist. Math.*, **47**, 225–235). In this paper, we present an alternate derivation of these results and also use this approach to establish some additional results. Extensions of these results to binary sequences of order h are also presented.

Key words and phrases: Probability generating function, waiting time, binary sequence of order k, geometric distribution of order k.

1. Introduction

The distribution of the waiting time for the first consecutive k successes in independent Bernoulli trials, called the *geometric distribution of order* k, has been discussed by Feller (1968), Philippou *et al.* (1983), and Johnson *et al.* (1992). Aki and Hirano (1988, 1994) studied the distributions of numbers of failures, successes and overlapping number of success runs. Recently, Aki and Hirano (1995) derived joint distributions of the numbers of failures, successes and success-runs of length less than k until the first consecutive k successes for a binary sequence of order k among other cases. In this paper, we derive these joint distributions by a direct algebraic method. We also use this approach to establish some additional results in this direction. We also present extensions of these results to binary sequences of order h. Mention should also be made here to the recent paper by Dhar and Jiang (1995) in which the distribution of the finite sum of a binary sequence of order k has been studied.

First of all, a sequence $\{X_i\}_{i=0}^{\infty}$ of $\{0,1\}$-valued random variables is said to be a *binary sequence of order* k if there exist a positive integer k and k real numbers $0 < p_1, p_2, \dots, p_k < 1$ such that $X_0 = 0$ almost surely and

$$\Pr(X_n = 1 \mid X_0 = x_0, X_1 = x_1, \dots, X_{n-1} = x_{n-1}) = p_j, \tag{1.1}$$

where $j = r - [\frac{r-1}{k}]k$, and r is the smallest positive integer such that $x_{n-r} = 0$. This sequence was defined by Aki (1985) and further discussed by Hirano and Aki (1987), Aki and Hirano (1988, 1994), and Aki (1992). As pointed out by Aki and Hirano (1995), this sequence is closely related to the cluster sampling scheme discussed by Philippou (1988) and Xekalaki and Panaretos (1989), and the $(k-1)$-step Markov dependence model studied by Fu (1986) in the context of consecutive-k-out-of-n: F system.

First of all, let us refer to "1" as "Success" (S) and "0" as "Failure" (F). Let τ denote the number of trials until the first consecutive k successes in $X_1, X_2, \ldots$. Let η be the number of failures among $X_1, X_2, \ldots, X_\tau$. Let ξ_i $(i = 1, 2, \ldots, k)$ be the number of overlapping success runs of length i until τ. Let ν_i $(i = 1, 2, \ldots, k)$ be the number of success runs of length at least i until τ. Finally, let μ_i $(i = 1, 2, \ldots, k)$ be the number of non-overlapping success runs of length i until τ.

2. Joint distribution of waiting time and the numbers of failures and overlapping success runs of length i

Let $\phi_1(r, s, t_1, \ldots, t_k)$ be the joint probability generating function of $(\tau, \eta, \xi_1, \xi_2, \ldots, \xi_k)$; that is,

$$\phi_1(r, s, t_1, \ldots, t_k) = E(r^\tau s^\eta t_1^{\xi_1} \cdots t_k^{\xi_k}). \tag{2.1}$$

In order to derive the joint probability generating function $\phi_1(r, s, t_1, \ldots, t_k)$, we first note that a typical sequence of outcomes leading to the first consecutive k successes among $X_1, X_2, \ldots$, with l failures, is given by

$$\begin{array}{cccccccccc} 0 & \overbrace{1\cdots1}^{\le k-1} & 0 & \overbrace{1\cdots1}^{\le k-1} & 0 & \cdots\overbrace{1\cdots1}^{\le k-1} & 0 & \overbrace{1\cdots1}^{k}. \\ X_0 & & 1 & & 2 & & l & \end{array} \tag{2.2}$$

Starting with $X_0 \equiv 0$, it is easy to observe that the contribution of $\overbrace{11\cdots1}^{\le k-1}0$ is

$$\begin{aligned} &q_1rs + p_1q_2r^2st_1 + p_1p_2q_3r^3st_1^2t_2 \\ &\quad + \cdots + p_1p_2\cdots p_{k-1}q_kr^kst_1^{k-1}t_2^{k-2}\cdots t_{k-1} \\ &= q_1rs + \sum_{i=1}^{k-1} p_1p_2\cdots p_iq_{i+1}r^{i+1}st_1^it_2^{i-1}\cdots t_i. \end{aligned} \tag{2.3}$$

Next, we observe that the contribution of $\overbrace{1\cdots1}^{k}$ following a 0 is

$$p_1p_2\cdots p_kr^kt_1^kt_2^{k-1}\cdots t_k. \tag{2.4}$$

Upon combining these two expressions, we obtain the net contribution of the sequence in (2.2) as

$$\begin{aligned} &\phi_{1,l}(r, s, t_1, \ldots, t_k) \\ &= \left\{q_1rs + \sum_{i=1}^{k-1} p_1p_2\cdots p_iq_{i+1}r^{i+1}st_1^it_2^{i-1}\cdots t_i\right\}^l \\ &\quad \cdot p_1p_2\cdots p_kr^kt_1^kt_2^{k-1}\cdots t_k, \end{aligned} \tag{2.5}$$

which readily yields

$$(2.6)\quad \phi_1(r,s,t_1,\dots,t_k) = \sum_{l=0}^{\infty} \phi_{1,l}(r,s,t_1,\dots,t_k)$$
$$= \frac{p_1p_2\cdots p_k r^k t_1^k t_2^{k-1}\cdots t_k}{1 - q_1rs - \sum_{i=1}^{k-1} p_1p_2\cdots p_i q_{i+1} r^{i+1} s t_1^i t_2^{i-1}\cdots t_i}.$$

(2.6) has been derived by Aki and Hirano (1995); also see Johnson *et al.* (1997).

From (2.6), we obtain the following expressions for the means, variances and covariances:

$$(2.7)\quad E(\tau) = \frac{kp_1p_2\cdots p_k}{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}} + \frac{p_1p_2\cdots p_k\{q_1 + \sum_{i=1}^{k-1}(i+1)p_1\cdots p_i q_{i+1}\}}{\{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^2},$$

$$(2.8)\quad E(\eta) = \frac{\{q_1 + \sum_{i=1}^{k-1} p_1p_2\cdots p_i q_{i+1}\}p_1p_2\cdots p_k}{\{1 - q_1 - \sum_{i=1}^{k-1} p_1p_2\cdots p_i q_{i+1}\}^2},$$

$$(2.9)\quad E(\xi_j) = \frac{(k-j+1)p_1p_2\cdots p_k}{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}} + \frac{p_1p_2\cdots p_k \sum_{i=j}^{k-1}(i-j+1)p_1p_2\cdots p_i q_{i+1}}{\{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^2},$$
$$j = 1, 2, \dots, k,$$

$$(2.10)\quad \mathrm{Var}(\tau) = \frac{k(k-1)p_1\cdots p_k}{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}} + \frac{p_1p_2\cdots p_k \sum_{i=1}^{k-1}(i+1)ip_1\cdots p_i q_{i+1}}{\{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^2}$$
$$+ \frac{2kp_1\cdots p_k\{q_1 + \sum_{i=1}^{k-1}(i+1)p_1\cdots p_i q_{i+1}\}}{\{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^2} + \frac{2p_1\cdots p_k\{q_1 + \sum_{i=1}^{k-1}(i+1)p_1\cdots p_i q_{i+1}\}^2}{\{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^3}$$
$$+ E(\tau) - \{E(\tau)\}^2,$$

$$(2.11)\quad \mathrm{Var}(\eta) = \frac{2p_1\cdots p_k\{q_1 + \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^2}{\{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^3} + E(\tau) - \{E(\tau)\}^2,$$

$$(2.12)\quad \mathrm{Var}(\xi_j) = \frac{(k-j+1)(k-j)p_1\cdots p_k}{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}} + \frac{p_1\cdots p_k \sum_{i=j-1}^{k-1}(i-j+1)(i-j)p_1\cdots p_i q_{i+1}}{\{1 - q_1 - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^2}$$

$$+\frac{2(k-j+1)p_1\cdots p_k\sum_{i=j}^{k-1}(i-j+1)p_1\cdots p_iq_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^2}$$
$$+\frac{2p_1\cdots p_k\{\sum_{i=j}^{k-1}(i-j+1)p_1\cdots p_iq_{i+1}\}^2}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^3}$$
$$+E(\xi_j)-\{E(\xi_j)\}^2, \qquad j=1,2,\ldots,k,$$

(2.13)
$$\mathrm{Cov}(\tau,\eta) = \frac{kp_1\cdots p_k\{q_1+\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^2}$$
$$+\frac{p_1\cdots p_k\{q_1+\sum_{i=1}^{k-1}(i+1)p_1\cdots p_iq_{i+1}\}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^2}$$
$$+\frac{2p_1\cdots p_k\{q_1+\sum_{i=1}^{k-1}(i+1)p_1\cdots p_iq_{i+1}\}\{q_1+\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^3}$$
$$-E(\tau)E(\eta),$$

(2.14)
$$\mathrm{Cov}(\tau,\xi_j) = \frac{k(k-j+1)p_1\cdots p_k}{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}}$$
$$+\frac{kp_1\cdots p_k\sum_{i=j}^{k-1}(i-j+1)p_1\cdots p_iq_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^2}$$
$$+\frac{(k-j+1)p_1\cdots p_k\{q_1+\sum_{i=1}^{k-1}(i+1)p_1\cdots p_iq_{i+1}\}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^2}$$
$$+\frac{p_1\cdots p_k\sum_{i=j}^{k-1}(i+1)(i-j+1)p_1\cdots p_iq_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^2}$$
$$+\frac{2p_1\cdots p_k\{q_1+\sum_{i=1}^{k-1}(i+1)p_1\cdots p_iq_{i+1}\}\sum_{i=j}^{k-1}(i-j+1)p_1\cdots p_iq_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^3}$$
$$-E(\tau)E(\xi_j), \qquad j=1,2,\ldots,k,$$

(2.15)
$$\mathrm{Cov}(\eta,\xi_j) = \frac{(k-j+1)p_1\cdots p_k\{q_1+\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^2}$$
$$+\frac{p_1\cdots p_k\sum_{i=j}^{k-1}(i-j+1)p_1\cdots p_iq_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^2}$$
$$+\frac{2p_1\cdots p_k\{q_1+\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}\sum_{i=j}^{k-1}(i-j+1)p_1\cdots p_iq_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}\}^3}$$
$$-E(\eta)E(\xi_j), \qquad j=1,2,\ldots,k,$$

and

$$
\begin{aligned}
(2.16)\quad & \operatorname{Cov}(\xi_j, \xi_{j'}) \\
&= \frac{(k-j+1)(k-j'+1)p_1\cdots p_k}{1-q_1-\sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}} \\
&\quad + \frac{(k-j+1)p_1\cdots p_k \sum_{i=j'}^{k-1}(i-j'+1)p_1\cdots p_i q_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^2} \\
&\quad + \frac{(k-j'+1)p_1\cdots p_k \sum_{i=j}^{k-1}(i-j+1)p_1\cdots p_i q_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^2} \\
&\quad + \frac{p_1\cdots p_k \sum_{i=j'}^{k-1}(i-j+1)(i-j'+1)p_1\cdots p_i q_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^2} \\
&\quad + \frac{2p_1\cdots p_k \sum_{i=j}^{k-1}(i-j+1)p_1\cdots p_i q_{i+1} \sum_{i=j'}^{k-1}(i-j'+1)p_1\cdots p_i q_{i+1}}{\{1-q_1-\sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1}\}^3} \\
&\quad - E(\xi_j)E(\xi_{j'}), \qquad 1 \le j < j' \le k.
\end{aligned}
$$

Furthermore, by rewriting (2.6) as

$$
\begin{aligned}
(2.17)\quad & \left\{1 - q_1 rs - \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1} r^{i+1} s t_1^i t_2^{i-1}\cdots t_i\right\} \phi_1(r, s, t_1, \dots, t_k) \\
&= p_1 p_2 \cdots p_k r^k t_1^k t_2^{k-1} \cdots t_k
\end{aligned}
$$

and comparing the coefficients of $r^a s^b t_1^{c_1} \cdots t_k^{c_k}$ on both sides of (2.17), we obtain a recurrence relation for the joint probability mass function as follows:

$$
(2.18)\quad P(k, 0, k, k-1, \dots, 1) = p_1 p_2 \cdots p_k
$$

and

$$
\begin{aligned}
(2.19)\quad & P(a, b, c_1, \dots, c_k) - q_1 P(a-1, b-1, c_1, \dots, c_k) \\
&\quad - \sum_{i=1}^{k-1} p_1 \cdots p_i q_{i+1} \\
&\quad \cdot P(a-i-1, b-1, c_1-i, c_2-i+1, \dots, c_i-1, c_{i+1}, \dots, c_k) \\
&= 0.
\end{aligned}
$$

In the above equations, $P(a, b, c_1, \dots, c_k)$ denotes the joint probability $\Pr(\tau = a, \eta = b, \xi_1 = c_1, \dots, \xi_k = c_k)$.

From (2.17), we also obtain the following recurrence relation for the joint moments:

$$
\begin{aligned}
(2.20)\quad & E[\tau^g \eta^h \xi_1^{l_1} \cdots \xi_k^{l_k}] - q_1 E[(\tau+1)^g (\eta+1)^h \xi_1^{l_1} \cdots \xi_k^{l_k}] \\
&\quad - \sum_{i=1}^{k-1} p_1 \cdots p_i q_{i+1} \\
&\quad \cdot E[(\tau+i+1)^g (\eta+1)^h (\xi_1+i)^{l_1} \cdots (\xi_i+1)^{l_i} \xi_{i+1}^{l_{i+1}} \cdots \xi_k^{l_k}] \\
&= p_1 p_2 \cdots p_k k^{g+l_1} (k-1)^{l_2} \cdots 1^{l_k} I(h=0),
\end{aligned}
$$

where $I(h-0)$ is the indicator function taking the value 1 if $h=0$ and the value 0 if $h \neq 0$. The recurrence relation in (2.20) can be used to derive the expressions of the means, variances and covariances presented in Eqs. (2.7)–(2.16). In addition, (2.20) can be applied recursively to determine the higher-order moments and joint moments as well.

3. Joint distribution of waiting time and the numbers of failures and success runs of length at least i

Let $\phi_2(r, s, t_1, \ldots, t_k)$ be the joint probability generating function of $(\tau, \eta, \nu_1, \ldots, \nu_k)$; that is,

$$\phi_2(r, s, t_1, \ldots, t_k) = E(r^\tau s^\eta t_1^{\nu_1} \cdots t_k^{\nu_k}). \tag{3.1}$$

In order to derive the joint probability generating function $\phi_2(r, s, t_1, \ldots, t_k)$, we first note that a typical sequence of outcomes leading to the first consecutive k successes among $X_1, X_2, \ldots$, with l failures, is given once again by (2.2). Starting with $X_0 \equiv 0$, it is easy to observe that the contribution of $\overbrace{11\cdots1}^{\leq k-1}0$ is

$$\begin{aligned} &q_1 rs + p_1 q_2 r^2 s t_1 + p_1 p_2 q_3 r^3 s t_1 t_2 + \cdots + p_1 p_2 \cdots p_{k-1} q_k r^k s t_1 t_2 \cdots t_k \\ &\quad = q_1 rs + \sum_{i=1}^{k-1} p_1 p_2 \cdots p_i q_{i+1} r^{i+1} s t_1 t_2 \cdots t_i. \end{aligned} \tag{3.2}$$

Next, we observe that the contribution of $\overbrace{11\cdots1}^{k}$ following a 0 is

$$p_1 p_2 \cdots p_k r^k t_1 t_2 \cdots t_k. \tag{3.3}$$

Upon combining these two expressions, we obtain the net contribution of the sequence in (2.2) as

$$\begin{aligned} &\phi_{2,l}(r, s, t_1, \ldots, t_k) \\ &\quad = \left\{ q_1 rs + \sum_{i=1}^{k-1} p_1 \cdots p_i q_{i+1} r^{i+1} s t_1 \cdots t_i \right\}^l p_1 \cdots p_k r^k t_1 \cdots t_k, \end{aligned} \tag{3.4}$$

which readily yields

$$\begin{aligned} \phi_2(r, s, t_1, \ldots, t_k) &= \sum_{l=0}^{\infty} \phi_{2,l}(r, s, t_1, \ldots, t_k) \\ &= \frac{p_1 p_2 \cdots p_k r^k t_1 t_2 \cdots t_k}{1 - q_1 rs - \sum_{i=1}^{k-1} p_1 \cdots p_i q_{i+1} r^{i+1} s t_1 \cdots t_i}. \end{aligned} \tag{3.5}$$

(3.5) has been derived by Aki and Hirano (1995).

Upon rewriting (3.5) as

$$(3.6)\qquad \left\{1 - q_1 rs - \sum_{i=1}^{k-1} p_1 \cdots p_i q_{i+1} r^{i+1} s t_1 \cdots t_i \right\} \phi_2(r, s, t_1, \ldots, t_k) = p_1 p_2 \cdots p_k r^k t_1 t_2 \cdots t_k,$$

we readily obtain the following recurrence relation:

$$(3.7)\qquad P(k, 0, 1, \ldots, 1) = p_1 p_2 \cdots p_k$$

and

$$(3.8)\qquad P(a, b, c_1, \ldots, c_k) - q_1 P(a-1, b-1, c_1, \ldots, c_k) - \sum_{i=1}^{k-1} p_1 \cdots p_i q_{i+1} \cdot P(a-i-1, b-1, c_1-1, \ldots, c_i-1, c_{i+1}, \ldots, c_k) = 0.$$

In the above equations, $P(a, b, c_1, \ldots, c_k)$ denotes the joint probability $\Pr(\tau = a, \eta = b, \nu_1 = c_1, \ldots, \nu_k = c_k)$.

From (3.6), we also obtain the following recurrence relation for the joint moments:

$$(3.9)\qquad E[\tau^g \eta^h \nu_1^{l_1} \cdots \nu_k^{l_k}] - q_1 E[(\tau+1)^g (\eta+1)^h \nu_1^{l_1} \cdots \nu_k^{l_k}] - \sum_{i=1}^{k-1} p_1 \cdots p_i q_{i+1} \cdot E[(\tau+i+1)^g (\eta+1)^h (\nu_1+1)^{l_1} \cdots (\nu_i+1)^{l_i} \nu_{i+1}^{l_{i+1}} \cdots \nu_k^{l_k}] = p_1 p_2 \cdots p_k k^g I(h=0),$$

where $I(h = 0)$ is the indicator function taking the value 1 if $h = 0$ and the value 0 if $h \neq 0$. The recurrence relation in (3.9) can be used to derive the expressions of the means, variances and covariances of the variables. In addition, (3.9) can be applied recursively to determine the higher-order moments and joint moments as well.

4. Joint distribution of waiting time and the numbers of failures and non-overlapping success runs of length i

Let $\phi_3(r, s, t_1, \ldots, t_k)$ be the joint probability generating function of $(\tau, \eta, \mu_1, \ldots, \mu_k)$; that is,

$$(4.1)\qquad \phi_3(r, s, t_1, \ldots, t_k) = E(r^\tau s^\eta t_1^{\mu_1} \cdots t_k^{\mu_k}).$$

In order to derive the joint probability generating function $\phi_3(r, s, t_1, \ldots, t_k)$, we first note that a typical sequence of outcomes leading to the first consecutive k

successes among $X_1, X_2, \dots,$ with l failures, is given once again by (2.2). Starting with $X_0 \equiv 0$, it is easy to observe that the contribution of $\overbrace{11\cdots1}^{\le k-1}0$ is

$$\text{(4.2)} \quad q_1rs + p_1q_2r^2st_1 + p_1p_2q_3r^3st_1^2t_2 + \cdots + p_1p_2\cdots p_{k-1}q_kr^kst_1^{k-1}t_2^{[(k-1)/2]}\cdots t_{k-1} = q_1rs + \sum_{i=1}^{k-1} p_1\cdots p_iq_{i+1}r^{i+1}st_1^it_2^{[i/2]}\cdots t_i.$$

Next, we observe that the contribution of $\overbrace{1\cdots1}^{k}$ following a 0 is

$$\text{(4.3)} \quad p_1p_2\cdots p_kr^kt_1^kt_2^{[k/2]}\cdots t_{k-1}^{[k/(k-1)]}t_k.$$

Upon combining these two expressions, we obtain the net contribution of the sequence in (2.2) as

$$\text{(4.4)} \quad \phi_{3,l}(r,s,t_1,\dots,t_k) = \left\{q_1rs + \sum_{i=1}^{k-1} p_1\cdots p_iq_{i+1}r^{i+1}st_1^it_2^{[i/2]}\cdots t_i\right\}^l \cdot p_1\cdots p_kr^kt_1^kt_2^{[k/2]}\cdots t_k,$$

which readily yields

$$\text{(4.5)} \quad \phi_3(r,s,t_1,\dots,t_k) = \sum_{l=0}^{\infty}\phi_{3,l}(r,s,t_1,\dots,t_k) = \frac{p_1p_2\cdots p_kr^kt_1^kt_2^{[k/2]}\cdots t_{k-1}^{[k/(k-1)]}t_k}{1-q_1rs-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}r^{i+1}st_1^it_2^{[i/2]}\cdots t_{i-1}^{[i/(i-1)]}t_i}.$$

(4.5) has been derived by Aki and Hirano (1995).

Upon rewriting (4.5) as

$$\text{(4.6)} \quad \left\{1-q_1rs-\sum_{i=1}^{k-1}p_1\cdots p_iq_{i+1}r^{i+1}st_1^it_2^{[i/2]}\cdots t_{i-1}^{[i/(i-1)]}t_i\right\}\phi_3(r,s,t_1,\dots,t_k) = p_1p_2\cdots p_kr^kt_1^kt_2^{[k/2]}\cdots t_{k-1}^{[k/(k-1)]}t_k,$$

we readily obtain the following recurrence relation:

$$\text{(4.7)} \quad P\left(k,0,k,\left[\frac{k}{2}\right],\dots,\left[\frac{k}{k-1}\right],1\right) = p_1p_2\cdots p_k$$

and

$$
\begin{aligned}
(4.8)\quad P(a,b,c_1,\ldots,c_k) &- q_1 P(a-1,b-1,c_1,\ldots,c_k)\\
&- \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1} P\Big(a-i-1, b-1, c_1-i, c_2-\left[\frac{i}{2}\right],\ldots,\\
&\qquad\qquad c_{i-1}-\left[\frac{i}{i-1}\right], c_i-1, c_{i+1},\ldots,c_k\Big)\\
&= 0.
\end{aligned}
$$

In the above equations, $P(a,b,c_1,\ldots,c_k)$ denotes the joint probability $\Pr(\tau = a, \eta = b, \mu_1 = c_1,\ldots,\mu_k = c_k)$.

From (4.6), we also obtain the following recurrence relation for the joint moments:

$$
\begin{aligned}
(4.9)\quad E[\tau^g\eta^h\mu_1^{l_1}\cdots\mu_k^{l_k}] &- q_1 E[(\tau+1)^g(\eta+1)^h\mu_1^{l_1}\cdots\mu_k^{l_k}]\\
&- \sum_{i=1}^{k-1} p_1\cdots p_i q_{i+1} E\Big[(\tau+i+1)^g(\eta+1)^h(\mu_1+i)^{l_1}\\
&\qquad \cdot\left(\mu_2+\left[\frac{i}{2}\right]\right)^{l_2}\cdots\left(\mu_{i-1}+\left[\frac{i}{i-1}\right]\right)^{l_{i-1}}\\
&\qquad \cdot(\mu_i+1)^{l_i}\mu_{i+1}^{l_{i+1}}\cdots\mu_k^{l_k}\Big]\\
&= p_1p_2\cdots p_k k^{g+l_1}\left[\frac{k}{2}\right]^{l_2}\cdots\left[\frac{k}{k-1}\right]^{l_{k-1}} I(h=0),
\end{aligned}
$$

where $I(h = 0)$ is the indicator function taking the value 1 if $h = 0$ and the value 0 if $h \neq 0$. The recurrence relation in (4.9) can be used to derive the expressions of the means, variances and covariances of the variables. In addition, (4.9) can be applied recursively to determine the higher-order moments and joint moments as well.

5. Extensions to binary sequences of order h

The results presented in Sections 2–4 can be extended to binary sequences of order h. First of all, it is easy to observe that the results do not change if $h > k$. However, for the case when $h < k$, the results do change. For convenience, let $[\frac{k-1}{h}] = a$ and $k - 1 = ah + b$, where $0 \leq b \leq h - 1$. Then, for the derivation of the joint probability generating function in (2.1), starting with $X_0 = 0$, we observe that the contribution of $\overbrace{11\cdots1}^{\leq k-1}0$ is

$$
\begin{aligned}
(5.1)\quad D = &\sum_{i=0}^{a-1}(p_1\cdots p_h)^i\sum_{j=0}^{h-1}(p_1\cdots p_j q_{j+1}) r^{ih+j+1} s t_1^{ih+j} t_2^{ih+j-1}\cdots t_{ih+j}\\
&+ (p_1\cdots p_h)^a\sum_{j=0}^{b} p_1\cdots p_j q_{j+1} r^{k-b+j} s t_1^{k-b+j-1} t_2^{k-b+j-2}\cdots t_{k-b+j-1}.
\end{aligned}
$$

Next, with $[\frac{k}{h}] = c$ and $k = ch + d$, we observe that the contribution of $\overbrace{11\cdots 1}^{k}$ following a 0 is

$$(5.2) \qquad (p_1 \cdots p_h)^c p_1 \cdots p_d r^k t_1^k t_2^{k-1} \cdots t_k.$$

We then obtain the joint probability generating function $\phi_1^*(r, s, t_1, \dots, t_k)$ as

$$(5.3) \qquad \phi_1^*(r, s, t_1, \dots, t_k) = \frac{(p_1 \cdots p_h)^c p_1 \cdots p_d r^k t_1^k t_2^{k-1} \cdots t_k}{1 - D},$$

where D is as given in (5.1).

It may be noted here that the expression in (5.3) may also be derived directly from (2.6) by setting $(p_1, \cdots, p_k) = (\overbrace{p_1, \dots, p_h, \dots, p_1, \dots, p_h}^{c \text{ times}}, p_1, \dots, p_d)$.

Similar extensions can be made to all the other results presented in Sections 2–4.

Further, the results presented in this section will be quite useful in developing inference procedures for the order h of a binary sequence. Work is currently in progress in this direction.

Acknowledgements

The author thanks Mrs. Debbie Iscoe for the excellent typing of the manuscript and the Natural Sciences and Engineering Research Council of Canada for funding this research. Thanks are also due to two referees for some valuable comments which led to an improvement in the presentation of this paper.

References

Aki, S. (1985). Discrete distributions of order k on a binary sequence, *Ann. Inst. Statist. Math.*, **37**, 205–224.

Aki, S. (1992). Waiting time problems for a sequence of discrete random variables, *Ann. Inst. Statist. Math.*, **44**, 363–378.

Aki, S. and Hirano, K. (1988). Some characteristics of the binomial distribution of order k and related distributions, *Statistical Theory and Data Analysis II*, *Proceedings of the 2nd Pacific Area Statistical Conference* (ed. K. Matusita), 211–222, North-Holland, Amsterdam.

Aki, S. and Hirano, K. (1994). Distributions of numbers of failures and successes until the first consecutive k successes, *Ann. Inst. Statist. Math.*, **46**, 193–202.

Aki, S. and Hirano, K. (1995). Joint distributions of numbers of success-runs and failures until the first consecutive k successes, *Ann. Inst. Statist. Math.*, **47**, 225–235.

Dhar, S. K. and Jiang, X. (1995). Probability bounds on the finite sum of the binary sequence of order k, *J. Appl. Probab.*, **32**, 1014–1027.

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, Vol. I, 3rd ed., Wiley, New York.

Fu, J. C. (1986). Reliability of consecutive-k-out-of-n: F systems with $(k-1)$-step Markov dependence, *IEEE Trans. Reliability*, **R-35**, 602–606.

Hirano, K. and Aki, S. (1987). Properties of the extended distributions of order k, *Statist. Probab. Lett.*, **6**, 67–69.

Johnson, N. L., Kotz, S. and Kemp, A. W. (1992). *Univariate Discrete Distributions*, 2nd ed., Wiley, New York.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1997). *Discrete Multivariate Distributions*, Wiley, New York.

Philippou, A. N. (1988). On multiparameter distributions of order k, *Ann. Inst. Statist. Math.*, **40**, 467–475.

Philippou, A. N., Georghiou, C. and Philippou, G. N. (1983). A generalized geometric distribution and some of its properties, *Statist. Probab. Lett.*, **1**, 171–175.

Xekalaki, E. and Panaretos, J. (1989). On some distributions arising in inverse cluster sampling, *Comm. Statist. Theory Methods*, **18**(1), 355–366.